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Preface

Here are my online notes for my Linear Algebra course that I teach here at Lamar University. Despite the fact that these are my "class notes" they should be accessible to anyone wanting to learn Linear Algebra or needing a refresher.

These notes do assume that the reader has a good working knowledge of basic Algebra. This set of notes is fairly self contained but there is enough Algebra type problems (arithmetic and occasionally solving equations) that can show up that not having a good background in Algebra can cause the occasional problem.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn Linear Algebra I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn't covered in class.
2. In general I try to work problems in class that are different from my notes. However, with a Linear Algebra course while I can make up the problems off the top of my head there is no guarantee that they will work out nicely or the way I want them to. So, because of that my class work will tend to follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head. Also, I often don't have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren't worked in class due to time restrictions.
3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these notes up, but the reality is that I can't anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I've not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.
4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.

Systems of Equations and Matrices

Introduction

We will start this chapter off by looking at the application of matrices that almost every book on Linear Algebra starts off with, solving systems of linear equations. Looking at systems of equations will allow us to start getting used to the notation and some of the basic manipulations of matrices that we'll be using often throughout these notes.

Once we've looked at solving systems of linear equations we'll move into the basic arithmetic of matrices and basic matrix properties. We'll also take a look at a couple of other ideas about matrices that have some nice applications to the solution to systems of equations.

One word of warning about this chapter, and in fact about this complete set of notes for that matter, we'll start out in the first section or to doing a lot of the details in the problems, but towards the end of this chapter and into the remaining chapters we will leave many of the details to you to check. We start off by doing lots of details to make sure you are comfortable working with matrices and the various operations involving them. However, we will eventually assume that you've become comfortable with the details and can check them on your own. At that point we will quit showing many of the details.

Here is a listing of the topics in this chapter.

Inverse Matrices and Elementary Matrices – Here we'll define the inverse and take a look at some of its properties. We'll also introduce the idea of Elementary Matrices.

Finding Inverse Matrices – In this section we'll develop a method for finding inverse matrices.

Special Matrices – We will introduce Diagonal, Triangular and Symmetric matrices in this section.

LU-Decompositions – In this section we'll introduce the LU-Decomposition a way of "factoring" certain kinds of matrices.

Systems Revisited – Here we will revisit solving systems of equations. We will take a look at how inverse matrices and LU-Decompositions can help with the solution process. We'll also take a look at a couple of other ideas in the solution of systems of equations.

Systems of Equations

Let's start off this section with the definition of a **linear equation**. Here are a couple of examples of linear equations.

$$6x - 8y + 10z = 3 \qquad\qquad 7x_1 - \frac{5}{9}x_2 = -1$$

In the second equation note the use of the subscripts on the variables. This is a common notational device that will be used fairly extensively here. It is especially useful when we get into the general case(s) and we won't know how many variables (often called unknowns) there are in the equation.

So, just what makes these two equations linear? There are several main points to notice. First, the unknowns only appear to the first power and there aren't any unknowns in the denominator of a fraction. Also notice that there are no products and/or quotients of unknowns. All of these ideas are required in order for an equation to be a linear equation. Unknowns only occur in numerators, they are only to the first power and there are no products or quotients of unknowns.

The most general linear equation is,

$$a_1 x_1 + a_2 x_2 + \cdots a_n x_n = b \qquad \textbf{(1)}$$

where there are *n* unknowns, $x_1, x_2, \ldots, x_n$, and $a_1, a_2, \ldots, a_n, b$ are all known numbers.

Next we need to take a look at the **solution set** of a single linear equation. A solution set (or often just solution) for (1) is a set of numbers $t_1, t_2, \ldots, t_n$ so that if we set $x_1 = t_1$, $x_2 = t_2$, … , $x_n = t_n$ then (1) will be satisfied. By satisfied we mean that if we plug these numbers into the left side of (1) and do the arithmetic we will get *b* as an answer.

The first thing to notice about the solution set to a single linear equation that contains at least two variables with non-zero coefficents is that we will have an infinite number of solutions. We will also see that while there are infinitely many possible solutions they are all related to each other in some way.

Note that if there is one or less variables with non-zero coefficients then there will be a single solution or no solutions depending upon the value of *b*.

Let's find the solution set's for the two linear equations given at the start of this section.

Example 1 Find the solution set for each of the following linear equations.

(a) $7x_1 - \frac{5}{9}x_2 = -1$

(b) $6x - 8y + 10z = 3$

Solution

(b) The first thing that we'll do here is solve the equation for one of the two unknowns. It doesn't matter which one we solve for, but we'll usually try to pick the one that will mean the least amount (or at least simpler) work. In this case it will probably be slightly easier to solve for x_1 so let's do that.

$$\begin{aligned} 7x_1 - \frac{5}{9}x_2 &= -1 \\ 7x_1 &= \frac{5}{9}x_2 - 1 \\ x_1 &= \frac{5}{63}x_2 - \frac{1}{7} \end{aligned}$$

Now, what this tells us is that if we have a value for x_2 then we can determine a corresponding value for x_1. Since we have a single linear equation there is nothing to restrict our choice of x_2 and so we we'll let x_2 be any number. We will usually write this as $x_2 = t$, where *t* is any number. Note that there is nothing special about the *t*, this is just the letter that I usually use in these cases. Others often use *s* for this letter and, of course, you could choose it to be just about anything as long as it's not a letter representing one of the unknowns in the equation (*x* in this case).

Once we've "chosen" x_2 we'll write the general solution set as follows,

$$x_1 = \frac{5}{63}t - \frac{1}{7} \qquad\qquad x_2 = t$$

So, just what does this tell us as far as actual number solutions go? We'll choose any value of *t* and plug in to get a pair of numbers x_1 and x_2 that will satisfy the equation. For instance picking a couple of values of *t* completely at random gives,

$$t = 0: \qquad x_1 = -\frac{1}{7},\ \ x_2 = 0$$

$$t = 27: \qquad x_1 = \frac{5}{63}(27) - \frac{1}{7} = 2,\ \ x_2 = 27$$

We can easily check that these are in fact solutions to the equation by plugging them back into the equation.

$$t = 0: \qquad 7\left(-\frac{1}{7}\right) - \frac{5}{9}(0) = -1$$

$$t = 27: \qquad 7(2) - \frac{5}{9}(27) = -1$$

So, for each case when we plugged in the values we got for x_1 and x_2 we got -1 out of the equation as we were supposed to.

Note that since there an infinite number of choices for *t* there are in fact an infinite number of possible solutions to this linear equation.

(b) We'll do this one with a little less detail since it works in essentially the same manner. The fact that we now have three unknowns will change things slightly but not overly much. We will first solve the equation for one of the variables and again it won't matter which one we chose to solve for.

$$10z = 3 - 6x + 8y$$

$$z = \frac{3}{10} - \frac{3}{5}x + \frac{4}{5}y$$

In this case we will need to know values for both *x* and *y* in order to get a value for *z*. As with the first case, there is nothing in this problem to restrict out choices of *x* and *y*. We can therefore let them be any number(s). In this case we'll choose $x = t$ and $y = s$. Note that we chose different letters here since there is no reason to think that both *x* and *y* will have exactly the same value (although it *is* possible for them to have the same value).

The solution set to this linear equation is then,

$$x = t \qquad y = s \qquad z = \frac{3}{10} - \frac{3}{5}t + \frac{4}{5}s$$

So, if we choose any values for *t* and *s* we can get a set of number solutions as follows.

$$x = 0 \qquad y = -2 \qquad z = \frac{3}{10} - \frac{3}{5}(0) + \frac{4}{5}(-2) = -\frac{13}{10}$$

$$x = -\frac{3}{2} \qquad y = 5 \qquad z = \frac{3}{10} - \frac{3}{5}\left(-\frac{3}{2}\right) + \frac{4}{5}(5) = \frac{26}{5}$$

As with the first part if we take either set of three numbers we can plug them into the equation to verify that the equation will be satisfied. We'll do one of them and leave the other to you to check.

$$6\left(\frac{-3}{2}\right)-8\left(5\right)+10\left(\frac{26}{5}\right)=-9-40+52=3$$

The variables that we got to choose for values for (x_2 in the first example and x and y in the second) are sometimes called **free variables**.

We now need to start talking about the actual topic of this section, systems of linear equations. A **system of linear equations** is nothing more than a collection of two or more linear equations. Here are some examples of systems of linear equations.

$$\begin{aligned}2x+3y&=9\\ x-2y&=-13\end{aligned}\qquad\qquad \begin{aligned}4x_1-5x_2+x_3&=9\\ -x_1+10x_3&=-2\\ 7x_1-x_2-4x_3&=5\end{aligned}\qquad\qquad \begin{aligned}6x_1+x_2&=9\\ -5x_1-3x_2&=7\\ 3x_1-10x_1&=-4\end{aligned}$$

$$\begin{aligned}x_1-x_2+x_3-x_4+x_5&=1\\ 3x_1+2x_2-x_4+9x_2&=0\\ 7x_1+10x_2+3x_3+6x_4-9x_5&=-7\end{aligned}$$

As we can see from these examples systems of equation can have any number of equations and/or unknowns. The system may have the same number of equations as unknowns, more equations than unknowns, or fewer equations than unknowns.

A solution set to a system with *n* unknowns, $x_1, x_2, \ldots, x_n$, is a set of numbers, $t_1, t_2, \ldots, t_n$, so that if we set $x_1 = t_1$, $x_2 = t_2$, … , $x_n = t_n$ then all of the equations in the system will be satisfied. Or, in other words, the set of numbers $t_1, t_2, \ldots, t_n$ is a solution to each of the individual equations in the system.

For example, $x=-3$, $y=5$ is a solution to the first system listed above,

$$\begin{aligned}2x+3y&=9\\ x-2y&=-13\end{aligned}\qquad\qquad \textbf{(2)}$$

because,

$$2(-3)+3(5)=9 \qquad\qquad \& \qquad\qquad (-3)-2(5)=-13$$

However, $x=-15$, $y=-1$ is not a solution to the system because,

$$2(-15)+3(-1)=-33\neq 9 \qquad\qquad \& \qquad\qquad (-15)-2(-1)=-13$$

We can see from these calculations that $x=-15$, $y=-1$ is NOT a solution to the first equation, but it IS a solution to the second equation. Since this pair of numbers is not a solution to both of the equations in (2) it is not a solution to the system. The fact that it's a solution to one of them isn't material. In order to be a solution to the system the set of numbers must be a solution to each and every equation in the system.

It is completely possible as well that a system will not have a solution at all. Consider the following system.

$$\begin{aligned} x - 4y &= 10 \\ x - 4y &= -3 \end{aligned} \qquad \textbf{(3)}$$

It is clear (hopefully) that this system of equations can't possibly have a solution. A solution to this system would have to be a pair of numbers x and y so that if we plugged them into each equation it will be a solution to each equation. However, since the left side is identical this would mean that we'd need an x and a y so that $x - 4y$ is both 10 and -3 for the exact same pair of numbers. This clearly can't happen and so (3) does not have a solution.

Likewise, it is possible for a system to have more than one solution, although we do need to be careful here as we'll see. Let's take a look at the following system.

$$\begin{aligned} -2x + y &= 8 \\ 8x - 4y &= -32 \end{aligned} \qquad \textbf{(4)}$$

We'll leave it to you to verify that all of the following are four of the infinitely many solutions to the first equation in this system.

$$x = 0, y = 8 \qquad x = -3, y = 2, \qquad x = -4, y = 0 \qquad x = 5, y = 18$$

Recall from our work above that there will be infinitely many solutions to a single linear equation.

We'll also leave it to you to verify that these four solutions are also four of the infinitely many solutions to the second equation in (4).

Let's investigate this a little more. Let's just find the solution to the first equation (we'll worry about the second equation in a second). Following the work we did in Example 1 we can see that the infinitely many solutions to the first equation in (4) are

$$x = t \qquad y = 2t + 8, \quad t \text{ is any number}$$

Now, if we also find just the solutions to the second equation in (4) we get

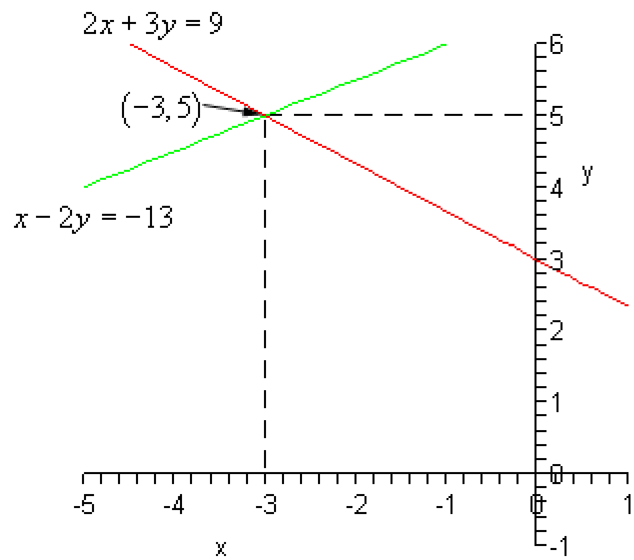
$$x = t \qquad y = 2t + 8, \quad t \text{ is any number}$$

These are exactly the same! So, this means that if we have an actual numeric solution (found by choosing t above…) to the first equation it will be guaranteed to also be a solution to the second equation and so will be a solution to the system (4). This means that we in fact have infinitely many solutions to (4).

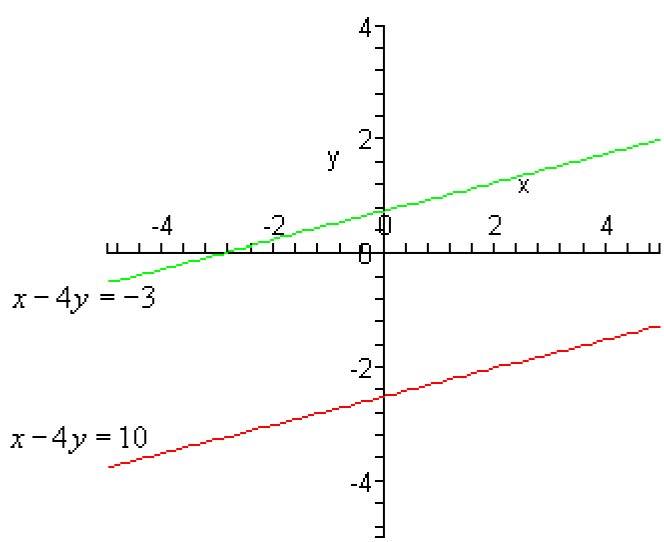
Let's take a look at the three systems we've been working with above in a little more detail. This will allow us to see a couple of nice facts about systems.

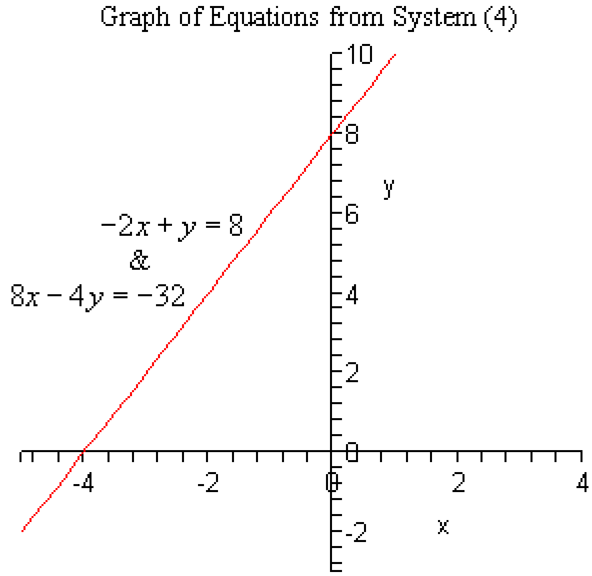
Since each of the equations in (2),(3), and (4) are linear in two unknowns (x and y) the graph of each of these equations is that of a line. Let's graph the pair of equations from each system on the same graph and see what we get.

Graph of Equations from System (2)



Graph of Equations from System (3)





From the graph of the equations for system (2) we can see that the two lines intersect at the point $(-3,5)$ and notice that, as a point, this is the solution to the system as well. In other words, in this case the solution to the system of two linear equations and two unknowns is simply the intersection point of the two lines.

Note that this idea is validated in the solution to systems (3) and (4). System (3) has no solution and we can see from the graph of these equations that the two lines are parallel and hence will never intersect. In system (4) we had infinitely many solutions and the graph of these equations shows us that they are in fact the same line, or in some ways the "intersect" at an infinite number of points.

Now, to this point we've been looking at systems of two equations with two unknowns but some of the ideas we saw above can be extended to general systems of *n* equations with *m* unknowns.

First, there is a nice geometric interpretation to the solution of systems with equations in two or three unknowns. Note that the number of equations that we've got won't matter the interpretation will be the same.

If we've got a system of linear equations in two unknowns then the solution to the system represents the point(s) where all (not some but ALL) the lines will intersect. If there is no solution then the lines given by the equations in the system will not intersect at a single point. Note in the no solution case if there are more than two equations it may be that any two of the equations will intersect, but there won't be a single point were all of the lines will intersect.

If we've got a system of linear equations in three unknowns then the graphs of the equations will be planes in 3D-space and the solution to the system will represent the point(s) where all the planes will intersect. If there is no solution then there are no point(s) where all the planes given by the equations of the system will intersect. As with lines, it may be in this case that any two of the planes will intersect, but there won't be any point where all of the planes intersect at that point.

On a side note we should point out that lines can intersect at a single point or if the equations give the same line we can think of them as intersecting at infinitely many points. Planes can intersect at a point or on a line (and so will have infinitely many intersection points) and if the equations give the same plane we can think of the planes as intersecting at infinitely many places.

We need to be a little careful about the infinitely many intersection points case. When we're dealing with equations in two unknowns and there are infinitely many solutions it means that the equations in the system all give the same line. However, when dealing with equations in three unknowns and we've got infinitely many solutions we can have one of two cases. Either we've got planes that intersect along a line, or the equations will give the same plane.

For systems of equations in more than three variables we can't graph them so we can't talk about a "geometric" interpretation, but we can still say that a solution to such a system will represent the point(s) where all the equations will "intersect" even if we can't visualize such an intersection point.

From the geometric interpretation of the solution to two equations in two unknowns we now that we have one of three possible solutions. We will have either no solution (the lines are parallel), one solution (the lines intersect at a single point) or infinitely many solutions (the equations are the same line). There is simply no other possible number of solutions since two lines that intersect will either intersect exactly once or will be the same line. It turns out that this is in fact the case for a general system.

Theorem 1 Given a system of n equations and m unknowns there will be one of three possibilities for solutions to the system.

1. There will be no solution.
2. There will be exactly one solution.
3. There will be infinitely many solutions.

If there is no solution to the system we call the system **inconsistent** and if there is at least one solution to the system we call it **consistent**.

Now that we've got some of the basic ideas about systems taken care of we need to start thinking about how to use linear algebra to solve them. Actually that's not quite true. We're not going to do any solving until the next section. In this section we just want to get some of the basic notation and ideas involved in the solving process out of the way before we actually start trying to solve them.

We're going to start off with a *simplified* way of writing the system of equations. For this we will need the following general system of n equations and m unknowns.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= b_n \end{aligned} \tag{5}$$

In this system the unknowns are $x_1, x_2, \ldots, x_m$ and the a_{ij} and b_i are known numbers. Note as well how we've subscripted the coefficients of the unknowns (the a_{ij}). The first subscript, i, denotes the equation that the subscript is in and the second subscript, j, denotes the unknown that it multiples. For instance, a_{36} would be in the coefficient of x_6 in the third equation.

Any system of equations can be written as an **augmented matrix**. A matrix is just a rectangular array of numbers and we'll be looking at these in great detail in this course so don't worry too much at this point about what a matrix is. Here is the augmented matrix for the general system in (5).

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2m} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} & b_n \end{bmatrix}$$

Each row of the augmented matrix consists of the coefficients and constant on the right of the equal sign form a given equation in the system. The first row is for the first equation, the second row is for the second equation *etc*. Likewise each of the first n columns of the matrix consists of the coefficients from the unknowns. The first column contains the coefficients of x_1, the second column contains the coefficients of x_2, etc. The final column (the n+1$^{\text{st}}$ column) contains all the constants on the right of the equal sign. Note that the augmented part of the name arises because we tack the b_i's onto the matrix. If we don't tack those on and we just have

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

and we call this the **coefficient matrix** for the system.

Example 2 Write down the augmented matrix for the following system.

$$
\begin{aligned}
3x_1 - 10x_2 + 6x_3 - x_4 &= 3 \\
x_1 + 9x_3 - 5x_4 &= -12 \\
-4x_1 + x_2 - 9x_3 + 2x_4 &= 7
\end{aligned}
$$

Solution

There really isn't too much to do here other than write down the system.

$$
\begin{bmatrix} 3 & -10 & 6 & -1 & 3 \\ 1 & 0 & 9 & -5 & -12 \\ -4 & 1 & -9 & 2 & 7 \end{bmatrix}
$$

Notice that the second equation did not contain an x_2 and so we consider its coefficient to be zero.

Note as well that given an augmented matrix we can always go back to a system of equations.

Example 3 For the given augmented matrix write down the corresponding system of equations.

$$
\begin{bmatrix} 4 & -1 & 1 \\ -5 & -8 & 4 \\ 9 & 2 & -2 \end{bmatrix}
$$

Solution

So since we know each row corresponds to an equation we have three equations in the system. Also, the first two columns represent coefficients of unknowns and so we'll have two unknowns while the third column consists of the constants to the right of the equal sign. Here's the system that corresponds to this augmented matrix.

$$
\begin{aligned}
4x_1 - x_2 &= 1 \\
-5x_1 - 8x_2 &= 4 \\
9x_1 + 2x_2 &= -2
\end{aligned}
$$

There is one final topic that we need to discuss in this section before we move onto actually solving systems of equation with linear algebra techniques. In the next section where we will actually be solving systems our main tools will be the three **elementary row operations**. Each of these operations will operate on a row (which shouldn't be too surprising given the name…) in the augmented matrix and since each row in the augmented matrix corresponds to an equation these operations have equivalent operations on equations.

Here are the three row operations, their equivalent equation operations as well as the notation that we'll be using to denote each of them.

Row Operation	**Equation Operation**	**Notation**
Multiply row *i* by the constant *c*	Multiply equation *i* by the constant *c*	cR_i

Interchange rows i and j	Interchange equations i and j	$R_i \leftrightarrow R_j$
Add c times row i to row j	Add c times equation i to equation j	$R_j + cR_i$

The first two operations are fairly self explanatory. The third is also a fairly simple operation however there are a couple things that we need to make clear about this operation. First, in this operation only row (equation) j actually changes. Even though we are multiplying row (equation) i by c that is done in our heads and the results of this multiplication are added to row (equation) j. Also, when we say that we add c time a row to another row we really mean that we add corresponding entries of each row.

Let's take a look at some examples of these operations in action.

Example 4 Perform each of the indicated row operations on given augmented matrix.

$$\begin{bmatrix} 2 & 4 & -1 & -3 \\ 6 & -1 & -4 & 10 \\ 7 & 1 & -1 & 5 \end{bmatrix}$$

(a) $-3R_1$

(b) $\frac{1}{2}R_2$

(c) $R_1 \leftrightarrow R_3$

(d) $R_2 + 5R_3$

(e) $R_1 - 3R_2$

Solution

In each of these we will actually perform both the row and equation operation to illustrate that they are actually the same operation and that the new augmented matrix we get is in fact the correct one. For reference purposes the system corresponding to the augmented matrix give for this problem is,

$$\begin{aligned} 2x_1 + 4x_2 - x_3 &= -3 \\ 6x_1 - x_2 - 4x_3 &= 10 \\ 7x_1 + x_2 - x_3 &= 5 \end{aligned}$$

Note that at each part we will go back to the original augmented matrix and/or system of equations to perform the operation. In other words, we won't be using the results of the previous part as a starting point for the current operation.

(a) Okay, in this case we're going to multiply the first row (equation) by -3. This means that we will multiply each element of the first row by -3 or each of the coefficients of the first equation by -3. Here is the result of this operation.

$$\begin{bmatrix} -6 & -12 & 3 & 9 \\ 6 & -1 & -4 & 10 \\ 7 & 1 & -1 & 5 \end{bmatrix} \qquad \Leftrightarrow \qquad \begin{aligned} -6x_1 - 12x_2 + 3x_3 &= 9 \\ 6x_1 - x_2 - 4x_3 &= 10 \\ 7x_1 + x_2 - x_3 &= 5 \end{aligned}$$

(b) This is similar to the first one. We will multiply each element of the second row by one-half or each coefficient of the second equation by one-half. Here are the results of this operation.

$$\begin{bmatrix} 2 & 4 & -1 & -3 \\ 3 & -\frac{1}{2} & -2 & 5 \\ 7 & 1 & -1 & 5 \end{bmatrix} \qquad \Leftrightarrow \qquad \begin{aligned} 2x_1 + 4x_2 - x_3 &= -3 \\ 3x_1 - \frac{1}{2}x_2 - 2x_3 &= 5 \\ 7x_1 + x_2 - x_3 &= 5 \end{aligned}$$

Do not get excited about the fraction showing up. Fractions are going to be a fact of life with much of the work that we're going to be doing so get used to seeing them.

Note that often in cases like this we will say that we divided the second row by 2 instead of multiplied by one-half.

(c) In this case were just going to interchange the first and third row or equation.

$$\begin{bmatrix} 7 & 1 & -1 & 5 \\ 6 & -1 & -4 & 10 \\ 2 & 4 & -1 & -3 \end{bmatrix} \qquad \Leftrightarrow \qquad \begin{aligned} 7x_1 + x_2 - x_3 &= 5 \\ 6x_1 - x_2 - 4x_3 &= 10 \\ 2x_1 + 4x_2 - x_3 &= -3 \end{aligned}$$

(d) Okay, we now need to work an example of the third row operation. In this case we will add 5 times the third row (equation) to the second row (equation).

So, for the row operation, in our heads we will multiply the third row times 5 and then add each entry of the results to the corresponding entry in the second row.

Here are the individual computations for this operation.

$$\begin{aligned} &1^{st} \text{ entry : } 6 + (5)(7) = 41 \\ &2^{nd} \text{ entry : } -1 + (5)(1) = 4 \\ &3^{rd} \text{ entry : } -4 + (5)(-1) = -9 \\ &4^{th} \text{ entry : } 10 + (5)(5) = 35 \end{aligned}$$

For the corresponding equation operation we will multiply the third equation by 5 to get,

$$35x_1 + 5x_2 - 5x_3 = 25$$

then add this to the second equation to get,

$$41x_1 - 4x_2 - 9x_3 = 35$$

Putting all this together gives and remembering that it's the second row (equation) that we're actually changing here gives,

$$\begin{bmatrix} 2 & 4 & -1 & -3 \\ 41 & -4 & -9 & 35 \\ 7 & 1 & -1 & 5 \end{bmatrix} \qquad \Leftrightarrow \qquad \begin{aligned} 2x_1 + 4x_2 - x_3 &= -3 \\ 41x_1 - 4x_2 - 9x_3 &= 35 \\ 7x_1 + x_2 - x_3 &= 5 \end{aligned}$$

It is important to remember that when multiplying the third row (equation) by 5 we are doing it in our head and don't actually change the third row (equation).

(e) In this case we'll not go into the detail that we did in the previous part. Most of these types of operations are done almost completely in our head and so we'll do that here as well so we can start getting used to it.

In this part we are going to subtract 3 times the second row (equation) from the first row (equation). Here are the results of this operation.

$$\begin{bmatrix} -16 & 7 & 11 & -33 \\ 6 & -1 & -4 & 10 \\ 7 & 1 & -1 & 5 \end{bmatrix} \qquad \Leftrightarrow \qquad \begin{aligned} -16x_1 + 7x_2 + 11x_3 &= -33 \\ 6x_1 - x_2 - 4x_3 &= 10 \\ 7x_1 + x_2 - x_3 &= 5 \end{aligned}$$

It is important when doing this work in our heads to be careful of minus signs. In operations such as this one there are often a lot of them and it easy to lose track of one or more when you get in a hurry.

Okay, we've not got most of the basics down that we'll need to start solving systems of linear equations using linear algebra techniques so it's time to move onto the next section.

Solving Systems of Equations

In this section we are going to take a look at using linear algebra techniques to solve a system of linear equations. Once we have a couple of definitions out of the way we'll see that the process is a fairly simple one. Well, it's fairly simple to write down the process anyway. Applying the process is fairly simple as well but for large systems can take quite a few steps. So, let's get the definitions out of the way.

A matrix (any matrix, not just an augmented matrix) is said to be in **reduced row-echelon form** if it satisfies all four of the following conditions.

1. If there are any rows of all zeros then they are at the bottom of the matrix.
2. If a row does not consist of all zeros then its first non-zero entry (*i.e.* the left most non-zero entry) is a 1. This 1 is called a **leading 1**.
3. In any two successive rows, neither of which consists of all zeroes, the leading 1 of the lower row is to the right of the leading 1 of the higher row.
4. If a column contains a leading 1 then all the other entries of that column are zero.

A matrix (again any matrix) is said to be in **row-echelon form** if it satisfies items 1 – 3 of the reduced row-echelon form definition.

Notice from these definitions that a matrix that is in reduced row-echelon form is also in row-echelon form while a matrix in row-echelon form may or may not be in reduced row-echelon form.

Example 1 The following matrices are all in row-echelon form.

$$\begin{bmatrix} 1 & -6 & \underline{9} & \underline{1} & 0 \\ 0 & 0 & 1 & \underline{-4} & -5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & \underline{5} \\ 0 & 1 & \underline{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \underline{-8} & 10 & \underline{5} & -3 \\ 0 & 1 & 13 & \underline{9} & 12 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

None of the matrices in the previous example are in reduced row-echelon form. The entries that are preventing these matrices from being in reduced row-echelon form are highlighted in red and underlined (for those without color printers...). In order for these matrices to be in reduced row-echelon form all of these highlighted entries would need to be zeroes.

Notice that we didn't highlight the entries above the 1 in the fifth column of the third matrix. Since this 1 is not a leading 1 (*i.e.* the leftmost non-zero entry) we don't need the numbers above it to be zero in order for the matrix to be in reduced row-echelon form.

Example 2 The following matrices are all in reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -7 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 9 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 16 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

In the second matrix on the first row we have all zeroes in the entries. This is perfectly acceptable and so don't worry about it. This matrix is in reduced row-echelon form, the fact that it doesn't have any non-zero entries does not change that fact since it satisfies the conditions. Also, in the second matrix of the second row notice that the last column does not have zeroes above the 1 in that column. That is perfectly acceptable since the 1 in that column is not a leading 1 for the fourth row.

Notice from Examples 1 and 2 that the only real difference between row-echelon form and reduced row-echelon form is that a matrix in row-echelon form is only required to have zeroes below a leading 1 while a matrix in reduced row-echelon from must have zeroes both below and above a leading 1.

Okay, let's now start thinking about how to use linear algebra techniques to solve systems of linear equations. The process is actually quite simple. To solve a system of equations we will first write down the augmented matrix for the system. We will then use elementary row operations to reduce the augmented matrix to either row-echelon form or to reduced row-echelon form. Any further work that we'll need to do will depend upon where we stop.

If we go all the way to reduced row-echelon form then in many cases we will not need to do any further work to get the solution and in those time where we do need to do more work we will generally not need to do much more work. Reducing the augmented matrix to reduced row-echelon form is called **Gauss-Jordan Elimination**.

If we stop at row-echelon form we will have a little more work to do in order to get the solution, but it is generally fairly simple arithmetic. Reducing the augmented matrix to row-echelon form and then stopping is called **Gaussian Elimination**.

At this point we should work a couple of examples.

Example 3 Use Gaussian Elimination and Gauss-Jordan Elimination to solve the following system of linear equations.

$$\begin{aligned} -2x_1 + x_2 - x_3 &= 4 \\ x_1 + 2x_2 + 3x_3 &= 13 \\ 3x_1 + x_3 &= -1 \end{aligned}$$

Solution

Since we're asked to use both solution methods on this system and in order to for a matrix to be in reduced row-echelon form the matrix must also be in row-echelon form. Therefore, we'll start off by putting the augmented matrix in row-echelon form, then stop to find the solution. This will be Gaussian Elimination. After doing that we'll go back and pick up from row-echelon form and further reduce the matrix to reduced row echelon form and at this point we'll have performed Gauss-Jordan Elimination.

So, let's start off by getting the augmented matrix for this system.

$$\begin{bmatrix} -2 & 1 & -1 & 4 \\ 1 & 2 & 3 & 13 \\ 3 & 0 & 1 & -1 \end{bmatrix}$$

As we go through the steps in this first example we'll mark the entry(s) that we're going to be looking at in each step in red so that we don't lose track of what we're doing. We should also point out that there are many different paths that we can take to get this matrix into row-echelon form and each path may well produce a different row-echelon form of the matrix. Keep this in mind as you work these problems. The path that you take to get this matrix into row-echelon form should be the one that you find the easiest and that may not be the one that the person next to you finds the easiest. Regardless of which path you take you are only allowed to use the three elementary row operations that we looked in the previous section.

So, with that out of the way we need to make the leftmost non-zero entry in the top row a one. In this case we could use any three of the possible row operations. We could divide the top row by -2 and this would certainly change the red "-2" into a one. However, this will also introduce fractions into the matrix and while we often can't avoid them let's not put them in before we need to.

Next, we could take row three and add it to row one, or we could take three times row 2 and add it to row one. Either of these would also change the red "-2" into a one. However, this row operation is the one that is most prone to arithmetic errors so while it would work let's not use it unless we need to.

This leaves interchanging any two rows. This is an operation that won't always work here to get a 1 into the spot we want, but when it does it will usually be the easiest operation to use. In this case we've already got a one in the leftmost entry of the second row so let's just interchange the first and second rows and we'll get a one in the leftmost spot of the first row pretty much for free. Here is this operation.

$$\begin{bmatrix} -2 & 1 & -1 & 4 \\ 1 & 2 & 3 & 13 \\ 3 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 & 3 & 13 \\ -2 & 1 & -1 & 4 \\ 3 & 0 & 1 & -1 \end{bmatrix}$$

Now, the next step we'll need to take is changing the two numbers in the first column under the leading 1 into zeroes. Recall that as we move down the rows the leading 1 MUST move off to the right. This means that the two numbers under the leading 1 in the first column will need to become zeroes. Again, there are often several row operations that can be done to do this. However, in most cases adding multiples of the row containing the leading 1 (the first row in this case) onto the rows we need to have zeroes is often the easiest. Here are the two row operations that we'll do in this step.

$$\begin{bmatrix} 1 & 2 & 3 & 13 \\ -2 & 1 & -1 & 4 \\ 3 & 0 & 1 & -1 \end{bmatrix} \begin{array}{c} R_2 + 2R_1 \\ R_3 - 3R_1 \\ \rightarrow \end{array} \begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 5 & 5 & 30 \\ 0 & -6 & -8 & -40 \end{bmatrix}$$

Notice that since each operation changed a different row we went ahead and performed both of them at the same time. We will often do this when multiple operations will all change different rows.

We now need to change the red "5" into a one. In this case we'll go ahead and divide the second row by 5 since this won't introduce any fractions into the matrix and it will give us the number we're looking for.

$$\begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 5 & 5 & 30 \\ 0 & -6 & -8 & -40 \end{bmatrix} \xrightarrow{\frac{1}{5}R_2} \begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & -6 & -8 & -40 \end{bmatrix}$$

Next, we'll use the third row operation to change the red "-6" into a zero so the leading 1 of the third row will move to the right of the leading 1 in the second row. This time we'll be using a multiple of the second row to do this. Here is the work in this step.

$$\begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & -6 & -8 & -40 \end{bmatrix} \xrightarrow{R_3 + 6R_2} \begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & -2 & -4 \end{bmatrix}$$

Notice that in both steps were we needed to get zeroes below a leading 1 we added multiples of the row containing the leading 1 to the rows in which we wanted zeroes. This will always work in this case. It may be possible to use other row operations, but the third can always be used in these cases.

The final step we need to get the matrix into row-echelon form is to change the red "-2" into a one. To do this we don't really have a choice here. Since we need the leading one in the third row to be in the third or fourth column (*i.e.* to the right of the leading one in the second column) we MUST retain the zeroes in the first and second column of the third row.

Interchanging the second and third row would definitely put a one in the third column of the third row, however, it would also change the zero in the second column which we can't allow. Likewise we could add the first row to the third row and again this would put a one in the third column of the third row, but this operation would also change both of the zeroes in front of it which can't be allowed.

Therefore, our only real choice in this case is to divide the third row by -2. This will retain the zeroes in the first and second column and change the entry in the third column into a one. Note that this step will often introduce fractions into the matrix, but at this point that can't be avoided. Here is the work for this step.

$$\begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & -2 & -4 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_3} \begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

At this point the augmented matrix is in row-echelon form. So if we're going to perform Gaussian Elimination on this matrix we'll stop and go back to equations. Doing this gives,

$$\begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 13 \\ x_2 + x_3 &= 6 \\ x_3 &= 2 \end{aligned}$$

At this point solving is quite simple. In fact we can see from this that $x_3 = 2$. Plugging this into the second equation gives $x_2 = 4$. Finally, plugging both of these into the first equation gives $x_1 = -1$. Summarizing up the solution to the system is,

$$x_1 = -1 \qquad x_2 = 4 \qquad x_3 = 2$$

This substitution process is called **back substitution**.

Now, let's pick back up at the row-echelon form of the matrix and further reduce the matrix into reduced row-echelon form. The first step in doing this will be to change the numbers above the leading 1 in the third row into zeroes. Here are the operations that will do that for us.

$$\begin{bmatrix} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \begin{array}{c} R_1 - 3R_3 \\ R_2 - R_3 \\ \rightarrow \end{array} \quad \begin{bmatrix} 1 & 2 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

The final step is then to change the red "2" above the leading one in the second row into a zero. Here is this operation.

$$\begin{bmatrix} 1 & 2 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \begin{array}{c} R_1 - 2R_2 \\ \rightarrow \end{array} \quad \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

We are now in reduced row-echelon form so all we need to do to perform Gauss-Jordan Elimination is to go back to equations.

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{l} x_1 = -1 \\ x_2 = 4 \\ x_3 = 2 \end{array}$$

We can see from this that one of the nice consequences to Gauss-Jordan Elimination is that when there is a single solution to the system there is no work to be done to find the solution. It is generally given to us for free. Note as well that it is the same solution as the one that we got by using Gaussian Elimination as we should expect.

Before we proceed with another example we need to give a quick fact. As was pointed out in this example there are many paths we could take to do this problem. It was also noted that the path we chose would affect the row-echelon form of the matrix. This will not be true for the reduced row-echelon form however. There is only one reduced row-echelon form of a given matrix no matter what path we chose to take to get to that point.

If we know ahead of time that we are going to go to reduced row-echelon form for a matrix we will often take a different path than the one used in the previous example. In the previous example we first got the matrix in row-echelon form by getting zeroes under the leading 1's and then went back and put the matrix in reduced row-echelon form by getting zeroes above the leading 1's. If we know ahead of time that we're going to want reduced row-echelon form we can just take care of the matrix in a column by column basis in the following manner. We first get a leading 1 in the correct column then instead of using this to convert only the numbers below it to zero we can use it to convert the numbers both above and below to zero. In this way once we reach the last column and take care of it of course we will be in reduced row-echelon form.

We should also point out the differences between Gauss-Jordan Elimination and Gaussian Elimination. With Gauss-Jordan Elimination there is more matrix work that needs to be performed in order to get the augmented matrix into reduced row-echelon form, but there will be less work required in order to get the solution. In fact, if there's a single solution then the solution will be given to us for free. We will see however, that if there are infinitely many solutions we will still have a little work to do in order to arrive at the solution. With Gaussian Elimination we have less matrix work to do since we are only reducing the augmented matrix to row-echelon form. However, we will always need to perform back substitution in order to get the solution. Which method you use will probably depend on which you find easier.

Okay let's do some more examples. Since we've done one example in excruciating detail we won't be bothering to put as much detail into the remaining examples. All operations will be shown, but the explanations of each operation will not be given.

Example 4 Solve the following system of linear equations.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -2 \\ -x_1 + x_2 - 2x_3 &= 3 \\ 2x_1 - x_2 + 3x_3 &= 1 \end{aligned}$$

Solution

First, the instructions to this problem did not specify which method to use so we'll need to make a decision. No matter which method we chose we will need to get the augmented matrix down to row-echelon form so let's get to that point and then see what we've got. If we've got something easy to work with we'll stop and do Gaussian Elimination and if not we'll proceed to reduced row-echelon form and do Gauss-Jordan Elimination.

So, let's start with the augmented matrix and then proceed to put it into row-echelon form and again we're not going to put in quite the detail in this example as we did with the first one. So, here is the augmented matrix for this system.

$$\begin{bmatrix} 1 & -2 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 3 & 1 \end{bmatrix}$$

and here is the work to put it into row-echelon form.

$$\begin{bmatrix} 1 & -2 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 3 & 1 \end{bmatrix} \begin{matrix} R_2 + R_1 \\ R_3 - 2R_1 \\ \rightarrow \end{matrix} \begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & -1 & 1 & 1 \\ 0 & 3 & -3 & 5 \end{bmatrix} \begin{matrix} -R_2 \\ \rightarrow \end{matrix} \begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 3 & -3 & 5 \end{bmatrix}$$

$$\begin{matrix} R_3 - 3R_2 \\ \rightarrow \end{matrix} \begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 8 \end{bmatrix} \begin{matrix} \frac{1}{8}R_3 \\ \rightarrow \end{matrix} \begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Okay, we're now in row-echelon form. Let's go back to equation and see what we've got.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -2 \\ x_2 - x_3 &= -1 \\ 0 &= 1 \end{aligned}$$

Hmmmm. That last equation doesn't look correct. We've got a couple of possibilities here. We've either just managed to prove that 0=1 (and we know that's not true), we've made a mistake (always possible, but we haven't in this case) or there's another possibility we haven't thought of yet.

Recall from Theorem 1 in the previous section that a system has one of three possibilities for a solution. Either there is no solution, one solution or infinitely many solutions. In this case we've got no solution. When we go back to equations and we get an equation that just clearly can't be true such as the third equation above then we know that we've got not solution.

Note as well that we didn't really need to do the last step above. We could have just as easily arrived at this conclusion by looking at the second to last matrix since 0=8 is just as incorrect as 0=1.

So, to close out this problem, the official answer that there is no solution to this system.

In order to see how a simple change in a system can lead to a totally different type of solution let's take a look at the following example.

Example 5 Solve the following system of linear equations.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -2 \\ -x_1 + x_2 - 2x_3 &= 3 \\ 2x_1 - x_2 + 3x_3 &= -7 \end{aligned}$$

Solution

The only difference between this system and the previous one is the -7 in the third equation. In the previous example this was a 1.

Here is the augmented matrix for this system.

$$\begin{bmatrix} 1 & -2 & 3 & -2 \\ -1 & 1 & -2 & 3 \\ 2 & -1 & 3 & -7 \end{bmatrix}$$

Now, since this is essentially the same augmented matrix as the previous example the first few steps are identical and so there is no reason to show them here. After taking the same steps as above (we won't need the last step this time) here is what we arrive at.

$$\begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For some good practice you should go through the steps above and make sure you arrive at this matrix.

In this case the last line converts to the equation

$$0 = 0$$

and this is a perfectly acceptable equation because after all zero is in fact equal to zero! In other words, we shouldn't get excited about it.

At this point we could stop convert the first two lines of the matrix to equations and find a solution. However, in this case it will actually be easier to do the one final step to go to reduced row-echelon form. Here is that step.

$$\begin{bmatrix} 1 & -2 & 3 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 + 2R_2} \begin{bmatrix} 1 & 0 & 1 & -4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We are now in reduced row-echelon form so let's convert to equations and see what we've got.

$$\begin{aligned} x_1 + x_3 &= -4 \\ x_2 - x_3 &= -1 \end{aligned}$$

Okay, we've got more unknowns than equations and in many cases this will mean that we have infinitely many solutions. To see if this is the case for this example let's notice that each of the equations has an x_3 in it and so we can solve each equation for the remaining variable in terms of x_3 as follows.

$$\begin{aligned} x_1 &= -4 - x_3 \\ x_2 &= -1 + x_3 \end{aligned}$$

So, we can choose x_3 to be any value we want to, and hence it is a **free variable** (recall we saw these in the previous section), and each choice of x_3 will give us a different solution to the system. So, just like in the previous section when we'll rename the x_3 and write the solution as follows,

$$x_1 = -4 - t \qquad x_2 = -1 + t \qquad x_3 = t \qquad t \text{ is any number}$$

We therefore get infinitely many solutions, one for each possible value of *t* and since *t* can be any real number there are infinitely many choices for *t*.

Before moving on let's first address the issue of why we used Gauss-Jordan Elimination in the previous example. If we'd used Gaussian Elimination (which we definitely could have used) the system of equations would have been.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -4 \\ x_2 - x_3 &= -1 \end{aligned}$$

To arrive at the solution we'd have to solve the second equation for x_2 first and then substitute this into the first equation before solving for x_1. In my mind this is more work and work that I'm more likely to make an arithmetic mistake than if we'd just gone to reduced row-echelon form in the first place as we did in the solution.

There is nothing wrong with using Gaussian Elimination on a problem like this, but the back substitution is definitely more work when we've got infinitely many solutions than when we've got a single solution.

Okay, to this point we've worked nothing but systems with the same number of equations and unknowns. We need to work a couple of other examples where this isn't the case so we don't get too locked into this kind of system.

Example 6 Solve the following system of linear equations.

$$\begin{aligned} 3x_1 - 4x_2 &= 10 \\ -5x_1 + 8x_2 &= -17 \\ -3x_1 + 12x_2 &= -12 \end{aligned}$$

Solution

So, let's start with the augmented matrix and reduce it to row-echelon form and see if what we've got is nice enough to work with or if we should go the extra step(s) to get to reduced row-echelon form. Let's start with the augmented matrix.

$$\begin{bmatrix} 3 & -4 & 10 \\ -5 & 8 & -17 \\ -3 & 12 & -12 \end{bmatrix}$$

Notice that this time in order to get the leading 1 in the upper left corner we're probably going to just have to divide the row by 3 and deal with the fractions that will arise. Do not go to great lengths to avoid fractions, they are a fact of life with these problems and so while it's okay to try to avoid them, sometimes it's just going to be easier to deal with it and work with them.

So, here's the work for reducing the matrix to row-echelon form.

$$\begin{bmatrix} 3 & -4 & 10 \\ -5 & 8 & -17 \\ -3 & 12 & -12 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & -\frac{4}{3} & \frac{10}{3} \\ -5 & 8 & -17 \\ -3 & 12 & -12 \end{bmatrix} \begin{matrix} R_2+5R_1 \\ R_3+3R_1 \\ \rightarrow \end{matrix} \begin{bmatrix} 1 & -\frac{4}{3} & \frac{10}{3} \\ 0 & \frac{4}{3} & -\frac{1}{3} \\ 0 & 8 & -2 \end{bmatrix}$$

$$\xrightarrow{\frac{3}{4}R_2} \begin{bmatrix} 1 & -\frac{4}{3} & \frac{10}{3} \\ 0 & 1 & -\frac{1}{4} \\ 0 & 8 & -2 \end{bmatrix} \xrightarrow{R_3-8R_2} \begin{bmatrix} 1 & -\frac{4}{3} & \frac{10}{3} \\ 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

Okay, we're in row-echelon form and it looks like if we go back to equations at this point we'll need to do one quick back substitution involving numbers and so we'll go ahead and stop here at this point and do Gaussian Elimination.

Here are the equations we get from the row-echelon form of the matrix and the back substitution.

$$x_1 - \frac{4}{3}x_2 = \frac{10}{3} \qquad \Rightarrow \qquad x_1 = \frac{10}{3} + \frac{4}{3}\left(-\frac{1}{4}\right) = 3$$
$$x_2 = -\frac{1}{4}$$

So, the solution to this system is,

$$x_1 = 3 \qquad x_2 = -\frac{1}{4}$$

Example 7 Solve the following system of linear equations.

$$\begin{aligned} 7x_1 + 2x_2 - 2x_3 - 4x_4 + 3x_5 &= 8 \\ -3x_1 - 3x_2 + 2x_4 + x_5 &= -1 \\ 4x_1 - x_2 - 8x_3 + 20x_5 &= 1 \end{aligned}$$

Solution

First, let's notice that we are guaranteed to have infinitely many solutions by the fact above since we've got more equations than unknowns. Here's the augmented matrix for this system.

$$\begin{bmatrix} 7 & 2 & -2 & -4 & 3 & 8 \\ -3 & -3 & 0 & 2 & 1 & -1 \\ 4 & -1 & -8 & 0 & 20 & 1 \end{bmatrix}$$

In this example we can avoid fractions in the first row simply by adding twice the second row to the first to get our leading 1 in that row. So, with that as our initial step here's the work that will put this matrix into row-echelon form.

$$\begin{bmatrix} 7 & 2 & -2 & -4 & 3 & 8 \\ -3 & -3 & 0 & 2 & 1 & -1 \\ 4 & -1 & -8 & 0 & 20 & 1 \end{bmatrix} \xrightarrow{R_1+2R_2} \begin{bmatrix} 1 & -4 & -2 & 0 & 5 & 6 \\ -3 & -3 & 0 & 2 & 1 & -1 \\ 4 & -1 & -8 & 0 & 20 & 1 \end{bmatrix}$$

$$\begin{array}{c} R_2+3R_1 \\ R_3-4R_1 \\ \rightarrow \end{array}\begin{bmatrix} 1 & -4 & -2 & 0 & 5 & 6 \\ 0 & -15 & -6 & 2 & 16 & 17 \\ 0 & 15 & 0 & 0 & 0 & -23 \end{bmatrix} \begin{array}{c} R_2 \leftrightarrow R_3 \\ \rightarrow \end{array} \begin{bmatrix} 1 & -4 & -2 & 0 & 5 & 6 \\ 0 & 15 & 0 & 0 & 0 & -23 \\ 0 & -15 & -6 & 2 & 16 & 17 \end{bmatrix}$$

$$\begin{array}{c} R_3+R_2 \\ \rightarrow \end{array}\begin{bmatrix} 1 & -4 & -2 & 0 & 5 & 6 \\ 0 & 15 & 0 & 0 & 0 & -23 \\ 0 & 0 & -6 & 2 & 16 & -6 \end{bmatrix} \begin{array}{c} \frac{1}{15}R_2 \\ -\frac{1}{6}R_3 \\ \rightarrow \end{array} \begin{bmatrix} 1 & -4 & -2 & 0 & 5 & 6 \\ 0 & 1 & 0 & 0 & 0 & -\frac{23}{15} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{8}{3} & 1 \end{bmatrix}$$

We are now in row-echelon form. Notice as well that in several of the steps above we took advantage of the form of several of the rows to simplify the work somewhat and in doing this we did several of the steps in a different order than we've done to this point. Remember that there are no set paths to take through these problems!

Because of the fractions that we've got here we're going to have some work to do regardless of whether we stop here and do Gaussian Elimination or go the couple of extra steps in order to do Gauss-Jordan Elimination. So with that in mind let's go all the way to reduced row-echelon form so we can say that we've got another example of that in the notes. Here's the remaining work.

$$\begin{bmatrix} 1 & -4 & -2 & 0 & 5 & 6 \\ 0 & 1 & 0 & 0 & 0 & -\frac{23}{15} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{8}{3} & 1 \end{bmatrix} \begin{array}{c} R_1+2R_3 \\ \rightarrow \end{array} \begin{bmatrix} 1 & -4 & 0 & -\frac{2}{3} & -\frac{1}{3} & 8 \\ 0 & 1 & 0 & 0 & 0 & -\frac{23}{15} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{8}{3} & 1 \end{bmatrix}$$

$$\begin{array}{c} R_1+4R_2 \\ \rightarrow \end{array} \begin{bmatrix} 1 & 0 & 0 & -\frac{2}{3} & -\frac{1}{3} & \frac{28}{15} \\ 0 & 1 & 0 & 0 & 0 & -\frac{23}{15} \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{8}{3} & 1 \end{bmatrix}$$

We're now in reduced row-echelon form and so let's go back to equations and see what we've got.

$$\begin{aligned} x_1-\frac{2}{3}x_4-\frac{1}{3}x_5=\frac{28}{15} \qquad &\Rightarrow \qquad x_1=\frac{28}{15}+\frac{2}{3}x_4+\frac{1}{3}x_5 \\ x_2=-\frac{23}{15} \qquad & \\ x_3-\frac{1}{3}x_4-\frac{8}{3}x_5=1 \qquad &\Rightarrow \qquad x_3=1+\frac{1}{3}x_4+\frac{8}{3}x_5 \end{aligned}$$

So, we've got two free variables this time, x_4 and x_5, and notice as well that unlike any of the other infinite solution cases we actually have a value for one of the variables here. That will happen on occasion so don't worry about it when it does. Here is the solution for this system.

$$x_1 = \frac{28}{15} + \frac{2}{3}t + \frac{1}{3}s \qquad x_2 = -\frac{23}{15} \qquad x_3 = 1 + \frac{1}{3}t + \frac{8}{3}s$$
$$x_4 = t \qquad x_5 = s \qquad s \text{ and } t \text{ are any numbers}$$

Now, with all the examples that we've worked to this point hopefully you've gotten the idea that there really isn't any one set path that you always take through these types of problems. Each system of equations is different and so may need a different solution path. Don't get too locked into any one solution path as that can often lead to problems.

Homogeneous Systems of Linear Equations
We've got one more topic that we need to discuss briefly in this section. A system of *n* linear equations in *m* unknowns in the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= 0 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= 0 \end{aligned}$$

is called a **homogeneous system**. The one characteristic that defines a homogeneous system is the fact that all the equations are set equal to zero unlike a general system in which each equation can be equal to a different (probably non-zero) number.

Hopefully, it is clear that if we take

$$x_1 = 0 \quad x_2 = 0 \quad x_3 = 0 \quad \cdots \quad x_m = 0$$

we will have a solution to the homogeneous system of equations. In other words, with a homogeneous system we are guaranteed to have at least one solution. This means that Theorem 1 from the previous section can then be reduced to the following for homogeneous systems.

Theorem 1 Given a homogeneous system of *n* equations and *m* unknowns there will be one of two possibilities for solutions to the system.

4. There will be exactly one solution, $x_1 = 0,\ x_2 = 0,\ x_3 = 0, \cdots,\ x_m = 0$. This solution is called the **trivial solution**.
5. There will be infinitely many non-zero solutions in addition to the trivial solution.

Note that when we say non-zero solution in the above fact we mean that at least one of the x_i's in the solution will not be zero. It is completely possible that some of them will still be zero, but at least one will not be zero in a non-zero solution.

We can make a further reduction to Theorem 1 from the previous section if we assume that there are more unknowns than equations in a homogeneous system as the following theorem shows.

Theorem 2 Given a homogeneous system of *n* linear equations in *m* unknowns if $m > n$ (*i.e.* there are more unknowns than equations) there will be infinitely many solutions to the system.

Matrices

In the previous section we used augmented matrices to denote a system of linear equations. In this section we're going to start looking at matrices in more generality. A matrix is nothing more than a rectangular array of numbers and each of the numbers in the matrix is called an **entry**. Here are some examples of matrices.

$$\begin{bmatrix} 4 & 3 & 0 & 6 & -1 & 0 \\ 0 & 2 & -4 & -7 & 1 & 3 \\ -6 & 1 & 15 & \frac{1}{2} & -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 7 & 10 & -1 \\ 8 & 0 & -2 \\ 9 & 3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 12 \\ -4 \\ 2 \\ -17 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 & 12 & 0 & -9 \end{bmatrix} \qquad \begin{bmatrix} -2 \end{bmatrix}$$

The **size** of a matrix with *n* rows and *m* columns is denoted by $n \times m$. In denoting the size of a matrix we always list the number of rows first and the number of columns second.

Example 1 Give the size of each of the matrices above.

Solution

$$\begin{bmatrix} 4 & 3 & 0 & 6 & -1 & 0 \\ 0 & 2 & -4 & -7 & 1 & 3 \\ -6 & 1 & 15 & \frac{1}{2} & -1 & 0 \end{bmatrix} \qquad \Rightarrow \qquad \text{size : } 3\times 6$$

$$\begin{bmatrix} 7 & 10 & -1 \\ 8 & 0 & -2 \\ 9 & 3 & 0 \end{bmatrix} \qquad \Rightarrow \qquad \text{size : } 3\times 3$$

In this matrix the number of rows is equal to the number of columns. Matrices that have the same number of rows as columns are called **square matrices**.

$$\begin{bmatrix} 12 \\ -4 \\ 2 \\ -17 \end{bmatrix} \qquad \Rightarrow \qquad \text{size : } 4\times 1$$

This matrix has a single column and is often called a **column matrix**.

$$\begin{bmatrix} 3 & -1 & 12 & 0 & -9 \end{bmatrix} \qquad \Rightarrow \qquad \text{size : } 1\times 5$$

This matrix has a single row and is often called a **row matrix**.

$$\begin{bmatrix} -2 \end{bmatrix} \qquad \Rightarrow \qquad \text{size} : 1 \times 1$$

Often when dealing with 1×1 matrices we will drop the surrounding brackets and just write -2.

Note that sometimes column matrices and row matrices are called **column vectors** and **row vectors** respectively. We do need to be careful with the word vector however as in later chapters the word vector will be used to denote something much more general than a column or row matrix. Because of this we will, for the most part, be using the terms column matrix and row matrix when needed instead of the column vector and row vector.

There are a lot of notational issues that we're going to have to get used to in this class. First, upper case letters are generally used to refer to matrices while lower case letters generally are used to refer to numbers. These are general rules, but as you'll see shortly there are exceptions to them, although it will usually be easy to identify those exceptions when they happen.

We will often need to refer to specific entries in a matrix and so we'll need a notation to take care of that. The entry in the i^{th} row and j^{th} column of the matrix *A* is denoted by,

$$a_{i\,j} \qquad \text{OR} \qquad \left(A\right)_{i\,j}$$

In the first notation the lower case letter we use to denote the entries of a matrix will always match with the upper case letter we use to denote the matrix. So the entries of the matrix *B* will be denoted by $b_{i\,j}$.

In both of these notations the first (left most) subscript will always give the row the entry is in and the second (right most) subscript will always give the column the entry is in. So, $c_{4\,9}$ will be the entry in the 4^{th} row and 9^{th} column of *C* (which is assumed to be a matrix since it's an upper case letter…).

Using the lower case notation we can denote a general $n \times m$ matrix, *A*, as follows,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n\,m} \end{bmatrix} \qquad \text{OR} \qquad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n\,m} \end{bmatrix}_{n \times m}$$

We don't generally subscript the size of the matrix as we did in the second case, but on occasion it may be useful to make the size clear and in those cases we tend to subscript it as shown in the second case.

The notation above for a general matrix is fairly cumbersome so we've also got some much more compact notation that we'll use when we can. When possible we'll use the following to denote a general matrix.

$$\left[a_{i\,j} \right] \qquad\qquad \left[a_{i\,j} \right]_{n\times m} \qquad\qquad A_{n\times m}$$

The first two we tend to use when we need to talk about the general entry of a matrix (such as certain formulas) but don't really care what that entry is. Also, we'll denote the size if it's important or needed for whatever we're doing, but otherwise we'll not bother with the size. The third notation is really nothing more than the standard notation with the size denoted. We'll use this only when we need to talk about a matrix and the size is important but the entries aren't. We won't run into this one too often, but we will on occasion.

We will be dealing extensively with column and row matrices in later chapters/sections so we need to take care of some notation for those. There are the main exception to the upper case/lower case convention we adopted earlier for matrices and their entries. Column and row matrices tend to be denoted with a lower case letter that has either been bolded or has an arrow over it as follows,

$$\mathbf{a} = \vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \qquad\qquad \mathbf{b} = \vec{b} = \begin{bmatrix} b_1 & b_2 & \cdots & b_m \end{bmatrix}$$

In written documents, such as this, column and row matrices tend to be in bold face while on the chalkboard of a classroom they tend to get arrows written over them since it's often difficult on a chalkboard to differentiate a letter that's in bold from one that isn't.

Also, notice with column and row matrices the entries are still denoted with lower case letters that match the letter that represents the matrix and in this case since there is either a single column or a single row there was no reason to double subscript the entries.

Next we need to get a quick definition out of the way for square matrices. Recall that a square matrix is a matrix whose size is $n \times n$ (*i.e.* it has the same number of rows as columns). In a square matrix the entries $a_{11}, a_{22}, \ldots, a_{nn}$ (see the shaded portion of the matrix below) are called the **main diagonal**.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

The next topic that we need to discuss in this section is that of partitioned matrices and submatrices. Any matrix can be **partitioned** into smaller **submatrices** simply by adding in horizontal and/or vertical lines between selected rows and/or columns.

Example 2 Here are several partitions of a general 5×3 matrix.
(a)

$$A=\left[\begin{array}{c|cc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \end{array}\right]=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

In this case we partitioned the matrix into four submatrices. Also notice that we simplified the matrix into a more compact form and in this compact form we've mixed and matched some of our notation. The partitioned matrix can be thought of as a smaller matrix with four entries, except this time each of the entries are matrices instead of numbers and so we used capital letters to represent the entries and subscripted each on with the location in portioned matrix.

Be careful not to confuse the location subscripts on each of the submatrices with the size of each submatrix. In this case A_{11} is a 2×1 sub matrix of *A*, A_{12} is a 2×2 sub matrix of *A*, A_{21} is a 3×1 sub matrix of *A*, and A_{22} is a 3×2 sub matrix of *A*.

(b)

$$A=\left[\begin{array}{c|c|c} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \end{array}\right]=\left[\mathbf{c}_1 \mid \mathbf{c}_2 \mid \mathbf{c}_3\right]$$

In this case we partitioned *A* into three column matrices each representing one column in the original matrix. Again, note that we used the standard column matrix notation (the bold face letters) and subscripted each on with the location in the partitioned matrix. The $\mathbf{c}_i$ in the partitioned matrix are sometimes called the **column matrices of *A***.

(c)

$$A=\left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ \hline a_{21} & a_{22} & a_{23} \\ \hline a_{31} & a_{32} & a_{33} \\ \hline a_{41} & a_{42} & a_{43} \\ \hline a_{51} & a_{52} & a_{53} \end{array}\right]=\left[\begin{array}{c} \mathbf{r}_1 \\ \hline \mathbf{r}_2 \\ \hline \mathbf{r}_3 \\ \hline \mathbf{r}_4 \\ \hline \mathbf{r}_5 \end{array}\right]$$

Just as we can partition a matrix into each of its columns as we did in the previous part we can also partition a matrix into each of its rows. The $\mathbf{r}_i$ in the partitioned matrix are sometimes called the **row matrices of *A***.

The previous example showed three of the many possible ways to partition up the matrix. There are, of course, many other ways to partition this matrix. We won't be partitioning up too many matrices here, but we will be doing it on occasion, so it's a useful idea to remember. Also note that when we do partition up a matrix into its column/row matrices we will generally put in the bars separating the columns/rows as we've done here to indicate that we've got a partitioned matrix.

To close out this section we're going to introduce a couple of special matrices that we'll see show up on occasion.

The first matrix is the **zero matrix**. The zero matrix is pretty much what the name implies. It is an $n \times m$ matrix whose entries are all zeroes. The notation we'll use for the zero matrix is $0_{n \times m}$ for a general zero matrix or $\mathbf{0}$ for a zero column or row matrix. Here are a couple of zero matrices just so we can say we have some in the notes.

$$0_{2\times 4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{0} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If the size of a column or row zero matrix is important we will sometimes subscript the size on those as well just to make it clear what the size is. Also, if the size of a full zero matrix is not important or implied from the problem we will drop the size from $0_{n \times m}$ and just denote it by 0.

The second special matrix we'll look at in this section is the **identity matrix**. The identity matrix is a square $n \times n$ matrix usually denoted by I_n or just *I* if the size is unimportant or clear from the context of the problem. The entries on the main diagonal of the identity matrix are all ones and all the other entries in the identity matrix are zeroes. Here are a couple of identity matrices.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

As we'll see identity matrices will arise fairly regularly. Here is a nice theorem about the reduced row-echelon form of a square matrix and how it relates to the identity matrix.

Theorem 1 If *A* is an $n \times n$ matrix then the reduced row-echelon form of the matrix will either contain at least one row of all zeroes or it will be I_n, the $n \times n$ identity matrix.

Proof : This is a simple enough theorem to prove that we may as well. Let's suppose that *B* is the reduced row-echelon form of the matrix. If *B* has at least one row of all zeroes we are done so let's suppose that *B* does not have a row of all zeroes. This means that every row has a leading 1 in it.

Now, we know that the leading 1 of a row must be the right of the leading 1 of the row immediately above it. Because we are assuming that *B* is square and doesn't have any rows of all zeroes we can actually locate each of the leading 1's in *B*.

First, let's suppose that the leading 1 in the first row is NOT b_{11} (*i.e.* $b_{11}=0$). The next possible location of the leading 1 in the first row would then be b_{12}. So, let's suppose that this is where the leading 1 is. So, upon assuming this we can say that *B* must have the following form.

$$B=\begin{bmatrix} 0 & 1 & b_{13} & \cdots & b_{1n} \\ 0 & 0 & b_{23} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & b_{n3} & \cdots & b_{nn} \end{bmatrix}$$

Now, let's assume the best possible scenario happens. That is the leading 1 of each of the lower rows is exactly one column to the right of the leading 1 above it. This however, leads us to instant problems. Because our first leading 1 is in the second column by the time we reach the n-1$^{\text{st}}$ row our leading 1 will be in the n^{th} column and this will in turn force the n^{th} row to be a row of all zeroes which contradicts our initial assumption. If you're not sure you believe this consider the 4×4 case.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Sure enough a row of all zeroes in the 4^{th} row.

Now, we assumed the best possible scenario for the leading 1's in the lower rows and ran into problems. If the leading 1 jumps to the right say 2 columns (or 3 or 4, *etc.*) we will run into the same kind of problem only we'll end up with more than one row of all zeroes.

Likewise if the leading 1 in the first row is in any of $b_{13}, b_{14}, \ldots, b_{1n}$ we will have the same problem. So, in order to meet the assumption that we don't have any rows of all zeroes we know that the leading 1 in the first row must be at b_{11}.

Using a similar argument to that above we can see that if the leading 1 on any of the lower rows jumps to the right more than one column we will have a leading 1 in the n^{th} column prior to hitting the n^{th} row. This will in turn force at least the n^{th} row to be a row of all zeroes which will again contradict our initial assumption.

Therefore we know that the leading one in the first row is at b_{11} and the only hope of not having a row of all zeroes at the bottom is to have the leading 1's of a row be exactly one column to the right of the leading 1 of the row above it. This means that the leading 1 in the second row must be at b_{22}, the leading 1 in the third row must be at b_{33}, *etc.* Eventually we'll hit the n^{th} row and in this row the leading 1 must be at b_{nn}.

Therefore the leading 1's of B must be on the diagonal and because B is the reduced row-echelon form of A we also know that all the entries above and below the leading 1's must be zeroes. This however, is exactly I_n. Therefore, if B does not have a row of all zeroes in it then we must have that $B = I_n$.

■

Matrix Arithmetic & Operations

One of the biggest impediments that some people have in learning about matrices for the first time is trying to take everything that they know about arithmetic of real numbers and translate that over to matrices. As you will eventually see much of what you know about arithmetic of real numbers will also be true here, but there is also a few ideas/facts that will no longer hold here. To make matters worse there are some rules of arithmetic of real numbers that will work occasionally with matrices but won't work in general. So, keep this in mind as you go through the next couple of sections and don't be too surprised when something doesn't quite work out as you expect it to.

This section is devoted mostly to developing the arithmetic of matrices as well as introducing as well as introducing a couple of operations on matrices that don't really have an equivalent operation in real numbers. We will see some of the differences between arithmetic of real numbers and matrices mentioned above in this section. We will also see more of them in the next section when we delve into the properties of matrix arithmetic in more detail.

Okay, let's start off matrix arithmetic by defining just what we mean when we say that two matrices are equal.

Definition 1 If A and B are both $n \times m$ matrices then we say that $A = B$ provided corresponding entries from each matrix are equal. Or in other words, $A = B$ provided $a_{ij} = b_{ij}$ for all i and j.

Matrices of different sizes cannot be equal.

Example 1 Consider the following matrices.

$$A = \begin{bmatrix} -9 & 123 \\ 3 & -7 \end{bmatrix} \qquad B = \begin{bmatrix} -9 & b \\ 3 & -7 \end{bmatrix} \qquad C = \begin{bmatrix} -9 \\ 3 \end{bmatrix}$$

For these matrices we have that $A \neq C$ and $B \neq C$ since they are different sizes and so can't be equal. The fact that C is essentially the first column of both A and B is not important to determining equality in this case. The size of the two matrices is the first thing we should look at in determining equality.

Next, $A = B$ provided we have $b = 123$. If $b \neq 123$ then we will have $A \neq B$.

Next we need to move on to addition and subtraction of two matrices.

Definition 2 If A and B are both $n \times m$ matrices then $A \pm B$ is a new $n \times m$ matrix that is found by adding/subtracting corresponding entries from each matrix. Or in other words,

$$A \pm B = \left[a_{ij} \pm b_{ij} \right]$$

Matrices of different sizes cannot be added or subtracted.

Example 2 For the following matrices perform the indicated operation, if possible.

$$A = \begin{bmatrix} 2 & 0 & -3 & 2 \\ -1 & 8 & 10 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -4 & -7 & 2 \\ 12 & 3 & 7 & 9 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 0 & 2 \\ -4 & 9 & 5 \\ 6 & 0 & -6 \end{bmatrix}$$

(a) $A + B$
(b) $B - A$
(c) $A + C$

Solution
(a) Both A and B are the same size and so we know the addition can be done in this case. Once we know the addition can be done there really isn't all that much to do here other than to just add the corresponding entries here to get the results.
.

$$A + B = \begin{bmatrix} 2 & -4 & -10 & 4 \\ 11 & 11 & 17 & 4 \end{bmatrix}$$

(b) Again, since A and B are the same size we can do the difference and as like the previous part there really isn't all that much to do. All that we need to be careful with is the order. Just like with real number arithmetic $B - A$ is different from $A - B$. So, in this case we'll subtract the entries of A from the entries of B.

$$B - A = \begin{bmatrix} -2 & -4 & -4 & 0 \\ 13 & -5 & -3 & 14 \end{bmatrix}$$

(c) In this case because A and C are different sizes the addition can't be done. Likewise, $A - C$, $C - A$, $B + C$. $C - B$, and $B - C$ can't be done for the same reason.

We now need to move into multiplication involving matrices. However, there are actually two kinds of multiplication to look at : Scalar Multiplication and Matrix Multiplication. Let's start with scalar multiplication.

Definition 3 If A is any matrix and c is any number then the product (or **scalar multiple)**, cA, is a new matrix of the same size as A and it's entries are found by multiplying the original entries of A by c. In other words $cA = \left[c\, a_{ij} \right]$ for all i and j.

Note that in the field of Linear Algebra a number is often called a **scalar** and hence the name scalar multiple since we are multiplying a matrix by a scalar (number). From this point on we will generally call numbers scalars.

Before doing an example we need to get another quick definition out of the way. If $A_1, A_2, \ldots, A_n$ are all matrices of the same size and $c_1, c_2, \ldots, c_n$ are scalars then the **linear combination** of $A_1, A_2, \ldots, A_n$ with **coefficients** $c_1, c_2, \ldots, c_n$ is,

$$c_1 A_1 + c_2 A_2 + \cdots + c_n A_n$$

This may seem like a silly thing to define but we'll be using linear combination in quite a few places in this class and so we need to get used to seeing them.

Example 3 Given the matrices

$$A = \begin{bmatrix} 0 & 9 \\ 2 & -3 \\ -1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 8 & 1 \\ -7 & 0 \\ 4 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 3 \\ -2 & 5 \\ 10 & -6 \end{bmatrix}$$

compute $3A + 2B - \frac{1}{2}C$.

Solution

So, we're really being asked to compute a linear combination here. We'll do that by first computing the scalar multiplies and the performing the addition and subtraction. Note as well that in the case of the third scalar multiple we are going to consider the scalar to be a positive $\frac{1}{2}$ and leave the minus sign out in front of the matrix. Here is the work for this problem.

$$3A + 2B - \frac{1}{2}C = \begin{bmatrix} 0 & 27 \\ 6 & -9 \\ -3 & 3 \end{bmatrix} + \begin{bmatrix} 16 & 2 \\ -14 & 0 \\ 8 & -2 \end{bmatrix} - \begin{bmatrix} 1 & \frac{3}{2} \\ -1 & \frac{5}{2} \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 15 & \frac{55}{2} \\ -7 & -\frac{23}{2} \\ 0 & 4 \end{bmatrix}$$

We now need to move into matrix multiplication, however before we do the general case let's look at a special case first since this will help with the general case.

Suppose that we have the following two matrices,

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

So, **a** is a row matrix and **b** is a column matrix and they have the same number of entries. Then the product of **a** and **b** is defined to be,

$$\mathbf{ab} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

It is important to note that this product can only be done if **a** and **b** have the same number of entries. If they have a different number of entries then this product is not defined.

Example 4 Compute **ab** given that,

$$\mathbf{a} = \begin{bmatrix} 4 & -10 & 3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -4 \\ 3 \\ 8 \end{bmatrix}$$

Solution

There is not really a whole lot to do here other than use the definition given above.

$$\mathbf{ab} = (4)(-4) + (-10)(3) + (3)(8) = -22$$

Now let's move onto general matrix multiplication.

Definition 4 If A is an $n \times p$ matrix and B is a $p \times m$ matrix then the product (or **matrix multiplication**) is a new matrix with size $n \times m$ whose ij^{th} entry is found by multiplying row i of A times column j of B.

So, just like with addition and subtraction, we need to be careful with the sizes of the two matrices we're dealing with. However, with multiplication we need to be a little more careful. This definition tells us that the product AB is only defined if A (*i.e.* the first matrix listed in the product) has the same number of columns as B (*i.e.* the second matrix listed in the product) has rows. If the number of columns of the first matrix listed is not the same as the number of rows of the second matrix listed then the product is not defined.

An easy way to check that a product is defined is to write down the two matrices in the order that we want to multiply them and underneath them write down the sizes as shown below.

$$\begin{matrix} A & B & = & AB \\ n \times p & p \times m & & n \times m \end{matrix}$$

If the two inner numbers are equal then the product is defined and the size of the product will be given by the outside numbers.

Example 5 Compute AC and CA for the following two matrices, if possible.

$$A = \begin{bmatrix} 1 & -3 & 0 & 4 \\ -2 & 5 & -8 & 9 \end{bmatrix} \qquad C = \begin{bmatrix} 8 & 5 & 3 \\ -3 & 10 & 2 \\ 2 & 0 & -4 \\ -1 & -7 & 5 \end{bmatrix}$$

Solution

Okay, let's first do AC. Here are the sizes for A and C.

$$\begin{matrix} A & C & = & AC \\ 2 \times 4 & 4 \times 3 & & 2 \times 3 \end{matrix}$$

So, the two inner numbers (4 and 4) are the same and so the multiplication can be done and we can see that the new size of the matrix is 2×3. Now, let's actually do the multiplication. We'll go through the first couple of entries in the product in detail and then do the remaining entries a little quicker.

To get the number in the first row and first column of *AC* we'll multiply the first row of *A* by the first column of *B* as follows,

$$(1)(8)+(-3)(-3)+(0)(2)+(4)(-1)=13$$

If we next want the entry in the first row and second column of *AC* we'll multiply the first row of *A* by the second column of *B* as follows,

$$(1)(5)+(-3)(10)+(0)(0)+(4)(-7)=-53$$

Okay, at this point, let's stop and insert these into the product so we can make sure that we've got our bearings. Here's the product so far,

$$\begin{bmatrix} 1 & -3 & 0 & 4 \\ -2 & 5 & -8 & 9 \end{bmatrix}\begin{bmatrix} 8 & 5 & 3 \\ -3 & 10 & 2 \\ 2 & 0 & -4 \\ -1 & -7 & 5 \end{bmatrix}=\begin{bmatrix} 13 & -53 & \square \\ \square & \square & \square \end{bmatrix}$$

As we can see we've got four entries left to compute. For these we'll give the row and column multiplications but leave it to you to make sure we used the correct row/column and put the result in the correct place. Here's the remaining work.

$$(1)(3)+(-3)(2)+(0)(-4)+(4)(5)=17$$
$$(-2)(8)+(5)(-3)+(-8)(2)+(9)(-1)=-56$$
$$(-2)(5)+(5)(10)+(-8)(0)+(9)(-7)=-23$$
$$(-2)(3)+(5)(2)+(-8)(-4)+(9)(5)=81$$

Here is the completed product.

$$\begin{bmatrix} 1 & -3 & 0 & 4 \\ -2 & 5 & -8 & 9 \end{bmatrix}\begin{bmatrix} 8 & 5 & 3 \\ -3 & 10 & 2 \\ 2 & 0 & -4 \\ -1 & -7 & 5 \end{bmatrix}=\begin{bmatrix} 13 & -53 & 17 \\ -56 & -23 & 81 \end{bmatrix}$$

Now let's do *CA*. Here are the sizes for this product.

$$\begin{array}{cccc} C & A & = & CA \\ 4\times 3 & 2\times 4 & & \text{N/A} \end{array}$$

Okay, in this case the two inner numbers (3 and 2) are NOT the same and so this product can't be done.

So, with this example we've now run across the first real difference between real number arithmetic and matrix arithmetic. When dealing with real numbers the order in which we write a product doesn't affect the actual result. For instance (2)(3)=6 and (3)(2)=6. We can flip the order and we get the same answer. With matrices however, we will have to be very careful and pay attention to the order in which the product is written down. As this example has shown the product *AC* could be computed while the product *CA* in not defined.

Now, do not take the previous example and assume that all products will work that way. It is possible for both *AC* and *CA* to be defined as we'll see in the next example.

Example 6 Compute *BD* and *DB* for the given matrices, if possible.

$$B = \begin{bmatrix} 3 & -1 & 7 \\ 10 & 1 & -8 \\ -5 & 2 & 4 \end{bmatrix} \qquad D = \begin{bmatrix} -1 & 4 & 9 \\ 6 & 2 & -1 \\ 7 & 4 & 7 \end{bmatrix}$$

Solution

First, notice that both of these matrices are 3×3 matrices and so both *BD* and *DB* are defined. Again, it's worth pointing out that this example differs from the previous example in that both the products are defined in this example rather than only one being defined as in the previous example. Also note that in both cases the product will be a new 3×3 matrix.

In this example we're going to leave the work of verifying the products to you. It is good practice so you should try and verify at least one of the following products.

$$BD = \begin{bmatrix} 3 & -1 & 7 \\ 10 & 1 & -8 \\ -5 & 2 & 4 \end{bmatrix} \begin{bmatrix} -1 & 4 & 9 \\ 6 & 2 & -1 \\ 7 & 4 & 7 \end{bmatrix} = \begin{bmatrix} 40 & 38 & 77 \\ -60 & 10 & 33 \\ 45 & 0 & -19 \end{bmatrix}$$

$$DB = \begin{bmatrix} -1 & 4 & 9 \\ 6 & 2 & -1 \\ 7 & 4 & 7 \end{bmatrix} \begin{bmatrix} 3 & -1 & 7 \\ 10 & 1 & -8 \\ -5 & 2 & 4 \end{bmatrix} = \begin{bmatrix} -8 & 23 & -3 \\ 43 & -6 & 22 \\ 26 & 11 & 45 \end{bmatrix}$$

This example leads us to yet another difference (although it's related to the first) between real number arithmetic and matrix arithmetic. In this example both *BD* and *DB* were defined. Notice however that the products were definitely not the same. There is nothing wrong with this so don't get excited about it when it does happen. Note however that this doesn't mean that the two products will never be the same. It is possible for them to be the same and we'll see at least one case where the two products are the same in a couple of sections.

For the sake of completeness if *A* is an $n \times p$ matrix and *B* is a $p \times m$ matrix then the entry in the i^{th} row and j^{th} column of *AB* is given by the following formula,

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ip}b_{pj}$$

This formula can be useful on occasion, but is really used mostly in proofs and computer programs that compute the product of matrices.

On occasion it can be convenient to know a single row or a single column from a product and not the whole product itself. The following theorem tells us how to get our hands on just that.

Theorem 1 Assuming that A and B are appropriately sized so that AB is defined then,

1. The i^{th} row of AB is given by the matrix product : [i^{th} row of A]B.
2. The j^{th} column of AB is given by the matrix product : A[j^{th} column of B].

Example 7 Compute the second row and third column of AC given the following matrices.

$$A = \begin{bmatrix} 1 & -3 & 0 & 4 \\ -2 & 5 & -8 & 9 \end{bmatrix} \qquad C = \begin{bmatrix} 8 & 5 & 3 \\ -3 & 10 & 2 \\ 2 & 0 & -4 \\ -1 & -7 & 5 \end{bmatrix}$$

Solution

These are the matrices from Example 5 and so we can verify the results of using this fact once we're done.

Let's find the second row first. So, according to the fact this means we need to multiply the second row of A by C. Here is that work.

$$\begin{bmatrix} -2 & 5 & -8 & 9 \end{bmatrix} \begin{bmatrix} 8 & 5 & 3 \\ -3 & 10 & 2 \\ 2 & 0 & -4 \\ -1 & -7 & 5 \end{bmatrix} = \begin{bmatrix} -56 & -23 & 81 \end{bmatrix}$$

Sure enough, this is the correct second row of the product AC.

Next, let's use the fact to get the third column. This means that we'll need to multiply A by the third column of B. Here is that work.

$$\begin{bmatrix} 1 & -3 & 0 & 4 \\ -2 & 5 & -8 & 9 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -4 \\ 5 \end{bmatrix} = \begin{bmatrix} 17 \\ 81 \end{bmatrix}$$

And sure enough, this also gives us the correct answer.

We can use this fact about how to get individual rows or columns of a product as well as the idea of a partitioned matrix that we saw in the previous section to derive a couple of new ways to find the product of two matrices.

Let's start by assuming we've got two matrices A (size $n \times p$) and B (size $p \times m$) so we know the product AB is defined.

Now, for the first new way of finding the product let's partition A into its row matrices as follows,

$$A = \left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1p} \\ \hline a_{21} & a_{22} & \cdots & a_{2p} \\ \hline \vdots & \vdots & & \vdots \\ \hline a_{n1} & a_{n2} & \cdots & a_{np} \end{array}\right] = \left[\begin{array}{c} \mathbf{r}_1 \\ \hline \mathbf{r}_2 \\ \hline \vdots \\ \hline \mathbf{r}_n \end{array}\right]$$

Now, from the fact we know that the i^{th} row of AB is [i^{th} row of A]B, or $\mathbf{r}_i B$. Using this idea the product AB can then be written as a new partitioned matrix as follows.

$$AB = \left[\begin{array}{c} \mathbf{r}_1 \\ \hline \mathbf{r}_2 \\ \hline \vdots \\ \hline \mathbf{r}_n \end{array}\right] B = \left[\begin{array}{c} \mathbf{r}_1 B \\ \hline \mathbf{r}_2 B \\ \hline \vdots \\ \hline \mathbf{r}_n B \end{array}\right]$$

For the second new way of finding the determinate we'll partition B into its column matrices as,

$$B = \left[\begin{array}{c|c|c|c} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pm} \end{array}\right] = \left[\, \mathbf{c}_1 \mid \mathbf{c}_2 \mid \cdots \mid \mathbf{c}_m \,\right]$$

We can then use the fact that t he j^{th} column of AB is given by A[j^{th} column of B] and so the product AB can be written as a new partitioned matrix as follows.

$$AB = A\left[\, \mathbf{c}_1 \mid \mathbf{c}_2 \mid \cdots \mid \mathbf{c}_m \,\right] = \left[\, A\mathbf{c}_1 \mid A\mathbf{c}_2 \mid \cdots \mid A\mathbf{c}_m \,\right]$$

Example 8 Use both of the new methods for computing products to find AC for the following matrices.

$$A = \begin{bmatrix} 1 & -3 & 0 & 4 \\ -2 & 5 & -8 & 9 \end{bmatrix} \qquad C = \begin{bmatrix} 8 & 5 & 3 \\ -3 & 10 & 2 \\ 2 & 0 & -4 \\ -1 & -7 & 5 \end{bmatrix}$$

Solution

So, once again we know the answer to this so we can use it to check our results against the answer from Example 5.

First, let's use the row matrices of A. Here are the two row matrices of A.

$$\mathbf{r}_1 = \begin{bmatrix} 1 & -3 & 0 & 4 \end{bmatrix} \qquad \mathbf{r}_2 = \begin{bmatrix} -2 & 5 & -8 & 9 \end{bmatrix}$$

and here are the rows of the product.

$$\mathbf{r}_1 C = \begin{bmatrix} 1 & -3 & 0 & 4 \end{bmatrix} \begin{bmatrix} 8 & 5 & 3 \\ -3 & 10 & 2 \\ 2 & 0 & -4 \\ -1 & -7 & 5 \end{bmatrix} = \begin{bmatrix} 13 & -53 & 17 \end{bmatrix}$$

$$\mathbf{r}_2 C = \begin{bmatrix} -2 & 5 & -8 & 9 \end{bmatrix} \begin{bmatrix} 8 & 5 & 3 \\ -3 & 10 & 2 \\ 2 & 0 & -4 \\ -1 & -7 & 5 \end{bmatrix} = \begin{bmatrix} -54 & -23 & 81 \end{bmatrix}$$

Putting these together gives,

$$AC = \begin{bmatrix} \mathbf{r}_1 C \\ \mathbf{r}_2 C \end{bmatrix} = \begin{bmatrix} 13 & -53 & 17 \\ -56 & -23 & 81 \end{bmatrix}$$

and this is the correct answer.

Now let's compute the product using columns. Here are the three column matrices for C.

$$\mathbf{c}_1 = \begin{bmatrix} 8 \\ -3 \\ 2 \\ -1 \end{bmatrix} \qquad \mathbf{c}_2 = \begin{bmatrix} 5 \\ 10 \\ 0 \\ -7 \end{bmatrix} \qquad \mathbf{c}_3 = \begin{bmatrix} 3 \\ 2 \\ -4 \\ 5 \end{bmatrix}$$

Here are the columns of the product.

$$A\mathbf{c}_1 = \begin{bmatrix} 1 & -3 & 0 & 4 \\ -2 & 5 & -8 & 9 \end{bmatrix} \begin{bmatrix} 8 \\ -3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 13 \\ -56 \end{bmatrix}$$

$$A\mathbf{c}_2 = \begin{bmatrix} 1 & -3 & 0 & 4 \\ -2 & 5 & -8 & 9 \end{bmatrix} \begin{bmatrix} 5 \\ 10 \\ 0 \\ -7 \end{bmatrix} = \begin{bmatrix} -53 \\ -23 \end{bmatrix}$$

$$A\mathbf{c}_3 = \begin{bmatrix} 1 & -3 & 0 & 4 \\ -2 & 5 & -8 & 9 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -4 \\ 5 \end{bmatrix} = \begin{bmatrix} 17 \\ 81 \end{bmatrix}$$

Putting all this together as follows gives the correct answer.

$$AB = \begin{bmatrix} A\mathbf{c}_1 & A\mathbf{c}_2 & A\mathbf{c}_3 \end{bmatrix} = \begin{bmatrix} 13 & -53 & 17 \\ -56 & -23 & 81 \end{bmatrix}$$

We can also write certain kinds of matrix products as a linear combination of column matrices. Consider A an $n \times p$ matrix and $\mathbf{x}$ a $p \times 1$ column matrix. We can easily compute this product directly as follows,

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1p}x_p \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2p}x_p \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{np}x_p \end{bmatrix}_{n\times 1}$$

Now, using matrix addition we can write the resultant $n \times 1$ matrix as follows,

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1p}x_p \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2p}x_p \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{np}x_p \end{bmatrix} = \begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{n1}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \\ \vdots \\ a_{n2}x_2 \end{bmatrix} + \cdots + \begin{bmatrix} a_{1p}x_p \\ a_{2p}x_p \\ \vdots \\ a_{np}x_p \end{bmatrix}$$

Now, each of the p column matrices on the right above can also be rewritten as a scalar multiple as follows.

$$\begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{n1}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \\ \vdots \\ a_{n2}x_2 \end{bmatrix} + \cdots + \begin{bmatrix} a_{1p}x_p \\ a_{2p}x_p \\ \vdots \\ a_{np}x_p \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \cdots + x_p \begin{bmatrix} a_{1p} \\ a_{2p} \\ \vdots \\ a_{np} \end{bmatrix}$$

Finally, the column matrices that are multiplied by the x_i's are nothing more than the column matrices of A. So, putting all this together gives us,

$$A\mathbf{x} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \cdots + x_p \begin{bmatrix} a_{1p} \\ a_{2p} \\ \vdots \\ a_{np} \end{bmatrix} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_p\mathbf{c}_p$$

where $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_p$ are the column matrices of A. Written in this matter we can see that $A\mathbf{x}$ can be written as the linear combination of the column matrices of A, $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_p$, with the entries of $\mathbf{x}$, $x_1, x_2, \ldots x_p$, as coefficients.

Example 9 Compute $A\mathbf{x}$ directly and as a linear combination for the following matrices.

$$A = \begin{bmatrix} 4 & 1 & 2 & -1 \\ -12 & 1 & 3 & 2 \\ 0 & -5 & -10 & 9 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 6 \\ 8 \end{bmatrix}$$

Solution

We'll leave it to you to verify that the direct computation of the product gives,

$$A\mathbf{x} = \begin{bmatrix} 4 & 1 & 2 & -1 \\ -12 & 1 & 3 & 2 \\ 0 & -5 & -10 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 11 \\ 9 \\ 17 \end{bmatrix}$$

Here is the linear combination method of computing the product.

$$\begin{aligned} A\mathbf{x} &= \begin{bmatrix} 4 & 1 & 2 & -1 \\ -12 & 1 & 3 & 2 \\ 0 & -5 & -10 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 6 \\ 8 \end{bmatrix} \\ &= 2\begin{bmatrix} 4 \\ -12 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix} + 6\begin{bmatrix} 2 \\ 3 \\ -10 \end{bmatrix} + 8\begin{bmatrix} -1 \\ 2 \\ 9 \end{bmatrix} \\ &= \begin{bmatrix} 8 \\ -24 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix} + \begin{bmatrix} 12 \\ 18 \\ -60 \end{bmatrix} + \begin{bmatrix} -8 \\ 16 \\ 72 \end{bmatrix} \\ &= \begin{bmatrix} 11 \\ 9 \\ 17 \end{bmatrix} \end{aligned}$$

This is the same result that we got by the direct computation.

Matrix multiplication also gives us a very nice and compact way of writing systems of equations. In fact we even saw most of it as we introduced the above idea. Let's start out with a general system of n equations and m unknowns.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= b_n \end{aligned}$$

Now, instead of thinking of these as a set of equations let's think of each side as a vector of size $n \times 1$ as follows,

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

In the work above we saw that the left side of this can be written as the following matrix product,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

If we now denote the coefficient matrix by *A*, the column matrix containing the unknowns by **x** and the column matrix containing the b_i's by **b**. we can write the system in the following **matrix form**,

$$A\mathbf{x} = \mathbf{b}$$

In many of the section to follow we'll write general systems of equations as $A\mathbf{x} = \mathbf{b}$ given its compact nature in order to save space.

Now that we've gotten the basics of matrix arithmetic out of the way we need to introduce a couple of matrix operations that don't really have any equivalent operations with real numbers.

Definition 5 If *A* is an $n \times m$ matrix then the **transpose** of *A*, denoted by A^T, is an $m \times n$ matrix that is obtained by interchanging the rows and columns of *A*. So, the first row of A^T is the first column of *A*, the second row of A^T is the second column of *A*, *etc*. Likewise, the first column of A^T is the first row of *A*, the second column of A^T is the second row of *A*, *etc*.

On occasion you'll see the transpose defined as follows,

$$A = \left[a_{ij} \right]_{n \times m} \qquad \Rightarrow \qquad A^T = \left[a_{ji} \right]_{m \times n} \quad \text{for all } i \text{ and } j$$

Notice the difference in the subscripts. Under this definition, the entry in the i^{th} row and j^{th} column of *A* will be in the j^{th} row and i^{th} column of A^T.

Notice that these two definitions are really the same definition, they just don't look like they are the same at first glance.

Definition 6 If *A* is a square matrix of size $n \times n$ then the **trace** of *A*, denoted by tr(*A*), is the sum of the entries on main diagonal. Or,

$$\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

If *A* is not square then the trace is not defined.

Example 10 Determine the transpose and trace (if it is defined) for each of the following matrices.

$$A = \begin{bmatrix} 4 & 10 & -7 & 0 \\ 5 & -1 & 3 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 2 & -6 \\ -9 & 1 & -7 \\ 5 & 0 & 12 \end{bmatrix} \qquad C = \begin{bmatrix} 9 \\ -1 \\ 8 \end{bmatrix}$$

$$D = \begin{bmatrix} 15 \end{bmatrix} \qquad E = \begin{bmatrix} -12 & -7 \\ -7 & 10 \end{bmatrix}$$

Solution
There really isn't all that much to do here other than to go through the definitions. Note as well that the trace will only not be defined for A and C since these matrices are not square.

$$A^T = \begin{bmatrix} 4 & 5 \\ 10 & -1 \\ -7 & 3 \\ 0 & -2 \end{bmatrix} \qquad \text{tr}\left(A\right): \text{ Not defined since } A \text{ is not square.}$$

$$B^T = \begin{bmatrix} 3 & -9 & 5 \\ 2 & 1 & 0 \\ -6 & -7 & 12 \end{bmatrix} \qquad \text{tr}\left(B\right) = 3 + 1 + 12 = 16$$

$$C^T = \begin{bmatrix} 9 & -1 & 8 \end{bmatrix} \qquad \text{tr}\left(c\right): \text{ Not defined since } C \text{ is not square.}$$

$$D^T = \begin{bmatrix} 15 \end{bmatrix} \qquad \text{tr}\left(D\right) = 15$$

$$E^T = \begin{bmatrix} -12 & -7 \\ -7 & 10 \end{bmatrix} \qquad \text{tr}\left(E\right) = -12 + 10 = -2$$

In the previous example note that $D^T = D$ and that $E^T = E$. In these cases the matrix is called **symmetric**. So, in the previous example D and E are symmetric while A, B, and C, are not symmetric.

Properties of Matrix Arithmetic and the Transpose

In this section we're going to take a quick look at some of the properties of matrix arithmetic and of the transpose of a matrix. As mentioned in the previous section most of the basic rules of real number arithmetic are still valid in matrix arithmetic. However, there are a few that are no longer valid in matrix arithmetic as we'll be seeing.

We've already seen one of the real number properties that doesn't hold in matrix arithmetic. If a and b are two real numbers then we know by the commutative law for multiplication of real numbers that $ab = ba$ (*i.e.* $(2)(3)=(3)(2)=6$). However, if *A* and *B* are two matrices such that *AB* is defined we saw an example in the previous section in which *BA* was not defined as well as an example in which *BA* was defined and yet $AB \neq BA$. In other words, we don't have a commutative law for matrix multiplication. Note that doesn't mean that we'll never have $AB = BA$ for some matrices *A* and *B*, it is possible for this to happen (as we'll see in the next section) we just can't guarantee that this will happen if both *AB* and *BA* are defined.

Now, let's take a quick look at the properties of real number arithmetic that are valid in matrix arithmetic.

Properties

In the following set of properties *a* and *b* are scalars and *A*, *B*, and *C* are matrices. We'll assume that the size of the matrices in each property are such that the operation in that property is defined.

1. $A+B=B+A$ Commutative law for addition
2. $A+(B+C)=(A+B)+C$ Associative law for addition
3. $A(BC)=(AB)C$ Associative law for multiplication
4. $A(B\pm C)=AB\pm AC$ Left distributive law
5. $(B\pm C)A=BA\pm CA$ Right distributive law
6. $a(B\pm C)=aB\pm aC$
7. $(a\pm b)C=aC\pm bC$
8. $(ab)C=a(bC)$
9. $a(BC)=(aB)C=B(aC)$

With real number arithmetic we didn't need both 4. and 5. since we've also got the commutative law for multiplication. However, since we don't have the commutative law for matrix multiplication we really do need both 4. and 5. Also, properties 6. – 9. are simply distributive or associative laws for dealing with scalar multiplication.

Now, let's take a look at couple of other idea from real number arithmetic and see if they have equivalent ideas in matrix arithmetic.

We'll start with the following idea. From real number arithmetic we know that $1\cdot a = a\cdot 1 = a$. Or, in other words, if we multiply a number by 1 (one) doesn't change the number. The identity matrix will give the same result in matrix multiplication. If *A* is an $n\times m$ matrix then we have,

$$I_n A = A I_m = A$$

Note that we really do need different identity matrices on each side of *A* that will depend upon the size of *A*.

Example 1 Consider the following matrix.

$$A = \begin{bmatrix} 10 & 0 \\ -3 & 8 \\ -1 & 11 \\ 7 & -4 \end{bmatrix}$$

Then,

$$I_4 A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 0 \\ -3 & 8 \\ -1 & 11 \\ 7 & -4 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ -3 & 8 \\ -1 & 11 \\ 7 & -4 \end{bmatrix}$$

$$A I_2 = \begin{bmatrix} 10 & 0 \\ -3 & 8 \\ -1 & 11 \\ 7 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ -3 & 8 \\ -1 & 11 \\ 7 & -4 \end{bmatrix}$$

Now, just like the identity matrix takes the place of the number 1 (one) in matrix multiplication, the zero matrix (denoted by 0 for a general matrix and **0** for a column/row matrix) will take the place of the number 0 (zero) in most of the matrix arithmetic. Note that we said most of the matrix arithmetic. There are a couple of properties involving 0 in real numbers that are not necessarily valid in matrix arithmetic.

Let's first start with the properties that are still valid.

Zero Matrix Properties

In the following properties A is a matrix and 0 is the zero matrix sized appropriately for the indicated operation to be valid.

1. $A + 0 = 0 + A = A$
2. $A - A = 0$
3. $0 - A = -A$
4. $0A = 0$ and $A0 = 0$

Now, in real number arithmetic we know that if $ab = ac$ and $a \neq 0$ then we must have $b = c$ (sometimes called the cancellation law). We also know that if $ab = 0$ then we have $a = 0$ and/or $b = 0$ (sometimes called the zero factor property). Neither of these properties of real number arithmetic are valid in general for matrix arithmetic.

Example 2 Consider the following three matrices.

$$A = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 4 \\ 6 & 1 \end{bmatrix}$$

We'll leave it to you to verify that,

$$AB = \begin{bmatrix} 9 & -10 \\ 18 & -20 \end{bmatrix} = AC$$

Clearly $A \neq 0$ and just as clearly $B \neq C$ and yet we do have $AB = AC$. So, at least in this case, the cancellation law does not hold.

We should be careful and not read too much into the results of the previous example. The cancellation law will not be valid in general for matrix multiplication. However, there are times when a variation of the cancellation law will be valid as we'll see in the next section.

Example 3 Consider the following two matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} -16 & 2 \\ 8 & -1 \end{bmatrix}$$

We'll leave it to you to verify that,

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So, we've got $AB = 0$ despite the fact that $A \neq 0$ and $B \neq 0$. So, in this case the zero factor property does not hold in this case.

Now, again, we need to be careful. There are times when we will have a variation of the zero factor property, however there will be no zero factor property for the multiplication of any two random matrices.

The next topic that we need to take a look at is that of powers of matrices. At this point we'll just work with positive exponents. We'll need the next section before we can deal with negative exponents. Let's start off with the following definitions.

Definition 1 If A is a square matrix then,

$$A^0 = I \qquad A^n = \underbrace{AA\cdots A}_{n \text{ times}}, \quad n > 0$$

We've also got several of the standard integer exponent properties that we are used to working with.

Properties of Matrix Exponents

If A is a square matrix and n and m are integers then,

$$A^n A^m = A^{n+m} \qquad \left(A^n\right)^m = A^{nm}$$

We can also talk about plugging matrices into polynomials using the following definition. If we have the polynomial,

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

and A is a square matrix then,

$$p(A) = a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I$$

where the identity matrix on the constant term a_0 has the same size as A.

Example 4 Evaluate each of the following for the give matrix.

$$A = \begin{bmatrix} -7 & 3 \\ 5 & 1 \end{bmatrix}$$

(a) A^2

(b) A^3

(c) $p(A)$ where $p(x) = -6x^3 + 10x - 9$

Solution

(a) There really isn't much to do with this problem. We'll leave it to you to verify the multiplication here.

$$A^2 = \begin{bmatrix} -7 & 3 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} -7 & 3 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 64 & -18 \\ -30 & 16 \end{bmatrix}$$

(b) In this case we may as well take advantage of the fact that we've got the result from the first part already. Again, we'll leave it to you to verify the multiplication.

$$A^3 = A^2 A = \begin{bmatrix} 64 & -18 \\ -30 & 16 \end{bmatrix}\begin{bmatrix} -7 & 3 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} -538 & 174 \\ 290 & -74 \end{bmatrix}$$

(c) In this case we'll need the result from the second part. Outside of that there really isn't much to do here.

$$\begin{aligned} p(A) &= -6A^3 + 10A - 9I \\ &= -6\begin{bmatrix} -538 & 174 \\ 290 & -74 \end{bmatrix} + 10\begin{bmatrix} -7 & 3 \\ 5 & 1 \end{bmatrix} - 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= -6\begin{bmatrix} -538 & 174 \\ 290 & -74 \end{bmatrix} + 10\begin{bmatrix} -7 & 3 \\ 5 & 1 \end{bmatrix} - 9\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3228 & -1044 \\ -1740 & 444 \end{bmatrix} + \begin{bmatrix} -70 & 30 \\ 50 & 10 \end{bmatrix} - \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 3149 & -1014 \\ -1690 & 445 \end{bmatrix} \end{aligned}$$

The last topic in this section that we need to take care of is some quick properties of the transpose of a matrix.

Properties of the Transpose

If A and B are matrices whose sizes are such that the given operations are defined and c is any scalar then,

1. $\left(A^T\right)^T = A$
2. $\left(A \pm B\right)^T = A^T \pm B^T$
3. $\left(cA\right)^T = cA^T$
4. $\left(AB\right)^T = B^T A^T$

The first three of these properties should be fairly obvious from the definition of the transpose. The fourth is a little trickier to see, but isn't that bad to verify.

Proof of #4 : We know that the entry in the i^{th} row and j^{th} column of AB is given by,

$$\left(AB\right)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ip}b_{pj}$$

We also know that the entry in the i^{th} row and j^{th} column of $\left(AB\right)^T$ is found simply by interchanging the subscripts i and j and so it is,

$$\left(\left(AB\right)^T\right)_{ij} = \left(AB\right)_{ji} = a_{j1}b_{1i} + a_{j2}b_{2i} + a_{j3}b_{3i} + \cdots + a_{jp}b_{pi}$$

Now, let's denote the entries of A^T and B^T as $\overline{a}_{ij}$ and $\overline{b}_{ij}$ respectively. Again, based on the definition of the transpose we also know that,

$$A^T = \left[\overline{a}_{ij}\right] = \left[a_{ji}\right] \qquad\qquad B^T = \left[\overline{b}_{ij}\right] = \left[b_{ji}\right]$$

and so from this we see that $\overline{a}_{ij} = a_{ji}$ and $\overline{b}_{ij} = b_{ji}$.

Finally, the entry in the i^{th} row and j^{th} column of $B^T A^T$ is given by,

$$\left(B^T A^T\right)_{ij} = \overline{b}_{i1}\overline{a}_{1j} + \overline{b}_{i2}\overline{a}_{2j} + \overline{b}_{i3}\overline{a}_{3j} + \cdots + \overline{b}_{ip}\overline{a}_{pj}$$

Now, plug in for $\overline{a}_{ij}$ and $\overline{b}_{ij}$ and we get that,

$$\begin{aligned}\left(B^T A^T\right)_{ij} &= \overline{b}_{i1}\overline{a}_{1j} + \overline{b}_{i2}\overline{a}_{2j} + \overline{b}_{i3}\overline{a}_{3j} + \cdots + \overline{b}_{ip}\overline{a}_{pj} \\ &= b_{1i}a_{j1} + b_{2i}a_{j2} + b_{3i}a_{j3} + \cdots + b_{pi}a_{jp} \\ &= a_{j1}b_{1i} + a_{j2}b_{2i} + a_{j3}b_{3i} + \cdots + a_{jp}b_{pi} \\ &= \left(\left(AB\right)^T\right)_{ij}\end{aligned}$$

So, just what have we done here? We've managed to show that the entry in the i^{th} row and j^{th} column of $\left(AB\right)^T$ is equal to the entry in the i^{th} row and j^{th} column of $B^T A^T$. Therefore, since each of the entries are equal the matrices must also be equal.

■

Note that #4 can be naturally extended to more than two matrices. For example,

$$\left(ABC\right)^T = C^T B^T A^T$$

Inverse Matrices and Elementary Matrices

Our main goal in this section is define inverse matrices and to take a look at some nice properties involving matrices. We won't actually be finding any inverse matrices in this section. That is the topic of the next section.

We'll also take a quick look at elementary matrices which as we'll see in the next section we can use to help us find inverse matrices. Actually, that's not totally true. We'll use them to help us devise a method for finding inverse matrices, but we won't be explicitly using them to find the inverse.

So, let's start off with the definition of the inverse matrix.

Definition 1 If A is a square matrix and we can find another matrix of the same size, say B, such that

$$AB = BA = I$$

then we call A **invertible** and we say that B is an **inverse** of the matrix A.

If we can't find such a matrix B we call A a **singular** matrix.

Note that we only talk about inverse matrices for square matrices. Also note that if A is invertible it will on occasion be called **non-singular**. We should also point out that we could also say that B is invertible and that A is the inverse of B.

Before proceeding we need to show that the inverse of a matrix is unique, that is for a given invertible matrix A there is exactly one inverse for the matrix.

Theorem 1 Suppose that A is invertible and that both B and C are inverses of A. Then $B = C$ and we will denote the inverse as A^{-1}.

Proof : Since B is an inverse of A we know that $AB = I$. Now multiply both sides of this by C to get $C\left(AB\right) = CI = C$. However, by the associative law of matrix multiplication we can also write $C\left(AB\right)$ as $C\left(AB\right) = \left(CA\right)B = I\,B = B$. Therefore, putting these two pieces together we see that $C = C\left(AB\right) = B$ or $C = B$.

■

So, the inverse for a matrix is unique. To denote this fact we now will denote the inverse of the matrix A as A^{-1} from this point on.

Example 1 Given the matrix A verify that the indicated matrix is in fact the inverse.

$$A = \begin{bmatrix} -4 & -2 \\ 5 & 5 \end{bmatrix} \qquad\qquad A^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{5} \\ \frac{1}{2} & \frac{2}{5} \end{bmatrix}$$

Solution

To verify that we do in fact have the inverse we'll need to check that

$$AA^{-1} = A^{-1}A = I$$

This is easy enough to do and so we'll leave it to you to verify the multiplication.

$$AA^{-1} = \begin{bmatrix} -4 & -2 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & -\frac{1}{5} \\ \frac{1}{2} & \frac{2}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{5} \\ \frac{1}{2} & \frac{2}{5} \end{bmatrix}\begin{bmatrix} -4 & -2 \\ 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

As the definition of an inverse matrix suggests, not every matrix will have an inverse. Here is an example of a matrix without an inverse.

Example 2 The matrix below does not have an inverse.

$$B = \begin{bmatrix} 3 & 9 & 2 \\ 0 & 0 & 0 \\ -4 & -5 & 1 \end{bmatrix}$$

This is fairly simple to see. If B has a matrix then it must be a 3×3 matrix. So, let's just take any old 3×3,

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

Now let's think about the product BC. We know that the 2nd row of BC can be found by looking at the following matrix multiplication,

$$\begin{bmatrix} 2^{nd} \text{ row of } B \end{bmatrix} C = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

So, the second row of BC is $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$, but if C is to be the inverse of B the product BC must be the identity matrix and this means that the second row must in fact be $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$.

Now, C was a general 3×3 matrix and we've shown that the second row of BC is all zeroes and hence the product will never be the identity matrix and so B can't have an inverse and so is a singular matrix.

In the previous section we introduced the idea of matrix exponentiation. However, we needed to restrict ourselves to positive exponents. We can now take a look at negative exponents.

Definition 2 If A is a square matrix and $n > 0$ then,

$$A^{-n} = \left(A^{-1}\right)^n = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{n \text{ times}}$$

Example 3 Compute A^{-3} for the matrix,

$$A = \begin{bmatrix} -4 & -2 \\ 5 & 5 \end{bmatrix}$$

Solution

From Example 1 we know that the inverse of A is,

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{5} \\ \frac{1}{2} & \frac{2}{5} \end{bmatrix}$$

So, this is easy enough to compute.

$$\begin{aligned} A^{-3} = \left(A^{-1}\right)^3 &= \begin{bmatrix} -\frac{1}{2} & -\frac{1}{5} \\ \frac{1}{2} & \frac{2}{5} \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & -\frac{1}{5} \\ \frac{1}{2} & \frac{2}{5} \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & -\frac{1}{5} \\ \frac{1}{2} & \frac{2}{5} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{20} & \frac{1}{50} \\ -\frac{1}{20} & \frac{3}{50} \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & -\frac{1}{5} \\ \frac{1}{2} & \frac{2}{5} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{13}{200} & -\frac{11}{500} \\ \frac{11}{200} & \frac{17}{500} \end{bmatrix} \end{aligned}$$

Next, let's take a quick look at some nice facts about the inverse matrix.

Theorem 2 Suppose that A and B are invertible matrices of the same size. Then,

(a) AB is invertible and $\left(AB\right)^{-1} = B^{-1}A^{-1}$.

(b) A^{-1} is invertible and $\left(A^{-1}\right)^{-1} = A$.

(c) For $n = 0, 1, 2, \ldots$ A^n is invertible and $\left(A^n\right)^{-1} = A^{-n} = \left(A^{-1}\right)^n$.

(d) If c is any non-zero scalar then cA is invertible and $\left(cA\right)^{-1} = \frac{1}{c}A^{-1}$

(e) A^T is invertible and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

Proof :

Note that in each case in order to prove that the given matrix is invertible all we need to do is show that the inverse is what we claim it to be. Also, don't get excited about showing that the inverse is what we claim it to be. In these cases all we need to do is show that the product (both left and right product) of the given matrix and what we claim is the inverse is the identity matrix. That's it.

Also, do not get excited about the inverse notation. For example, in the first one we state that $\left(AB\right)^{-1} = B^{-1}A^{-1}$. Remember that the $\left(AB\right)^{-1}$ is just the notation that we use to denote the inverse of *AB*. This notation will not be used in the proof except in the final step to denote the inverse.

(a) Now, as suggested above showing this is not really all that difficult. All we need to do is show that $\left(AB\right)\left(B^{-1}A^{-1}\right) = I$ and $\left(B^{-1}A^{-1}\right)\left(AB\right) = I$. Here is that work.

$$\left(AB\right)\left(B^{-1}A^{-1}\right) = A\left(BB^{-1}\right)A^{-1} = AIA^{-1} = AA^{-1} = I$$
$$\left(B^{-1}A^{-1}\right)\left(AB\right) = B^{-1}\left(A^{-1}A\right)B = B^{-1}IB = B^{-1}B = I$$

So, we've shown both and so we now know that *AB* is in fact invertible (since we've found the inverse!) and that $\left(AB\right)^{-1} = B^{-1}A^{-1}$.

(b) Now, we know from the fact that *A* is invertible that

$$AA^{-1} = A^{-1}A = I$$

But this is telling us that if we multiply A^{-1} by *A* on both sides then we'll get the identity matrix. But this is exactly what we need to show that A^{-1} is invertible and that its inverse is *A*.

(c) The best way to prove this part is by a proof technique called induction. However, there's a chance that a good many of you don't know that and that isn't the point of this class. Luckily, for this part anyway, we can at least outline another way to prove this.

To officially prove this part we'll need to show that $\left(A^{n}\right)\left(A^{-n}\right) = \left(A^{-n}\right)\left(A^{n}\right) = I$. We'll show one of the inequalities and leave the other to you to verify since the work is pretty much identical.

$$\begin{aligned}
\left(A^{n}\right)\left(A^{-n}\right) &= \left(\underbrace{AA\cdots A}_{n\text{ times}}\right)\left(\underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{n\text{ times}}\right) \\
&= \left(\underbrace{AA\cdots A}_{n-1\text{ times}}\right)\left(AA^{-1}\right)\left(\underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{n-1\text{ times}}\right) \qquad \text{but } AA^{-1}=I \text{ so,} \\
&= \left(\underbrace{AA\cdots A}_{n-1\text{ times}}\right)\left(\underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{n-1\text{ times}}\right) \\
&= etc. \\
&= \left(AA\right)\left(A^{-1}A^{-1}\right) \\
&= A\left(AA^{-1}\right)A^{-1} \qquad \text{again } AA^{-1}=I \text{ so,} \\
&= AA^{-1} \\
&= I
\end{aligned}$$

Again, we'll leave the second product to you to verify, but the work is identical. After doing this product we can see that A^n is invertible and $\left(A^n\right)^{-1} = A^{-n} = \left(A^{-1}\right)^n$.

(d) To prove this part we'll need to show that $\left(cA\right)\left(\frac{1}{c}A^{-1}\right) = \left(\frac{1}{c}A^{-1}\right)\left(cA\right) = I$. As with the last part we'll do half the work and leave the other half to you to verify.

$$\left(cA\right)\left(\frac{1}{c}A^{-1}\right) = \left(c\cdot\frac{1}{c}\right)\left(AA^{-1}\right) = \left(1\right)\left(I\right) = I$$

Upon doing the second product we can see that cA is invertible and $\left(cA\right)^{-1} = \frac{1}{c}A^{-1}$.

(e) The part will require us to show that $A^T\left(A^{-1}\right)^T = \left(A^{-1}\right)^T A^T = I$ and in keeping with tradition of the last couple parts we'll do the first one and leave the second one to you to verify.

This one is a little tricky at first, but once you realize the correct formula to use it's not too bad. Let's start with $A^T\left(A^{-1}\right)^T$ and then remember that $\left(C\,D\right)^T = D^T C^T$. Using this fact (backwards) on $A^T\left(A^{-1}\right)^T$ gives us,

$$A^T\left(A^{-1}\right)^T = \left(A^{-1}A\right)^T = I^T = I$$

Note that we used the fact that $I^T = I$ here which we'll leave to you to verify.

So, upon showing the second product we'll have that A^T is invertible and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

■

Note that the first part of this theorem can be easily extended to more than two matrices as follows,

$$\left(ABC\right)^{-1} = C^{-1}B^{-1}A^{-1}$$

Now, in the previous section we saw that in general we don't have a cancellation law or a zero factor property. However, if we restrict ourselves just a little we can get variations of both of these.

Theorem 3 Suppose that *A* is an invertible matrix and that *B*, *C*, and *D* are matrices of the same size as *A*.

(a) If $AB = AC$ then $B = C$

(b) If $AD = 0$ then $D = 0$

Proof :

(a) Since we know that *A* is invertible we know that A^{-1} exists so multiply on the left by A^{-1} to get,

$$\begin{aligned} A^{-1}AB &= A^{-1}AC \\ IB &= IC \\ B &= C \end{aligned}$$

(b) Again we know that A^{-1} exists so multiply on the left by A^{-1} to get,

$$\begin{aligned} A^{-1}AD &= A^{-1}0 \\ ID &= 0 \\ D &= 0 \end{aligned}$$

■

Note that this theorem only required that *A* be invertible, it is completely possible that the other matrices are singular.

Note as well with the first one that we've got to remember that matrix multiplication is not commutative and so if we have $AB = CA$ then there is no reason to think that $B = C$ even if *A* is invertible. Because we don't know that $CA = AC$ we've got to leave this as is. Also when we multiply both sides of the equation by A^{-1} we've got multiply each side on the left or each side on the right, which is again because we don't have the commutative law with matrix multiplication. So, if we tried the above proof on $AB = CA$ we'd have,

$$\begin{aligned} A^{-1}AB &= A^{-1}CA \\ B &= A^{-1}CA \end{aligned} \qquad \text{OR} \qquad \begin{aligned} ABA^{-1} &= CAA^{-1} \\ ABA^{-1} &= C \end{aligned}$$

In either case we don't have $B = C$.

Okay, it is now time to take a quick look at Elementary matrices.

Definition 3 A square matrix is called an **elementary matrix** if it can be obtained by applying a single elementary row operation to the identity matrix of the same size.

Here are some examples of elementary matrices and the row operations that produced them.

Example 4 The following matrices are all elementary matrices. Also give is the row operation on the appropriately sized identity matrix.

$$\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \qquad 9R_1 \text{ on } I_2$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad R_1 \leftrightarrow R_4 \text{ on } I_4$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad R_2 - 7R_3 \text{ on } I_4$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad 1 \cdot R_2 \text{ on } I_3$$

Note that the fourth example above shows that any identity matrix is also an elementary matrix since we can think of arriving at that matrix by taking one times any row (not just the second as we used) of the identity matrix.

Here's a really nice theorem about elementary matrices that we'll be using extensively to develop a method for finding the inverse of a matrix.

Theorem 4 Suppose E is an elementary matrix that was found by applying an elementary row operation to I_n. Then if A is an $n \times m$ matrix EA is the matrix that will result by applying the same row operation to A.

Example 5 For the following matrix perform the row operation $R_1 + 4R_2$ on it and then find the elementary matrix, E, for this operation and verify that EA will give the same result.

$$A = \begin{bmatrix} 4 & 5 & -6 & 1 & -1 \\ -1 & 2 & -1 & 10 & 3 \\ 3 & 0 & 4 & -4 & 7 \end{bmatrix}$$

Solution

Performing the row operation is easy enough.

$$\begin{bmatrix} 4 & 5 & -6 & 1 & -1 \\ -1 & 2 & -1 & 10 & 3 \\ 3 & 0 & 4 & -4 & 7 \end{bmatrix} \begin{array}{c} R_1 + 4R_2 \\ \to \end{array} \begin{bmatrix} 0 & 13 & -10 & 41 & 11 \\ -1 & 2 & -1 & 10 & 3 \\ 3 & 0 & 4 & -4 & 7 \end{bmatrix}$$

Now, we can find E simply by applying the same operation to I_3 and so we have,

$$E = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We just need to verify that EA is then the same matrix that we got above.

$$EA = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 5 & -6 & 1 & -1 \\ -1 & 2 & -1 & 10 & 3 \\ 3 & 0 & 4 & -4 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 13 & -10 & 41 & 11 \\ -1 & 2 & -1 & 10 & 3 \\ 3 & 0 & 4 & -4 & 7 \end{bmatrix}$$

Sure enough the same matrix as the theorem predicted.

Now, let's go back to Example 4 for a second and notice that we can apply a second row operation to get the given elementary matrix back to the original identity matrix.

Example 6 Give the operation that will take the elementary matrices from Example 4 back to the original identity matrix.

$$\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{array}{c} \frac{1}{9} R_1 \\ \to \end{array} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} R_1 \leftrightarrow R_4 \\ \to \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -7 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{c} R_2 + 7R_3 \\ \to \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{c} 1 \cdot R_2 \\ \to \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

These kinds of operations are called **inverse operations** and each row operation will have an inverse operation associated with it. The following table gives the inverse operation for each row operation.

Row operation	Inverse Operation
Multiply row *i* by $c \neq 0$	Multiply row *i* by $\frac{1}{c}$
Interchange rows *i* and *j*	Interchange rows *i* and *j*
Add *c* times row *i* to row *j*	Add $-c$ times row *i* to row *j*

Now that we've got inverse operations we can give the following theorem.

Theorem 5 Suppose that *E* is the elementary matrix associated with a particular row operation and that E_0 is the elementary matrix associated with the inverse operation. Then *E* is invertible and $E^{-1} = E_0$

Proof : This is actually a really simple proof. Let's start with $E_0 E$. We know from Theorem 4 that this is the same as if we'd applied the inverse operation to *E*, but we also know that inverse operations will take an elementary matrix back to the original identity matrix. Therefore we have,

$$E_0 E = I$$

Likewise, if we look at EE_0 this will be the same as applying the original row operation to E_0. However, if you think about it this will only undo what the inverse operation did to the identity matrix and so we also have,

$$EE_0 = I$$

Therefore, we've proved that $EE_0 = E_0 E = I$ and so *E* is invertible and $E^{-1} = E_0$.

■

Now, suppose that we've got two matrices of the same size *A* and *B*. If we can reach *B* by applying a finite number of row operations to *A* then we call the two matrices **row equivalent**. Note that this will also mean that we can reach *A* from *B* by applying the inverse operations in the reverse order.

Example 7 Consider

$$A = \begin{bmatrix} 4 & 3 & -2 \\ -1 & 5 & 8 \end{bmatrix}$$

then

$$B = \begin{bmatrix} 4 & 3 & -2 \\ 14 & -1 & -22 \end{bmatrix}$$

is row equivalent to *A* because we reached *B* by first multiplying row 2 of *A* by -2 and the adding 3 times row 1 onto row 2.

For the practice let's do these operations using elementary matrices. Here are the elementary matrices (and their inverses) for the operations on A.

$$-2R_2 \quad : \quad E_1 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \qquad E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

$$R_2 + 3R_1 \quad : \quad E_2 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \qquad E_2^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

Now, to reach B Theorem 4 tells us that we need to multiply the left side of A by each of these in the same order as we applied the operations.

$$\begin{aligned} E_2E_1A &= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 4 & 3 & -2 \\ -1 & 5 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 & -2 \\ 2 & -10 & -16 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 3 & -2 \\ 14 & -1 & -22 \end{bmatrix} = B \end{aligned}$$

Sure enough we get B as we should.

Now, since A and B are row equivalent this means that we should be able to get to A from B by applying the inverse operations in the reverse order. Let's see if that does in fact work.

$$\begin{aligned} E_1^{-1}E_2^{-1}B &= \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 & -2 \\ 14 & -1 & -22 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 4 & 3 & -2 \\ 2 & -10 & -16 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 3 & -2 \\ -1 & 5 & 8 \end{bmatrix} = A \end{aligned}$$

So, we sure enough end up with the correct matrix and again remember that each time we multiplied the left side by an elementary matrix Theorem 4 tells us that is the same thing as applying the associated row operation to the matrix.

Finding Inverse Matrices

In the previous section we introduced the idea of inverse matrices and elementary matrices. In this section we need to devise a method for actually finding the inverse of a matrix and as we'll see this method will, in some way, involve elementary matrices, or at least the row operations that they represent.

The first thing that we'll need to do is take care of a couple of theorems.

Theorem 1 If A is an $n \times n$ matrix then the following statements are equivalent.

(a) A is invertible.

(b) The only solution to the system $A\mathbf{x} = 0$ is the trivial solution.

(c) A is row equivalent to I_n.

(d) A is expressible as a product of elementary matrices.

Before we get into the proof let's say a couple of words about just what this theorem tells us and how we go about proving something like this. First, when we have a set of statements and when we say that they are equivalent then what we're really saying is that either they are all true or they are all false. In other words, if you know one of these statements is true about a matrix A then they are all true for that matrix. Likewise, if one of these statements is false for a matrix A then they are all false for that matrix.

To prove a set of equivalent statements we need to prove a string of implications. This string has to be able to get from any one statement to any other through a finite number of steps. In this case we'll prove the following chain $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$. By doing this if we know one of them to be true/false then we can follow this chain to get to any of they others.

The actual proof will involve four parts, one for each implication. To prove a given implication we'll assume the statement on the left is true and show that this must in some way also force the statement on the right to also be true. So, let's get going.

Proof :

$(a) \Rightarrow (b)$: So we'll assume that A is invertible and we need to show that this assumption also implies that $A\mathbf{x} = 0$ will have only the trivial solution. That's actually pretty easy to do. Since A is invertible we know that A^{-1} exists. So, start by assuming that $\mathbf{x}_0$ is any solution to the system, plug this into the system and then multiply (on the left) both sides by A^{-1} to get,

$$\begin{aligned} A^{-1}A\mathbf{x}_0 &= A^{-1}0 \\ I\mathbf{x}_0 &= 0 \\ \mathbf{x}_0 &= 0 \end{aligned}$$

So, $A\mathbf{x} = 0$ has only the trivial solution and we've managed to prove this implication.

$(b) \Rightarrow (c)$: Here we're assuming that $A\mathbf{x} = 0$ will have only the trivial solution and we'll need to show that A is row equivalent to I_n. Recall that two matrices are row equivalent if we can get from one to the other by applying a finite set of elementary row operations.

Let's start off by writing down the augmented matrix for this system.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 \end{bmatrix}$$

Now, if we were going to solve this we would use elementary row operations to reduce this to reduced row-echelon form, Now we know that the solution to this system must be,

$$x_1 = 0, x_2 = 0, \ldots, x_n = 0$$

by assumption. Therefore, we also know what the reduced row-echelon form of the augmented matrix must be since that must give the above solution. The reduced-row echelon form of this augmented matrix must be,

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Now, the entries in the last column do not affect the values in the entries in the first *n* columns and so if we take the same set of elementary row operations and apply them to *A* we will get I_n and so *A* is row equivalent to I_n since we can get to I_n by applying a finite set of row operations to *A*. Therefore this implication has been proven.

$(c) \Rightarrow (d)$: In this case we're going to assume that *A* is row equivalent to I_n and we'll need to show that *A* can be written as a product of elementary matrices.

So, since *A* is row equivalent to I_n we know there is a finite set of elementary row operations that we can apply to *A* that will give us I_n. Let's suppose that these row operations are represented by the elementary matrices $E_1, E_2, \ldots, E_k$. Then by Theorem 4 of the previous section we know that applying each row operation to *A* is the same thing as multiplying the left side of *A* by each of the corresponding elementary matrices in the same order. So, we then know that we will have the following.

$$E_k \cdots E_2 E_1 A = I_n$$

Now, by Theorem 5 from the previous section we know that each of these elementary matrices is invertible and their inverses are also elementary matrices. So multiply the above equation (on the left) by $E_k^{-1}, \ldots, E_2^{-1}, E_1^{-1}$ (in that order) to get,

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

So, we see that *A* is a product of elementary matrices and this implication is proven.

$(d) \Rightarrow (a)$: Here we'll be assuming that *A* is a product of elementary matrices and we need to show that *A* is invertible. This is probably the easiest implication to prove.

First, A is a product of elementary matrices. Now, by Theorem 5 from the previous section we know each of these elementary matrices is invertible and by Theorem 2(a) also from the previous section we know that a product of invertible matrices is also invertible. Therefore, A is invertible since it can be written as a product of invertible matrices and we've proven this implication.

■

This theorem can actually be extended to include a couple more equivalent statements, but to do that we need another theorem.

Theorem 2 Suppose that A is a square matrix then

(a) If B is a square matrix such that $BA = I$ then A is invertible and $A^{-1} = B$.

(b) If B is a square matrix such that $AB = I$ then A is invertible and $A^{-1} = B$.

Proof :

(a) This proof will need part (b) of Theorem 1. If we can show that $A\mathbf{x} = 0$ has only the trivial solution then by Theorem 1 we will know that A is invertible. So, let $\mathbf{x}_0$ be any solution to $A\mathbf{x} = 0$. Plug this into the equation and then multiply both sides (on the left by B.

$$\begin{aligned} A\mathbf{x}_0 &= 0 \\ BA\mathbf{x}_0 &= B0 \\ I\mathbf{x}_0 &= 0 \\ \mathbf{x}_0 &= 0 \end{aligned}$$

So, this shows that any solution to $A\mathbf{x} = 0$ must be the trivial solution and so by Theorem 1 if one statement is true they all are and so A is invertible. We know from the previous section that inverses are unique and because $BA = I$ we must then also have $A^{-1} = B$.

(b) In this case let's let $\mathbf{x}_0$ be any solution to $B\mathbf{x} = 0$. Then multiplying both sides (on the left) of this by A we can use a similar argument to that used in **(a)** to show that $\mathbf{x}_0$ must be the trivial solution and so B is an invertible matrix and that in fact $B^{-1} = A$. Now, this isn't quite what we were asked to prove, but it does in fact give us the proof. Because B is invertible and its inverse is A (by the above work) we know that,

$$AB = BA = I$$

but this is exactly what it means for A to be invertible and that $A^{-1} = B$. So, we are done.

■

So, what's the big deal with this theorem? We'll recall in the last section that in order to show that a matrix, B, was the inverse of A we needed to show that $AB = BA = I$. In other words, we needed to show that both of these products were the identity matrix. Theorem 2 tells us that all we really need to do is show one of them and we get the other one for free.

This theorem gives us is the ability to add two equivalent statements to Theorem 1. Here is the improved Theorem 1.

Theorem 3 If A is an $n \times n$ matrix then the following statements are equivalent.

(a) A is invertible.
(b) The only solution to the system $A\mathbf{x} = 0$ is the trivial solution.
(c) A is row equivalent to I_n.
(d) A is expressible as a product of elementary matrices.
(e) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
(f) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

Note that **(e)** and **(f)** appear to be the same on the surface, but recall that consistent only says that there is at least one solution. If a system is consistent there may be infinitely many solutions. What this part is telling us is that if the system is consistent for any choice of **b** that we choose to put into the system then we will in fact only get a single solution. If even one **b** gives infinitely many solutions the **(f)** is false, which in turn makes all the other statements false.

Okay so how do we go about proving this? We've already proved that the first four statements are equivalent above so there's no reason to redo that work. This means that all we need to do is prove that one of the original statements implies the new two new statements and these in turn imply one of the four original statements. We'll do this by proving the following implications $(a) \Rightarrow (e) \Rightarrow (f) \Rightarrow (a)$.

Proof :
$(a) \Rightarrow (e)$: Okay with this implication we'll assume that A is invertible and we'll need to show that $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$. This is actually very simple to do. Since A is invertible we know that A^{-1} so we'll do the following.

$$\begin{aligned} A^{-1}A\mathbf{x} &= A^{-1}\mathbf{b} \\ I\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b} \end{aligned}$$

So, if A is invertible we've shown that the solution to the system will be $\mathbf{x} = A^{-1}\mathbf{b}$ and since matrix multiplication is unique (*i.e.* we aren't going to get two different answers from the multiplication) the solution must also be unique and so there is exactly one solution to the system.

$(e) \Rightarrow (f)$: This implication is trivial. We'll start off by assuming that the system $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$ but that also means that the system is consistent every $n \times 1$ matrix $\mathbf{b}$ and so we're done with the proof of this implication.

$(f) \Rightarrow (a)$: Here we'll start off by assuming that $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix **b** and we'll need to show that this implies *A* is invertible. So, if $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix **b** it is consistent for the following *n* systems.

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1} \qquad A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1} \qquad \cdots \qquad A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{n\times 1}$$

Since we know each of these systems have solutions let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ be those solutions and form a new matrix, *B*, with these solutions as its columns. In other words,

$$B = \left[\mathbf{x}_1 \mid \mathbf{x}_2 \mid \cdots \mid \mathbf{x}_n \right]$$

Now let's take a look at the product *AB*. We know from the matrix arithmetic section that the i^{th} column of *AB* will be given by $A\mathbf{x}_i$ and we know what each of these products will be since $\mathbf{x}_i$ is a solution to one of the systems above. So, let's use all this knowledge to see what the product *AB* is.

$$AB = \left[A\mathbf{x}_1 \mid A\mathbf{x}_2 \mid \cdots \mid A\mathbf{x}_n \right] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I$$

So, we've shown that $AB = I$, but by Theorem 2 this means that *A* must be invertible and so we're done with the proof.

■

Before proceeding let's notice that part (c) of this theorem is also telling us that if we reduced *A* down to reduced row-echelon form then we'd have I_n. This can also be seen in the proof in Theorem 1 of the implication $(b) \Rightarrow (c)$.

So, just how does this theorem help us to determine the inverse of a matrix? Well, first let's assume that *A* is in fact invertible and so all the statements in Theorem 3 are true. Now, go back to the proof of the implication $(c) \Rightarrow (d)$. In this proof we saw that there were elementary matrices, $E_1, E_2, \ldots, E_k$, so that we'd get the following,

$$E_k \cdots E_2 E_1 A = I_n$$

Since we know *A* is invertible we know that A^{-1} exists and so multiply (on the right) each side of this to get,

$$E_k \cdots E_2 E_1 A A^{-1} = I_n A^{-1} \qquad \Rightarrow \qquad A^{-1} = E_k \cdots E_2 E_1 I_n$$

What this tell us is that we need to find a series of row operation that will reduce *A* to I_n and then apply the same set of operations to I_n and the result will be the inverse, A^{-1}.

Okay, all this is fine. We can write down a bunch of symbols to tell us how to find the inverse, but that doesn't always help to actually find the inverse. The work above tells us that we need to identify a series of elementary row operations that will reduce *A* to I_n and then apply those operations to I_n. We'll it turns out that we can do both of these steps simultaneously and we don't need to mess around with the elementary matrices.

Let's start off by supposing that *A* is an invertible $n \times n$ matrix and then form the following new matrix.

$$\left[A \;\vdots\; I_n \right]$$

Note that all we did here was tack on I_n to the original matrix *A*. Now, if we apply a row operation to this it will be equivalent to applying it simultaneously to both *A* and to I_n. So, all we need to do is find a series of row operations that will reduce the "*A*" portion of this to I_n, making sure to apply the operations to the whole matrix. Once we've done this we will have,

$$\left[I_n \;\vdots\; A^{-1} \right]$$

provided *A* is in fact invertible of course. We'll deal with singular matrices in a bit.

Let's take a look at a couple of examples.

Example 1 Determine the inverse of the following matrix given that it is invertible.

$$A = \begin{bmatrix} -4 & -2 \\ 5 & 5 \end{bmatrix}$$

Solution

Note that this is the 2×2 we looked at in Example 1 of the previous section. In that example stated (and proved) that the inverse was,

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{5} \\ \frac{1}{2} & \frac{2}{5} \end{bmatrix}$$

We can now show how we arrived at this for the inverse.

We'll first form the new matrix

$$\left[\begin{array}{cc:cc} -4 & -2 & 1 & 0 \\ 5 & 5 & 0 & 1 \end{array} \right]$$

Next we'll find row operations that will convert the first two columns into I_2 and the third and fourth columns should then contain A^{-1}. Here is that work,

$$\left[\begin{array}{rr|rr} -4 & -2 & 1 & 0 \\ 5 & 5 & 0 & 1 \end{array}\right] \underset{\rightarrow}{R_1+R_2} \left[\begin{array}{rr|rr} 1 & 3 & 1 & 1 \\ 5 & 5 & 0 & 1 \end{array}\right] \underset{\rightarrow}{R_2-5R_1} \left[\begin{array}{rr|rr} 1 & 3 & 1 & 1 \\ 0 & -10 & -5 & -4 \end{array}\right]$$

$$\underset{\rightarrow}{-\tfrac{1}{10}R_2} \left[\begin{array}{rr|rr} 1 & 3 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & \frac{2}{5} \end{array}\right] \underset{\rightarrow}{R_1-3R_2} \left[\begin{array}{rr|rr} 1 & 0 & -\frac{1}{2} & -\frac{1}{5} \\ 0 & 1 & \frac{1}{2} & \frac{2}{5} \end{array}\right]$$

So, the first two columns are in fact I_2 and in the third and fourth columns we've got the inverse,

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{5} \\ \frac{1}{2} & \frac{2}{5} \end{bmatrix}$$

Example 2 Determine the inverse of the following matrix given that it is invertible.

$$C = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 2 & 2 \\ 5 & 0 & -1 \end{bmatrix}$$

Solution

Okay we'll first form the new matrix,

$$\left[\begin{array}{rrr|rrr} 3 & 1 & 0 & 1 & 0 & 0 \\ -1 & 2 & 2 & 0 & 1 & 0 \\ 5 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

and we'll use elementary row operations to reduce the first three rows to I_3 and then the last three rows will be the inverse of *C*. Here is that work.

$$\left[\begin{array}{rrr|rrr} 3 & 1 & 0 & 1 & 0 & 0 \\ -1 & 2 & 2 & 0 & 1 & 0 \\ 5 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \underset{\rightarrow}{R_1+2R_2} \left[\begin{array}{rrr|rrr} 1 & 5 & 4 & 1 & 2 & 0 \\ -1 & 2 & 2 & 0 & 1 & 0 \\ 5 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} R_2+R_1 \\ R_3-5R_1 \\ \rightarrow \end{array} \left[\begin{array}{rrr|rrr} 1 & 5 & 4 & 1 & 2 & 0 \\ 0 & 7 & 6 & 1 & 3 & 0 \\ 0 & -25 & -21 & -5 & -10 & 1 \end{array}\right] \underset{\rightarrow}{\tfrac{1}{7}R_2} \left[\begin{array}{rrr|rrr} 1 & 5 & 4 & 1 & 2 & 0 \\ 0 & 1 & \frac{6}{7} & \frac{1}{7} & \frac{3}{7} & 0 \\ 0 & -25 & -21 & -5 & -10 & 1 \end{array}\right]$$

$$\underset{\rightarrow}{R_3+25R_2} \left[\begin{array}{rrr|rrr} 1 & 5 & 4 & 1 & 2 & 0 \\ 0 & 1 & \frac{6}{7} & \frac{1}{7} & \frac{3}{7} & 0 \\ 0 & 0 & \frac{3}{7} & -\frac{10}{7} & \frac{5}{7} & 1 \end{array}\right] \underset{\rightarrow}{\tfrac{7}{3}R_3} \left[\begin{array}{rrr|rrr} 1 & 5 & 4 & 1 & 2 & 0 \\ 0 & 1 & \frac{6}{7} & \frac{1}{7} & \frac{3}{7} & 0 \\ 0 & 0 & 1 & -\frac{10}{3} & \frac{5}{3} & \frac{7}{3} \end{array}\right]$$

$$\begin{array}{c} R_2-\frac{6}{7}R_3 \\ R_1-4R_3 \\ \rightarrow \end{array} \left[\begin{array}{rrr|rrr} 1 & 5 & 0 & \frac{43}{3} & -\frac{14}{3} & -\frac{28}{3} \\ 0 & 1 & 0 & 3 & -1 & -2 \\ 0 & 0 & 1 & -\frac{10}{3} & \frac{5}{3} & \frac{7}{3} \end{array}\right] \underset{\rightarrow}{R_1-5R_2} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ 0 & 1 & 0 & 3 & -1 & -2 \\ 0 & 0 & 1 & -\frac{10}{3} & \frac{5}{3} & \frac{7}{3} \end{array}\right]$$

So, we've gotten the first three columns reduced to I_3 and that means the last three must be the inverse.

$$C^{-1} = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ 3 & -1 & -2 \\ -\frac{10}{3} & \frac{5}{3} & \frac{7}{3} \end{bmatrix}$$

We'll leave it to you to verify that $CC^{-1} = C^{-1}C = I_3$.

Okay, so far we've seen how to use the method above to determine an inverse, but what happens if a matrix doesn't have an inverse? We'll it turns out that we can also use this method to determine that as well and it generally doesn't take quite as much work as it does to actually find the inverse (if it exists of course….).

Let's take a look at an example of that.

Example 3 Show that the following matrix does not have an inverse, *i.e.* show the matrix is singular.

$$B = \begin{bmatrix} 3 & 3 & 6 \\ 0 & 1 & 2 \\ -2 & 0 & 0 \end{bmatrix}$$

Solution

Okay, the problem statement says that the matrix is singular, but let's pretend that we didn't know that and work the problem as we did in the previous two examples. That means we'll need the new matrix,

$$\left[\begin{array}{ccc|ccc} 3 & 3 & 6 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Now, let's get started on getting the first three columns reduced to I_3.

$$\left[\begin{array}{ccc|ccc} 3 & 3 & 6 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 6 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 + 2R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 6 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 6 & 12 & 2 & 0 & 3 \end{array}\right] \xrightarrow{R_3 - 6R_2} \left[\begin{array}{ccc|ccc} 1 & 3 & 6 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & -6 & 3 \end{array}\right]$$

At this point let's stop and examine the third row in a little more detail. In order for the first three columns to be I_3 the first three entries of the last row MUST be $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ which we clearly don't have. We could use a multiple of row 1 or row 2 to get a 1 in the third spot, but that would in turn change at least one of the first two entries away from 0. That's a problem since they must remain zeroes.

In other words, there is on way to make the third entry in the third row a 1 without also changing on or both of the first two entries into something other than zero and so we will never be able to make the first three columns into I_3.

So, there are no sets of row operations that will reduce B to I_3 and hence B is NOT row equivalent to I_3. Now, go back to Theorem 3. This was a set of equivalent statements and if one is false they are all false. We've just managed to show that part (c) is false and that means that part (a) must also be false. Therefore, B must be a singular matrix.

The idea used in this last example to show that B was singular can be used in general. If, in the course of reducing the new matrix, we ever end up with a row in which all the entries to the left of the dashed line are zeroes we will know that the matrix must be singular.

We'll leave this section off with a quick formula that can always be used to find the inverse of an invertible 2×2 matrix as well as a way to quickly determine if the matrix is invertible. The above method is nice in that it always works, but it can be cumbersome to use so the following formula can help to make things go quicker for 2×2 matrices.

Theorem 4 The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

will be invertible if $ad - bc \neq 0$ and singular if $ad - bc = 0$. If the matrix is invertible its inverse will be,

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Let's do a quick example or two of this fact.

Example 4 Use the fact to show that

$$A = \begin{bmatrix} -4 & -2 \\ 5 & 5 \end{bmatrix}$$

is an invertible matrix and find its inverse.

Solution

We've already looked at this one above, but let's do it here so we can contrast the work between the two methods. First, we need,

$$ad - bc = (-4)(5) - (5)(-2) = -10 \neq 0$$

So, the matrix is in fact invertible by the fact and here is the inverse,

$$A^{-1} = \frac{1}{-10} \begin{bmatrix} 5 & 2 \\ -5 & -4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{5} \\ \frac{1}{2} & \frac{2}{5} \end{bmatrix}$$

Example 5 Determine if the following matrix is singular.

$$B = \begin{bmatrix} -4 & -2 \\ 6 & 3 \end{bmatrix}$$

Solution

Not much to do with this one.

$$(-4)(3) - (-2)(6) = 0$$

So, by the fact the matrix is singular.

If you'd like to see a couple more example of finding inverses check out the section on Special Matrices, there are a couple more examples there.

Special Matrices

This section is devoted to a couple of special matrices that we could have talked about pretty much anywhere, but due to the desire to keep most of these sections as small as possible they just didn't fit in anywhere. However, we'll need a couple of these in the next section and so we now need to get them out of the way.

Diagonal Matrix

This first one that we're going to take a look at is a diagonal matrix. A square matrix is called **diagonal** if it has the following form.

$$D = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}_{n \times n}$$

In other words, in a diagonal matrix is any matrix in which the only potentially non-zero entries are one the main diagonal. Any entry off the main diagonal must be zero and note that it is possible to have one or more of the main diagonal entries be zero.

We've also been dealing with a diagonal matrix already to this point if you think about it a little. The identity matrix is a diagonal matrix.

Here is a nice theorem about diagonal matrices.

Theorem 1 Suppose D is a diagonal matrix and $d_1, d_2, \ldots d_n$ are the entries on the main diagonal. If one or more of the d_i's are zero then the matrix is singular. On the other hand if $d_i \neq 0$ for all i then the matrix is invertible and the inverse is,

$$D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{d_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}$$

Proof : First, recall Theorem 3 from the previous section. This theorem tells us that if *D* is row equivalent to the identity matrix then *D* is invertible and if *D* is not row equivalent to the identity then *D* is singular.

If none of the d_i's are zero then we can reduce *D* to the identity simply dividing each of the rows its diagonal entry (which we can do since we've assumed none of them are zero) and so in this case *D* will be row equivalent to the identity. Therefore, in this case *D* is invertible. We'll leave it to you to verify that the inverse is what we claim it to be. You can either compute this directly using the method from the previous section or you can verify that $DD^{-1} = D^{-1}D = I$.

Now, suppose that at least one of the d_i is equal to zero. In this case we will have a row of all zeroes, and because *D* is a diagonal matrix all the entries above the main diagonal entry in this row will also be zero and so there is no way for us to use elementary row operations to put a 1 into the main diagonal and so in this case *D* will not be row equivalent to the identity and hence must be singular.

■

Powers of diagonal matrices are also easy to compute. If *D* is a diagonal matrix and *k* is any integer then

$$D^k = \begin{bmatrix} d_1^k & 0 & 0 & \cdots & 0 \\ 0 & d_2^k & 0 & \cdots & 0 \\ 0 & 0 & d_3^k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

Triangular Matrix

The next kind of matrix we want to take a look at will be triangular matrices. In fact there are actually two kinds of triangular matrix. For an **upper triangular** matrix the matrix must be square and all the entries below the main diagonal are zero and the main diagonal entries and the entries above it may or may not be zero. A **lower triangular** matrix is just the opposite. The matrix is still a square matrix and all the entries of a lower triangular matrix above the main diagonal are zero and the main diagonal entries and those below it may or may not be zero.

Here are the general forms of an upper and lower triangular matrix.

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix}_{n\times n} \qquad \text{Upper Triangular}$$

$$L = \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix}_{n\times n} \qquad \text{Lower Triangular}$$

In these forms the u_{ij} and l_{ij} may or may not be zero.

If we do not care if the matrix is upper or lower triangular we will generally just call it **triangular**.

Note as well that a diagonal matrix can be thought of as both an upper triangular matrix and a lower triangular matrix.

Here's a nice theorem about the invertibility of a triangular matrix.

Theorem 2 If A is a triangular matrix with main diagonal entries $a_{11}, a_{22}, \ldots, a_{nn}$ then if one or more of the a_{ii}'s are zero the matrix will be singular. On the other hand if $a_{ii} \neq 0$ for all i then the matrix is invertible.

Here is the outline of the proof.

Proof Outline : First assume that $a_{ii} \neq 0$ for all i. In this case we can divide each row by a_{ii} (since it's not zero) and that will put a 1 in the main diagonal entry for each row. Now use the third row operation to eliminate all the non-zero entries above the main diagonal entry for an upper triangular matrix or below it for a lower triangular matrix. When done with these operations we will have reduced A to the identity matrix. Therefore, in this case A is row equivalent to the identity and so must be invertible.

Now assume that at least one of the a_{ii} are zero. In this case we can't get a 1 in the main diagonal entry just be dividing by a_{ii} as we did in the first place. Now, for a second let's suppose we have an upper triangular matrix. In this case we could use the third row operation using one of the rows above this to get a 1 into the main diagonal entry, however, this will also put non-zero entries into the entries to the left of this as well. In other words, we're not going to be able to reduce *A* to the identity matrix. The same type of problem will arise if we've got a lower triangular matrix.

In this case, *A* will not be row equivalent to the identity and so will be singular.

■

Here is another set of theorems about triangular matrices that we aren't going to prove.

Theorem 3

(a) The product of lower triangular matrices will be a lower triangular matrix.
(b) The product of upper triangular matrices will be an upper triangular matrix.
(c) The inverse of an invertible lower triangular matrix will be a lower triangular matrix.
(d) The inverse of an invertible upper triangular matrix will be an upper triangular matrix.

The proof of these will pretty much follow from how products and inverses are found and so well be left to you to verify.

The final kind of matrix that we want to look at in this section is that of a symmetric matrix. In fact we've already seen these in a previous section we just didn't have the space to investigate them in more detail in that section so we're going to do it here.

For completeness sake we'll give the definition here again. Suppose that *A* is an $n \times m$ matrix, then *A* will be called **symmetric** if $A = A^T$.

Note that the first requirement for a matrix to be symmetric is that the matrix must be square. Since the size of A^T will be $m \times n$ there is no way *A* and A^T can be equal if *A* is not square since they won't have the same size.

Example 1 The following matrices are all symmetric.

$$A = \begin{bmatrix} 4 & 6 \\ 6 & -7 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & -10 & 3 & 0 \\ -10 & 0 & 1 & -4 \\ 3 & 1 & 12 & 8 \\ 0 & -4 & 8 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} 10 \end{bmatrix}$$

We'll leave it to you to compute the transposes of each of these and verity that they are in fact symmetric. Notice with the second matrix (*B*) above that you can always quickly identify a symmetric matrix by looking at the diagonals off the main diagonal. The diagonals right above and below the main diagonal consists of the entries -10, 1, 8 are identical. Likewise, the diagonals two above and below the main diagonal consists of the entries 3, -4 and again are identical. Finally, the "diagonals" that are three above and below the main diagonal is identical as well.

This idea we see in the second matrix above will be true in any symmetric matrix.

Here is a nice set of facts about arithmetic with symmetric matrices.

Theorem 4 If A and B are symmetric matrices of the same size and c is any scalar then,

(a) $A \pm B$ is symmetric.
(b) cA is symmetric.
(c) A^T is symmetric.

Note that the product of two symmetric matrices is probably not symmetric. To see why this is consider the following. Suppose both A and B are symmetric matrices of the same size then,

$$\left(AB\right)^T = B^T A^T = BA$$

Notice that we used one of the properties of transposes we found earlier in the first step and the fact that A and B are symmetric in the last step.

So what this tells us is that unless A and B commute we won't have $\left(AB\right)^T = AB$ and the product won't be symmetric. If A and B do commute then the product will be symmetric.

Now, if A is any $n \times m$ matrix then because A^T will have size $m \times n$ both AA^T and $A^T A$ will be defined and in fact will be square matrices where AA^T has size $n \times n$ and $A^T A$ has size $m \times m$.

Here are a couple of quick facts about symmetric matrices.

Theorem 5

(a) For any matrix A both AA^T and $A^T A$ are symmetric.
(b) If A is an invertible symmetric matrix then A^{-1} is symmetric.
(c) If A is invertible then AA^T and $A^T A$ are both invertible.

Proof :

(a) We'll show that AA^T is symmetric and leave the other to you to verify. To show that AA^T is symmetric we'll need to show that $\left(AA^T\right)^T = AA^T$. This is actually quite simple if we recall the various properties of transpose matrices that we've got..

$$\left(AA^T\right)^T = \left(A^T\right)^T A^T = \left(A\right) A^T = AA^T$$

(b) In this case all we need is a theorem from a previous section to show that $\left(A^{-1}\right)^T = A^{-1}$. Here is the work,

$$\left(A^{-1}\right)^{T}=\left(A^{T}\right)^{-1}=\left(A\right)^{-1}=A^{-1}$$

(c) If A is invertible then we also know that A^T is invertible and we since the product of invertible matrices is invertible both AA^T and A^TA are invertible.

■

Let's finish this section with an example or two illustrating the results of some of the theorems above.

Example 2 Given the following matrices compute the indicated quantities.

$$A=\begin{bmatrix}4&-2&1\\0&9&-6\\0&0&-1\end{bmatrix}\qquad B=\begin{bmatrix}-2&0&3\\0&7&-1\\0&0&5\end{bmatrix}\qquad C=\begin{bmatrix}3&0&0\\0&2&0\\9&5&4\end{bmatrix}$$

$$D=\begin{bmatrix}-2&0&-4&1\\1&0&-1&6\\8&2&1&-1\end{bmatrix}\qquad E=\begin{bmatrix}1&-2&0\\-2&3&1\\0&1&0\end{bmatrix}$$

(a) AB
(b) C^{-1}
(c) D^TD
(d) E^{-1}

Solution

(a) There really isn't much to do here other than the multiplication and we'll leave it to you to verify the actual multiplication.

$$A=\begin{bmatrix}-8&-14&19\\0&63&-39\\0&0&-5\end{bmatrix}$$

So, as suggested by Theorem 3 the product of upper triangular matrices is in fact an upper triangular matrix.

(b) Here's the work for finding C^{-1}.

$$\left[\begin{array}{ccc|ccc}3&0&0&1&0&0\\0&2&0&0&1&0\\9&5&4&0&0&1\end{array}\right]\begin{array}{c}\frac{1}{3}R_1\\\frac{1}{2}R_2\\\rightarrow\end{array}\left[\begin{array}{ccc|ccc}1&0&0&\frac{1}{3}&0&0\\0&1&0&0&\frac{1}{2}&0\\9&5&4&0&0&1\end{array}\right]$$

$$\begin{array}{c}R_3-9R_1\\\rightarrow\end{array}\left[\begin{array}{ccc|ccc}1&0&0&\frac{1}{3}&0&0\\0&1&0&0&\frac{1}{2}&0\\0&5&4&-3&0&1\end{array}\right]\begin{array}{c}R_3-5R_2\\\rightarrow\end{array}\left[\begin{array}{ccc|ccc}1&0&0&\frac{1}{3}&0&0\\0&1&0&0&\frac{1}{2}&0\\0&0&4&-3&-\frac{5}{2}&1\end{array}\right]$$

$$\begin{array}{c}\frac{1}{4}R_3\\\rightarrow\end{array}\left[\begin{array}{ccc|ccc}1&0&0&\frac{1}{3}&0&0\\0&1&0&0&\frac{1}{2}&0\\0&0&1&-\frac{3}{4}&-\frac{5}{8}&\frac{1}{4}\end{array}\right]\quad\Rightarrow\quad C^{-1}=\begin{bmatrix}\frac{1}{3}&0&0\\0&\frac{1}{2}&0\\-\frac{3}{4}&-\frac{5}{8}&\frac{1}{4}\end{bmatrix}$$

So, again as suggested by Theorem 3 the inverse of a lower triangular matrix is also a lower triangular matrix.

(c) Here's the transpose and the product.

$$D^T = \begin{bmatrix} -2 & 1 & 8 \\ 0 & 0 & 2 \\ -4 & -1 & 1 \\ 1 & 6 & -1 \end{bmatrix}$$

$$D^T D = \begin{bmatrix} -2 & 1 & 8 \\ 0 & 0 & 2 \\ -4 & -1 & 1 \\ 1 & 6 & -1 \end{bmatrix} \begin{bmatrix} -2 & 0 & -4 & 1 \\ 1 & 0 & -1 & 6 \\ 8 & 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 69 & 16 & 15 & -4 \\ 16 & 4 & 2 & -2 \\ 15 & 2 & 18 & -11 \\ -4 & -2 & -11 & 38 \end{bmatrix}$$

So, as suggested by Theorem 5 this product is symmetric even though *D* was not symmetric (or square for that matter).

(d) Here is the work for finding E^{-1}.

$$E = \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ -2 & 3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 + 2R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 2 & 1 & 0 \end{array}\right] \xrightarrow[R_3 + R_1]{R_1 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 & 1 \end{array}\right]$$

So, the inverse is

$$E^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

and as suggested by Theorem 5 the inverse is symmetric.

LU-Decomposition

In this section we're going to discuss a method for *factoring* a square matrix *A* into a product of a lower triangular matrix, *L*, and an upper triangular matrix, *U*. Such a factorization can be used to solve systems of equations as we'll see in the next section when we revisit that topic.

Let's start the section out with a definition and a theorem.

Definition 1 If *A* is a square matrix and it can be factored as $A = LU$ where *L* is a lower triangular matrix and U is an upper triangular matrix, then we say that A has an **LU-Decomposition** of LU.

Theorem 1 If A is a square matrix and it can be reduced to a row-echelon form, U, without interchanging any rows then A can be factored as $A = LU$ where L is a lower triangular matrix.

We're not going to prove this theorem but let's examine it in some detail and we'll find a way to determine a way of determining L. Let's start off by assuming that we've got a square matrix A and that we are able to reduce it row-echelon form U without interchanging any rows. We know that each row operation that we used has a corresponding elementary matrix, so let's suppose that the elementary matrices corresponding to the row operations we used are $E_1, E_2, \ldots, E_k$.

We know from Theorem 4 in a previous section that multiplying these to the left side of A in the same order we applied the row operations will be the same as actually applying the operations. So, this means that we've got,

$$E_k \cdots E_2 E_1 A = U$$

We also know that elementary matrices are invertible so let's multiply each side by the inverses, $E_k^{-1}, \ldots, E_2^{-1}, E_1^{-1}$, in that order to get,

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} U$$

Now, it can be shown that provided we avoid interchanging rows the elementary row operations that we needed to reduce A to U will all have corresponding elementary matrices that are lower triangular matrices. We also know from the previous section that inverses of lower triangular matrices are lower triangular matrices and products of lower triangular matrices are lower triangular matrices. In other words, $L = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$ is a lower triangular matrix and so using this we get the LU-Decomposition for A of $A = LU$.

Let's take a look at an example of this.

Example 1 Determine an LU-Decomposition for the following matrix.

$$A = \begin{bmatrix} 3 & 6 & -9 \\ 2 & 5 & -3 \\ -4 & 1 & 10 \end{bmatrix}$$

Solution

So, first let's go through the row operations to get this into row-echelon form and remember that we aren't allowed to do any interchanging of rows. Also, we'll do this step by step so that we can keep track of the row operations that we used since we're going to need to write down the elementary matrices that are associated with them eventually.

$$\begin{bmatrix} 3 & 6 & -9 \\ 2 & 5 & -3 \\ -4 & 1 & 10 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -3 \\ -4 & 1 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -3 \\ -4 & 1 & 10 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 3 \\ -4 & 1 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 3 \\ -4 & 1 & 10 \end{bmatrix} \xrightarrow{R_3 + 4R_1} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 3 \\ 0 & 9 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 3 \\ 0 & 9 & -2 \end{bmatrix} \xrightarrow{R_3 - 9R_2} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & -29 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & -29 \end{bmatrix} \xrightarrow{-\frac{1}{29}R_3} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Okay so, we've got our hands on U.

$$U = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we need to get L. This is going to take a little more work. We'll need the elementary matrices for each of these, or more precisely their inverses. Recall that we can get the elementary matrix for a particular row operation by applying that operation to the appropriately sized identity matrix (3×3 in this case). Also recall that the inverse matrix can be found by applying the inverse operation to the identity matrix.

Here are the elementary matrices and their inverses for each of the operations above.

$$\frac{1}{3}R_1 \qquad E_1 = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_1^{-1} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 - 2R_1 \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 + 4R_1 \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \qquad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

$$R_3 - 9R_2 \qquad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -9 & 1 \end{bmatrix} \qquad E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 9 & 1 \end{bmatrix}$$

$$-\frac{1}{29}R_3 \qquad E_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{29} \end{bmatrix} \qquad E_5^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -29 \end{bmatrix}$$

Okay, we know can compute *L*.

$$\begin{aligned} L &= E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1} \\ &= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 9 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -29 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & 9 & -29 \end{bmatrix} \end{aligned}$$

Finally, we can verify that we've gotten an LU-Decomposition with a quick computation.

$$\begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & 9 & -29 \end{bmatrix}\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & -9 \\ 2 & 5 & -3 \\ -4 & 1 & 10 \end{bmatrix} = A$$

So we did all the work correctly.

That was a lot of work to determine *L*. There is an easier way to do it however. Let's start off with a general *L* with "*" in place of the potentially non-zero terms.

$$L = \begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix}$$

Let's start with the main diagonal and go back and look at the operations that was required to get 1's on the diagonal when we were computing *U*. To get a 1 in the first row we had to multiply that row by $\frac{1}{3}$. We didn't need to do anything to get a 1 in the second row, but for the sake argument let's say that we actually multiplied that row by 1. Finally, we multiplied the third row by $-\frac{1}{29}$ to get a 1 in the main diagonal entry in that row.

Next go back and look at the *L* that we had for this matrix. The main diagonal entries are 3, 1, and -29. In other words, they are the reciprocal of the numbers we used in computing *U*. This will always be the case. The main diagonal of *L* then using this idea is,

$$L = \begin{bmatrix} 3 & 0 & 0 \\ * & 1 & 0 \\ * & * & -29 \end{bmatrix}$$

Now, let's take a look at the two entries under the 3 in the first column. Again go back to the operations used to find *U* and take a look at the operations we used to get zeroes in these two spots. To get a zero in the second row we added $-2R_1$ onto R_2 and to get a zero in the third row we added $4R_1$ onto R_3.

Again, go back to the *L* we found and notice that these two entries are 2 and -4. Or, they are the negative of the multiple of the first row that we added onto that particular row to get that entry to be zero. Filling these in we now arrive at,

$$L = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & * & -29 \end{bmatrix}$$

Finally, in determining *U* we $-9R_2$ onto R_3 to get the entry in the third row and second column to be zero and in the *L* we found this entry is 9. Again, it's the negative of the multiple of the second row we used to make this entry zero. This gives us the final entry in *L*.

$$L = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & 9 & -29 \end{bmatrix}$$

This process we just went through will always work in determining *L* for our LU-Decomposition provided we follow the process above to find *U*. In fact that is the one drawback to this process. We need to find *U* using exactly the same steps we used in this example. In other words, multiply/divide the first row by an appropriate scalar to get a 1 in the first column then zero out the entries below that one. Next, multiply/divide the second row by an appropriate scalar to get a 1 in the main diagonal entry of the second row and then zero out all the entries below this. Continue in this fashion until you've dealt with all the columns. This will sometimes lead to some messy fractions.

Let's take a look at another example and this time we'll use the procedure outlined above to find *L* instead of dealing with all the elementary matrices.

Example 2 Determine an LU-Decomposition for the following matrix.

$$B=\begin{bmatrix} 2 & 3 & -4 \\ 5 & 4 & 4 \\ -1 & 7 & 0 \end{bmatrix}$$

Solution

So, we first need to reduce B to row-echelon form without using row interchanges. Also, if we're going to use the process outlined above to find L we'll need to do the reduction in the same manner as the first example. Here is that work.

$$\begin{bmatrix} 2 & 3 & -4 \\ 5 & 4 & 4 \\ -1 & 7 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & \frac{3}{2} & -2 \\ 5 & 4 & 4 \\ -1 & 7 & 0 \end{bmatrix} \xrightarrow{\substack{R_2-5R_1 \\ R_3+R_1}} \begin{bmatrix} 1 & \frac{3}{2} & -2 \\ 0 & -\frac{7}{2} & 14 \\ 0 & \frac{17}{2} & -2 \end{bmatrix}$$

$$\xrightarrow{-\frac{2}{7}R_2} \begin{bmatrix} 1 & \frac{3}{2} & -2 \\ 0 & 1 & -4 \\ 0 & \frac{17}{2} & -2 \end{bmatrix} \xrightarrow{R_3-\frac{17}{2}R_2} \begin{bmatrix} 1 & \frac{3}{2} & -2 \\ 0 & 1 & -4 \\ 0 & 0 & 32 \end{bmatrix} \xrightarrow{\frac{1}{32}R_3} \begin{bmatrix} 1 & \frac{3}{2} & -2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

So, U is,

$$U=\begin{bmatrix} 1 & \frac{3}{2} & -2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, let's get L. Again, we'll start with a general L and the main diagonal entries will be the reciprocal of the scalars we needed to multiply each row by to get a one in the main diagonal entry. This gives,

$$L=\begin{bmatrix} 2 & 0 & 0 \\ * & -\frac{7}{2} & 0 \\ * & * & 32 \end{bmatrix}$$

Now, for the remaining entries, go back to the process and look for the multiple that was needed to get a zero in that spot and this entry will be the negative of that multiple. This gives us our final L.

$$L=\begin{bmatrix} 2 & 0 & 0 \\ 5 & -\frac{7}{2} & 0 \\ -1 & \frac{17}{2} & 32 \end{bmatrix}$$

As a final check we can always do a quick multiplication to verify that we do in fact get B from this factorization.

$$\begin{bmatrix} 2 & 0 & 0 \\ 5 & -\frac{7}{2} & 0 \\ -1 & \frac{17}{2} & 32 \end{bmatrix}\begin{bmatrix} 1 & \frac{3}{2} & -2 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 2 & 3 & -4 \\ 5 & 4 & 4 \\ -1 & 7 & 0 \end{bmatrix}=B$$

So, it looks like we did all the work correctly.

We'll leave this section by pointing out a couple of facts about LU-Decompositions.

First, given a random square matrix, A, the only way we can guarantee that A will have an LU-Decomposition is if we can reduce it to row-echelon form without interchanging any rows. If we do have to interchange rows then there is a good chance that the matrix will NOT have an LU-Decomposition.

Second, notice that every time we've talked about an LU-Decomposition of a matrix we've used the word "an" and not "the" LU-Decomposition. This choice of words is intentional. As the choice suggests there is no single unique LU-Decomposition for A.

To see that LU-Decompositions are not unique go back to the first example. In that example we computed the following LU-Decomposition.

$$\begin{bmatrix} 3 & 6 & -9 \\ 2 & 5 & -3 \\ -4 & 1 & 10 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & 9 & -29 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

However, we've also got the following LU-Decomposition.

$$\begin{bmatrix} 3 & 6 & -9 \\ 2 & 5 & -3 \\ -4 & 1 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ -\frac{4}{3} & 9 & 1 \end{bmatrix} \begin{bmatrix} 3 & 6 & -9 \\ 0 & 1 & 3 \\ 0 & 0 & -29 \end{bmatrix}$$

This is clearly an LU-Decomposition since the first matrix is lower triangular and the second is upper triangular and you should verify that upon multiplying they do in fact give the shown matrix.

If you would like to see a further example of an LU-Decomposition worked out there is an example in the next section.

Systems Revisited

We opened up this chapter talking about systems of equations and we spent a couple of sections on them and then we moved away from them and haven't really talked much about them since. It's time to come back to systems and see how some of the ideas we've been talking about since then can be used to help us solve systems. We'll also take a quick look at a couple of other ideas about systems that we didn't look at earlier.

First let's recall that any system of n equations and m unknowns,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= b_n \end{aligned}$$

can be written in matrix form as follows.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$
$$A\mathbf{x} = \mathbf{b}$$

In the matrix form A is called the **coefficient matrix** and each row contains the coefficients of the corresponding equations, $\mathbf{x}$ is a column matrix that contains all the unknowns from the system of equations and finally $\mathbf{b}$ is a column matrix containing the constants on the right of the equal sign.

Now, let's see how inverses can be used to solve systems. First, we'll need to assume that the coefficient matrix is a square $n \times n$ matrix. In other words there are the same number of equations as unknowns in our system. Let's also assume that A is invertible. In this case we actually saw in the proof of Theorem 3 in the section on finding inverses that the solution to $A\mathbf{x} = \mathbf{b}$ is unique (*i.e.* only a single solution exists) and that it's given by,

$$\mathbf{x} = A^{-1}\mathbf{b}$$

So, if we've got the inverse of the coefficient matrix in hand (not always an easy thing to find of course…) we can get the solution based on a quick matrix multiplication. Let's see an example of this.

Example 1 Use the inverse of the coefficient matrix to solve the following system.

$$\begin{aligned} 3x_1 + x_2 &= 6 \\ -x_1 + 2x_2 + 2x_3 &= -7 \\ 5x_1 - x_3 &= 10 \end{aligned}$$

Solution

Okay, let's first write down the matrix form of this system.

$$\begin{bmatrix} 3 & 1 & 0 \\ -1 & 2 & 2 \\ 5 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \\ 10 \end{bmatrix}$$

Now, we found the inverse of the coefficient matrix back in Example 2 of the Finding Inverses section so here is the coefficient matrix and its inverse.

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 2 & 2 \\ 5 & 0 & -1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ 3 & -1 & -2 \\ -\frac{10}{3} & \frac{5}{3} & \frac{7}{3} \end{bmatrix}$$

The solution to the system in matrix form is then,

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ 3 & -1 & -2 \\ -\frac{10}{3} & \frac{5}{3} & \frac{7}{3} \end{bmatrix} \begin{bmatrix} 6 \\ -7 \\ 10 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 5 \\ -\frac{25}{3} \end{bmatrix}$$

Now since each of the entries of **x** are one of the unknowns in the original system above the system to the original system is then,

$$x_1 = \frac{1}{3} \qquad x_2 = 5 \qquad x_3 = -\frac{25}{3}$$

So, provided we have a square coefficient matrix that is invertible and we just happen to have our hands on the inverse of the coefficient matrix we can find the solution to the system fairly easily.

Next, let's look at how the topic of the previous section (LU-Decompositions) can be used to solve systems of equations. First let's recall how LU-Decompositions work. If we have a square matrix, A, (so we'll again be working the same number of equations as unknowns) then if we can reduce it to row-echelon form without using any row interchanges then we can write it as $A = LU$ where L is a lower triangular matrix and U is an upper triangular matrix.

So, let's start with a system $A\mathbf{x} = \mathbf{b}$ where the coefficient matrix, A, is an $n \times n$ square and has an LU-Decomposition of $A = LU$. Now, substitute this into the system for A to get,

$$LU\mathbf{x} = \mathbf{b}$$

Next, let's just take a look at $U\mathbf{x}$. This will be an $n \times 1$ column matrix and let's call it $\mathbf{y}$. So, we've got $U\mathbf{x} = \mathbf{y}$.

So, just what does this do for us? Well let's write the system in the following manner.

$$L\mathbf{y} = \mathbf{b} \qquad \text{where} \qquad U\mathbf{x} = \mathbf{y}$$

As we'll see it's very easy to solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$ and once we know $\mathbf{y}$ it will be very easy to solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$ which will be the solution to the original system.

It's probably easiest to see how this method works with an example so let's work one.

Example 2 Use the LU-Decomposition method to find the solution to the following system of equations.

$$\begin{aligned} 3x_1 + 6x_2 - 9x_3 &= 0 \\ 2x_1 + 5x_2 - 3x_3 &= -4 \\ -4x_1 + x_2 + 10x_3 &= 3 \end{aligned}$$

Solution

First let's write down the matrix form of the system.

$$\begin{bmatrix} 3 & 6 & -9 \\ 2 & 5 & -3 \\ -4 & 1 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}$$

Now, we found an LU-Decomposition to this coefficient matrix in Example 1 of the previous section. From that example we see that,

$$\begin{bmatrix} 3 & 6 & -9 \\ 2 & 5 & -3 \\ -4 & 1 & 10 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & 9 & -29 \end{bmatrix}\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

According to the method outlined above this means that we actually need to solve the following two systems.

$$\begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & 9 & -29 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix} \qquad\qquad \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

in order.

So, let's get started on the first one. Notice that we don't really need to do anything other than write down the equations that are associated with this system and solve using **forward substitution**. The first equation will give us y_1 for free and once we know that the second equation will give us y_2. Finally, with these two values in hand the third equation will give us y_3. Here is that work.

$$\begin{aligned} 3y_1 &= 0 & &\Rightarrow & y_1 &= 0 \\ 2y_1 + y_2 &= -4 & &\Rightarrow & y_2 &= -4 \\ -4y_1 + 9y_2 - 29y_3 &= 3 & &\Rightarrow & y_3 &= -\frac{39}{29} \end{aligned}$$

The second system that we need to solve is then,

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ -\frac{39}{29} \end{bmatrix}$$

Again, notice that to solve this all we need to do is write down the equations and do **back substitution**. The third equation will give us x_3 for free and plugging this into the second equation will give us x_2, *etc*. Here's the work for this.

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 & &\Rightarrow & x_1 &= -\frac{119}{29} \\ x_2 + 3x_3 &= -4 & &\Rightarrow & x_2 &= \frac{1}{29} \\ x_3 &= -\frac{39}{29} & &\Rightarrow & x_3 &= -\frac{39}{29} \end{aligned}$$

The solution to the original system is then shown above. Notice that while the final answers where a little messy the work was nothing more than a little arithmetic and wasn't terribly difficult.

Let's work one more of these since there's a little more work involved in this than the inverse matrix method of solving a system.

Example 3 Use the LU-Decomposition method to find a solution to the following system of equations.

$$\begin{aligned} -2x_1 + 4x_2 - 3x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 17 \\ -4x_2 + 3x_3 &= -9 \end{aligned}$$

Solution

Once again, let's first get the matrix form of the system.

$$\begin{bmatrix} -2 & 4 & -3 \\ 3 & -2 & 1 \\ 0 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 17 \\ -9 \end{bmatrix}$$

Now let's get an LU-Decomposition for the coefficient matrix. Here's the work that will reduce it to row-echelon form. Remember that the result of this will be *U*.

$$\begin{bmatrix} -2 & 4 & -3 \\ 3 & -2 & 1 \\ 0 & -4 & 3 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_1} \begin{bmatrix} 1 & -2 & \frac{3}{2} \\ 3 & -2 & 1 \\ 0 & -4 & 3 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 1 & -2 & \frac{3}{2} \\ 0 & 4 & -\frac{7}{2} \\ 0 & -4 & 3 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{4}R_2} \begin{bmatrix} 1 & -2 & \frac{3}{2} \\ 0 & 1 & -\frac{7}{8} \\ 0 & -4 & 3 \end{bmatrix} \xrightarrow{R_3 + 4R_2} \begin{bmatrix} 1 & -2 & \frac{3}{2} \\ 0 & 1 & -\frac{7}{8} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \xrightarrow{-2R_3} \begin{bmatrix} 1 & -2 & \frac{3}{2} \\ 0 & 1 & -\frac{7}{8} \\ 0 & 0 & 1 \end{bmatrix}$$

So, *U* is then,

$$U = \begin{bmatrix} 1 & -2 & \frac{3}{2} \\ 0 & 1 & -\frac{7}{8} \\ 0 & 0 & 1 \end{bmatrix}$$

Now, to get *L* remember that we start off with a general lower triangular matrix and on the main diagonals we put the reciprocal of the scalar used in the work above to get a one in that spot. Then, in the entries below the main diagonal we put the negative of the multiple used to get a zero in that spot above. *L* is then,

$$L = \begin{bmatrix} -2 & 0 & 0 \\ 3 & 4 & 0 \\ 0 & -4 & -\frac{1}{2} \end{bmatrix}$$

We'll leave it to you to verify that $A = LU$. Now let's solve the system. This will mean we need to solve the following two systems.

$$\begin{bmatrix} -2 & 0 & 0 \\ 3 & 4 & 0 \\ 0 & -4 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 17 \\ -9 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & \frac{3}{2} \\ 0 & 1 & -\frac{7}{8} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Here's the work for the first system.

$$\begin{aligned} -2y_1 &= -1 & &\Rightarrow & y_1 &= \frac{1}{2} \\ 3y_1 + 4y_2 &= 17 & &\Rightarrow & y_2 &= \frac{31}{8} \\ -4y_2 - \frac{1}{2}y_3 &= -9 & &\Rightarrow & y_3 &= -13 \end{aligned}$$

Now let's get the actual solution by solving the second system.

$$\begin{bmatrix} 1 & -2 & \frac{3}{2} \\ 0 & 1 & -\frac{7}{8} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{31}{8} \\ -13 \end{bmatrix}$$

Here is the substitution work for this system.

$$\begin{aligned} x_1 - 2x_2 + \frac{3}{2}x_3 &= \frac{1}{2} & &\Rightarrow & x_1 &= 5 \\ x_2 - \frac{7}{8}x_3 &= \frac{31}{8} & &\Rightarrow & x_2 &= -\frac{15}{2} \\ x_3 &= -13 & &\Rightarrow & x_3 &= -13 \end{aligned}$$

So there's the solution to this system.

Before moving onto the next topic of this section we should probably address why we even bothered with this method. It seems like a lot of work to solve a system of equations and when solving systems by hand it can be a lot of work. However, because the method for finding *L* and *U* is a fairly straightforward process and once those are found the method for solving the system is also very straightforward this is a perfect method for use in computer systems when programming the solution to systems. So, while it seems like a lot of work, it is a method that is very easy to program and so is a very useful method.

The remaining topics in this section don't really rely on previous sections as the first part of this section has. Instead we just need to look at a couple of ideas about solving systems that we didn't have room to put into the section on solving systems of equations.

First we want to take a look at the following scenario. Suppose that we need so solve a system of equations only there are two or more sets of the b_i's that we need to look at. For instance suppose we wanted to solve the following systems of equations.

$$A\mathbf{x} = \mathbf{b}_1 \qquad A\mathbf{x} = \mathbf{b}_2 \qquad \cdots \qquad A\mathbf{x} = \mathbf{b}_k$$

Again, the coefficient matrix is the same for all these systems and the only thing that is different is the $\mathbf{b}_i$'s. We could use any of the methods looked at so far to solve these systems. However, each of the methods we've looked at so far would require us to do each system individually and that could potentially lead to a lot of work.

There is one method however that can be easily extended to solve multiple systems simultaneously provided they all have the same coefficient matrix. In fact the method is the very first one we looked at. In that method we solved systems by adding the column matrix **b**, onto the coefficient matrix and then reducing it to row-echelon or reduced row-echelon form. For the systems above this would require working with the following augmented matrices.

$$\left[A \mid \mathbf{b}_1 \right] \qquad \left[A \mid \mathbf{b}_2 \right] \qquad \cdots \qquad \left[A \mid \mathbf{b}_k \right]$$

However, if you think about it almost the whole reduction process revolves around the columns in the augmented matrix that are associated with A and not the **b** column. So, instead of doing these individually let's add all of them onto the coefficient matrix as follows.

$$\left[A \mid \mathbf{b}_1 \mid \mathbf{b}_2 \mid \cdots \mid \mathbf{b}_k \right]$$

All we need to do this is reduce this to reduced row-echelon form and we'll have the answer to each of the systems. Let's take a look at an example of this.

Example 4 Find the solution to each of the following systems.

$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= 12 \\ 2x_1 - x_2 - 2x_3 &= -1 \\ 5x_1 - 2x_2 - 3x_3 &= 3 \end{aligned} \qquad\qquad \begin{aligned} x_1 - 3x_2 + 4x_3 &= 0 \\ 2x_1 - x_2 - 2x_3 &= 5 \\ 5x_1 - 2x_2 - 3x_3 &= -8 \end{aligned}$$

Solution

So, we've got two systems with the same coefficient matrix so let's form the following matrix. Note that we'll leave the vertical bars in to make sure we remember the last two columns are really **b**'s for the systems we're solving.

$$\left[\begin{array}{rrr|r|r} 1 & -3 & 4 & 12 & 0 \\ 2 & -1 & -2 & -1 & 5 \\ 5 & -2 & -3 & 3 & -8 \end{array}\right]$$

Now, we just need to reduce this to reduced row-echelon form. Here is the work for that.

$$\left[\begin{array}{rrr|r|r} 1 & -3 & 4 & 12 & 0 \\ 2 & -1 & -2 & -1 & 5 \\ 5 & -2 & -3 & 3 & -8 \end{array}\right] \begin{array}{c} R_2 - 2R_1 \\ R_3 - 5R_1 \\ \rightarrow \end{array} \left[\begin{array}{rrr|r|r} 1 & -3 & 4 & 12 & 0 \\ 0 & 5 & -10 & -25 & 5 \\ 0 & 13 & -23 & -57 & -8 \end{array}\right]$$

$$\xrightarrow{\frac{1}{5}R_2}\left[\begin{array}{ccc|c|c} 1 & -3 & 4 & 12 & 0 \\ 0 & 1 & -2 & -5 & 1 \\ 0 & 13 & -23 & -57 & -8 \end{array}\right] \xrightarrow{R_3 - 13R_2}\left[\begin{array}{ccc|c|c} 1 & -3 & 4 & 12 & 0 \\ 0 & 1 & -2 & -5 & 1 \\ 0 & 0 & 3 & 8 & -21 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_3}\left[\begin{array}{ccc|c|c} 1 & -3 & 4 & 12 & 0 \\ 0 & 1 & -2 & -5 & 1 \\ 0 & 0 & 1 & \frac{8}{3} & -7 \end{array}\right] \xrightarrow[R_1 - 4R_3]{R_2 + 2R_3}\left[\begin{array}{ccc|c|c} 1 & -3 & 0 & \frac{4}{3} & 28 \\ 0 & 1 & 0 & \frac{1}{3} & -13 \\ 0 & 0 & 1 & \frac{8}{3} & -7 \end{array}\right]$$

$$\xrightarrow{R_1 + 3R_2}\left[\begin{array}{ccc|c|c} 1 & 0 & 0 & \frac{7}{3} & -11 \\ 0 & 1 & 0 & \frac{1}{3} & -13 \\ 0 & 0 & 1 & \frac{8}{3} & -7 \end{array}\right]$$

Okay from the solution to the first system is in the fourth column since that is the **b** for the first system and likewise the solution to the second system is in the fifth column. Therefore, the solution to the first system is,

$$x_1 = \frac{7}{3} \qquad x_2 = \frac{1}{3} \qquad x_3 = \frac{8}{3}$$

and the solution to the second system is,

$$x_1 = -11 \qquad x_2 = -13 \qquad x_3 = -7$$

The remaining topic to discuss in this section gives us a method for answering the following question.

Given an $n \times m$ matrix A determine all the $m \times 1$ matrices, **b**, for which $A\mathbf{x} = \mathbf{b}$ is consistent, that is $A\mathbf{x} = \mathbf{b}$ has at least one solution. This is a question that can arise fairly often and so we should take a look at how to answer it.

Of course if A is invertible (and hence square) this answer is that $A\mathbf{x} = \mathbf{b}$ is consistent for all **b** as we saw in an earlier section. However, what if A isn't square or isn't invertible? The method we're going to look at doesn't really care about whether or not A is invertible but it really should be pointed out that we do know the answer for invertible matrices.

It's easiest to see how these work with an example so let's jump into one.

Example 5 Determine the conditions (if any) on b_1, b_2, and b_3 in order for the following system to be consistent.

$$\begin{aligned} x_1 - 2x_2 + 6x_3 &= b_1 \\ -x_1 + x_2 - x_3 &= b_2 \\ -3x_1 + x_2 + 8x_3 &= b_3 \end{aligned}$$

Solution

Okay, we're going to use the augmented matrix method we first looked at here and reduce the matrix down to reduced row-echelon form. The final form will be a little messy because of the presence of the b_i's but other than that the work is identical to what we've been doing to this point.

Here is the work.

$$\left[\begin{array}{ccc|c} 1 & -2 & 6 & b_1 \\ -1 & 1 & -1 & b_2 \\ -3 & 1 & 8 & b_3 \end{array}\right] \begin{array}{c} R_2 + R_1 \\ R_3 + 3R_1 \\ \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 6 & b_1 \\ 0 & -1 & 5 & b_2 + b_1 \\ 0 & -5 & 26 & b_3 + 3b_1 \end{array}\right]$$

$$\begin{array}{c} -R_2 \\ \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 6 & b_1 \\ 0 & 1 & -5 & -b_2 - b_1 \\ 0 & -5 & 26 & b_3 + 3b_1 \end{array}\right] \begin{array}{c} R_3 + 5R_2 \\ \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 6 & b_1 \\ 0 & 1 & -5 & -b_2 - b_1 \\ 0 & 0 & 1 & b_3 - 5b_2 - 2b_1 \end{array}\right]$$

$$\begin{array}{c} R_2 + 5R_3 \\ R_1 - 6R_3 \\ \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 0 & -6b_3 + 30b_2 + 13b_1 \\ 0 & 1 & 0 & 5b_3 - 26b_2 - 11b_1 \\ 0 & 0 & 1 & b_3 - 5b_2 - 2b_1 \end{array}\right] \begin{array}{c} R_1 + 2R_2 \\ \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4b_3 - 22b_2 - 9b_1 \\ 0 & 1 & 0 & 5b_3 - 26b_2 - 11b_1 \\ 0 & 0 & 1 & b_3 - 5b_2 - 2b_1 \end{array}\right]$$

Okay, just what does this all mean? Well go back to equations and let's see what we've got.

$$\begin{aligned} x_1 &= 4b_3 - 22b_2 - 9b_1 \\ x_2 &= 5b_3 - 26b_2 - 11b_1 \\ x_3 &= b_3 - 5b_2 - 2b_1 \end{aligned}$$

So, what this says is that no matter what our choice of b_1, b_2, and b_3 we can find a solution using the general solution above and in fact there will always be exactly one solution to the system for a given choice of **b**.

Therefore, there are no conditions on b_1, b_2, and b_3 in order for the system to be consistent.

Note that the result of the previous example shouldn't be too surprising given that the coefficient matrix is invertible.

Now, we need to see what happens if the coefficient matrix is singular (*i.e.*not invertible).

Example 6 Determine the conditions (if any) on b_1, b_2, and b_3 in order for the following system to be consistent.

$$\begin{aligned} x_1 + 3x_2 - 2x_3 &= b_1 \\ -x_1 - 5x_2 + 3x_3 &= b_2 \\ 2x_1 - 8x_2 + 3x_3 &= b_3 \end{aligned}$$

Solution

We'll do this one in the same manner as the previous one. So, convert to an augmented matrix and start the reduction process. As we'll see in this case we won't need to go all the way to reduced row-echelon form to get the answer however.

$$\left[\begin{array}{rrr|c} 1 & 3 & -2 & b_1 \\ -1 & -5 & 3 & b_2 \\ 2 & -8 & 3 & b_3 \end{array}\right] \begin{array}{c} R_2 + R_1 \\ R_3 - 2R_1 \\ \rightarrow \end{array} \left[\begin{array}{rrr|r} 1 & 3 & -2 & b_1 \\ 0 & -2 & 1 & b_2 + b_1 \\ 0 & -14 & 7 & b_3 - 2b_1 \end{array}\right]$$

$$\begin{array}{c} R_3 - 7R_2 \\ \rightarrow \end{array} \left[\begin{array}{rrr|r} 1 & 3 & -2 & b_1 \\ 0 & -2 & 1 & b_2 + b_1 \\ 0 & 0 & 0 & b_3 - 7b_2 - 9b_1 \end{array}\right]$$

Okay, let's stop here and see what we've got. The last row corresponds to the following equation.

$$0 = b_3 - 7b_2 - 9b_1$$

If the right side of this equation is NOT zero then this equation will not make any sense and so the system won't have a solution. If however, it is zero then this equation will not be a problem and since we can take the first two rows and finish out the process to find a solution for any given values of b_1 and b_2 we'll have a solution.

This then gives us our condition that we're looking for. In order for the system to have a solution, and hence be consistent, we must have

$$b_3 = 7b_2 + 9b_1$$

Determinants

Introduction

By this point in your mathematical career you should have run across functions. The functions that you've probably seen to this point have had the form $f(x)$ where *x* is a real number and the output of the function is also a real number. Some examples of functions are $f(x) = x^2$ and $f(x) = \cos(3x) - \sin(x)$.

Not all functions however need to take a real number as an argument. For instance we could have a function $f(X)$ that takes a matrix *X* and outputs a real number. In this chapter we are going to be looking at one such function, the determinant function. The determinant function is a function that will associate a real number with a square matrix.

The determinant function is a function that won't be seeing all that often in the rest of this course, but it will show up on occasion.

Here is a listing of the topics in this chapter.

The Determinant Function – We will give the formal definition of the determinant in this section. We'll also give formulas for computing determinants of 2×2 and 3×3 matrices.

Properties of Determinants – Here we will take a look at quite a few properties of the determinant function. Included are formulas for determinants of triangular matrices.

The Method of Cofactors – In this section we'll take a look at the first of two methods form computing determinants of general matrices.

Using Row Reduction to Find Determinants – Here we will take a look at the second method for computing determinants in general.

Cramer's Rule – We will take a look at yet another method for solving systems. This method will involve the use of determinants.

The Determinant Function

We'll start off the chapter by defining the determinant function. This is not such an easy thing however as it involves some ideas and notation that you probably haven't run across to this point. So, before we actually define the determinant function we need to get some preliminaries out of the way.

First, a **permutation** of the set of integers $\{1,2,\ldots,n\}$ is an arrangement of all the integers in the list without omission or repetitions. A permutation of $\{1,2,\ldots,n\}$ will typically be denoted by $(i_1,i_2,\ldots,i_n)$ where i_1 is the first number in the permutation, i_2 is the second number in the permutation, *etc.*

Example 1 List all permutations of $\{1,2\}$.

Solution
This one isn't too bad because there are only two integers in the list. We need to come up with all the possible ways to arrange these two numbers. Here they are.

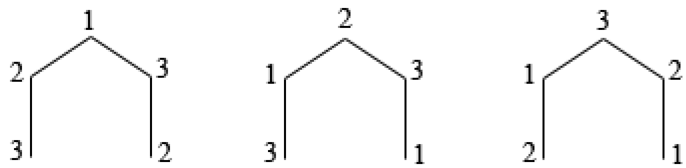
$$(1,2) \qquad\qquad (2,1)$$

Example 2 List all the permutations of $\{1,2,3\}$

Solution
This one is a little harder to do, but still isn't too bad. We need all the arrangements of these three numbers in which no number is repeated or omitted. Here they are.

$$(1,2,3) \qquad (1,3,2) \qquad (2,1,3) \qquad (2,3,1) \qquad (3,1,2) \qquad (3,2,1)$$

From this point on it can be somewhat difficult to find permutations for lists of numbers with more than 3 numbers in it. One way to make sure that you get all of them is to write down a **permutation tree**. Here is the permutation tree for $\{1,2,3\}$.



At the top we list all the numbers in the list and from this top number we'll branch out with each of the remaining numbers in the list. At the second level we'll again branch out with each of the numbers from the list not yet written down along that branch. Then each branch will represent a permutation of the given list of numbers

As you can see the number of permutations for a list will quick grow as we add numbers to the list. In fact it can be shown that there are $n!$ permutations of the list $\{1,2,\ldots,n\}$, or any list containing n distinct numbers, but we're going to be working with $\{1,2,\ldots,n\}$ so that's the one we'll reference. So, the list $\{1,2,3,4\}$ will have $4! = (4)(3)(2)(1) = 24$ permutations, the list $\{1,2,3,4,5\}$ will have $5! = (5)(4)(3)(2)(1) = 120$ permutations, *etc*.

Next we need to discuss inversions in a permutation. An **inversion** will occur in the permutation $(i_1, i_2, \ldots, i_n)$ whenever a larger number precedes a smaller number. Note as well we don't mean that the smaller number is immediately to the right of the larger number, but anywhere to the right of the larger number.

Example 3 Determine the number of inversions in each of the following permutations.

(a) $(3,1,4,2)$

(b) $(1,2,4,3)$

(c) $(4,3,2,1)$

(d) $(1,2,3,4,5)$

(e) $(2,5,4,1,3)$

Solution

(a) Okay, to count the number of inversions we will start at the left most number and count the number of numbers to the right that are smaller. We then move to the second number and do the same thing. We continue in this fashion until we get to the end. The total number of inversions are then the sum of all these.

We'll do this first one in detail and then do the remaining ones much quicker. We'll mark the number we're looking at in red and to the side give the number of inversions for that particular number.

$$(3,1,4,2) \qquad 2 \text{ inversions}$$
$$(3,1,4,2) \qquad 0 \text{ inversions}$$
$$(3,1,4,2) \qquad 1 \text{ inversion}$$

In the first case there are two numbers to the right of 3 that are smaller than 3 so there are two inversions there. In the second case we're looking at the smallest number in the list and so there won't be any inversions there. Then with 4 there is one number to the right that is smaller than 4 and so we pick up another inversion. There is no reason to look at the last number in the permutation since there are no numbers to the right of it and so won't introduce any inversions.

The permutation $(3,1,4,2)$ has a total of 3 inversions.

(b) We'll do this one much quicker. There are $0+0+1=1$ inversions in $(1,2,4,3)$. Note that each number in the sum above represents the number of inversion for the number in that position in the permutation.

(c) There are $3+2+1=6$ inversions in $(4,3,2,1)$.

(d) There are no inversions in $(1,2,3,4,5)$.

(e) There are $1+3+2+0=6$ in $(2,5,4,1,3)$.

Next, a permutation is called **even** if the number of inversions is even and **odd** if the number of inversions is odd.

Example 4 Classify as even or odd all the permutations of the following lists.

(a) $\{1,2\}$

(b) $\{1,2,3\}$

Solution

(a) Here's a table giving all the permutations, the number of inversions in each and the classification.

Permutation	# Inversions	Classification
$(1,2)$	0	even
$(2,1)$	1	odd

(b) We'll do the same thing here

Permutation	# Inversions	Classification

$(1,2,3)$	0	even
$(1,3,2)$	1	odd
$(2,1,3)$	1	odd
$(2,3,1)$	2	even
$(3,1,2)$	2	even
$(3,2,1)$	3	odd

We'll need these results later in the section.

Alright, let's move back into matrices. We still have some definitions to get out of the way before we define the determinant function, but at least we're back dealing with matrices.

Suppose that we have an $n \times n$ matrix, *A*, then an **elementary product** from this matrix will be a product of *n* entries from *A* and none of the entries in the product can be from the same row or column.

Example 5 Find all the elementary products for,

(a) a 2×2 matrix

(b) a 3×3 matrix.

Solution

(a) Okay let's first write down the general 2×2 matrix.

$$A = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

Each elementary product will contain two terms and since each term must come from different rows we know that each elementary product must have the form,

$$a_{1\square}\, a_{2\square}$$

All we need to do is fill in the column subscripts and remember in doing so that they must come from different columns. There are really only two possible ways to fill in the blanks in the product above. The two ways of filling in the blanks are $(1,2)$ and $(2,1)$ and yes we did mean to use the permutation notation there since that is exactly what we need. We will fill in the blanks with all the possible permutations of the list of column numbers, $\{1,2\}$ in this case.

So, the elementary products for a 2×2 matrix are

$$a_{11}a_{22} \qquad\qquad a_{12}a_{21}$$

(b) Again, let's start off with a general 3×3 matrix for reference purposes.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Each of the elementary products in this case will involve three terms and again since the must all come from different rows we can again write down the form they must take.

$$a_{1\square}\, a_{2\square}\, a_{3\square}$$

Again, each of the column subscripts will need to come from different columns and like the 2×2 case we can get all the possible choices for these by filling in the blanks will all the possible permutations of $\{1,2,3\}$.

So, the elementary products of the 3×3 are,

$$\begin{array}{ll} a_{11}a_{22}a_{33} & a_{11}a_{23}a_{32} \\ a_{12}a_{21}a_{33} & a_{12}a_{23}a_{31} \\ a_{13}a_{21}a_{32} & a_{13}a_{22}a_{31} \end{array}$$

For a general an $n\times n$ matrix *A*, will have $n!$ elementary products of the form

$$a_{1\,i_1}a_{2\,i_2}\cdots a_{n\,i_n}$$

where $(i_1, i_2, \ldots, i_n)$ ranges over all the permutations of $\{1, 2, \ldots, n\}$.

We can now take care of the final preliminary definition that we need for the determinant function. A **signed elementary product** from *A* will be the elementary product $a_{1i_1}a_{2i_2}\cdots a_{ni_n}$ that is multiplied by "+1" if $(i_1, i_2, \ldots, i_n)$ is an even permutation or multiplied by "-1" if $(i_1, i_2, \ldots, i_n)$ is an odd permutation.

Example 6 Find all the signed elementary products for,

(a) a 2×2 matrix

(b) a 3×3 matrix.

Solution

We listed out all the elementary products in Example 5 and we classified all the permutations used in them as even or odd in Example 4. So, all we need to do is put all this information together for each matrix.

(a) Here are the signed elementary products for the 2×2 matrix.

Elementary Product	Permutation	Signed Elementary Product
$a_{11}a_{12}$	$(1,2)$ - even	$a_{11}a_{12}$
$a_{12}a_{21}$	$(2,1)$ - odd	$-a_{12}a_{21}$

(b) Here are the signed elementary products for the 3×3 matrix.

Elementary Product	Permutation	Signed Elementary Product
$a_{11}a_{22}a_{33}$	$(1,2,3)$ - even	$a_{11}a_{22}a_{33}$
$a_{11}a_{23}a_{32}$	$(1,3,2)$ - odd	$-a_{11}a_{23}a_{32}$
$a_{12}a_{21}a_{33}$	$(2,1,3)$ - odd	$-a_{12}a_{21}a_{33}$
$a_{12}a_{23}a_{31}$	$(2,3,1)$ - even	$a_{12}a_{23}a_{31}$
$a_{13}a_{21}a_{32}$	$(3,1,2)$ - even	$a_{13}a_{21}a_{32}$
$a_{13}a_{22}a_{31}$	$(3,2,1)$ - odd	$-a_{13}a_{22}a_{31}$

Okay, we can now give the definition of the determinant function.

Definition 1 If *A* is square matrix then the **determinant function** is denoted by **det** and det(A) is defined to be the sum of all the signed elementary matrices of *A*.

Note that often we will call the number det(A) the **determinant of *A***. Also, there is some alternate notation that is sometimes used for determinants. We will sometimes denote determinants as $\det(A)=|A|$ and this is most often done with the actual matrix instead of the letter representing the matrix. For instance for a 2×2 matrix *A* we will use any of the following to denote the determinant,

$$\det(A)=|A|=\begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix}$$

So, now that we have the definition of the determinant function in hand we can actually start writing down some formulas. We'll give the formula for 2×2 and 3×3 matrices only because for any matrix larger than that the formula becomes very long and messy and at those sizes there are alternate methods for computing determinants that will be easier.

So, with that said, we've got all the signed elementary products for 2×2 and 3×3 matrices listed in Example 6 so let's write down the determinant function for these matrices.

First the determinant function for a 2×2 matrix.

$$\det(A)=\begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}$$

Now the determinant function for a 3×3 matrix.

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$
$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31}$$

Okay, the formula for a 2×2 matrix isn't too bad, but the formula for a 3×3 is messy and would not be fun to memorize. Fortunately, there is an easy way to quickly "derive" both of these formulas.

Before we give this quick trick to "derive" the formulas we should point out that what where going to do ONLY works for 2×2 and 3×3 matrices. There is no corresponding trick for larger matrices!

Okay, let's start with a 2×2 matrix. Let's examine the determinant below.

$$\begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix}$$

Notice the two diagonals that we've put on this determinant. The diagonal that runs from left to right also covers the positive elementary product in the formula. Likewise, the diagonal that runs from right to left covers the negative elementary product.

So, for a 2×2 matrix all we need to do is write down the determinant, sketch in the diagonals multiply along the diagonals then add the product if the diagonal runs from left to right and subtract the product if the diagonal runs from right to left.

Now let's take a look at a 3×3 matrix. There is a similar trick that will work here, but in order to get it to work we'll first need to tack copies the first 2 columns onto the right side of the determinant as shown below.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

With the addition of the two extra columns we can see that we've got three diagonals running in each direction and that each will cover one of the elementary products for this matrix. Also, the diagonals that run from left to right cover the positive elementary products and those that run from right to left cover the negative elementary product. So, as with the 2×2 matrix, we can quickly write down the determinant function formula here by simply multiplying along each diagonal and then adding it if the diagonal runs left to right or subtracting it if the diagonal runs right to left.

Let's take a quick look at a couple of examples with numbers just to make sure we can do these.

Example 7 Compute the determinant of each of the following matrices.

(a) $A = \begin{bmatrix} 3 & 2 \\ -9 & 5 \end{bmatrix}$

(b) $B = \begin{bmatrix} 3 & 5 & 4 \\ -2 & -1 & 8 \\ -11 & 1 & 7 \end{bmatrix}$

(c) $C = \begin{bmatrix} 2 & -6 & 2 \\ 2 & -8 & 3 \\ -3 & 1 & 1 \end{bmatrix}$

Solution

(a) We don't really need to sketch in the diagonals for 2×2 matrices. The determinant is simply the product of the diagonal running left to right minus the product of the diagonal running from right to left. So, here is the determinant for this matrix. The only thing we need to worry about is paying attention to minus signs. It is easy to make a mistake with minus signs in these computations if you aren't paying attention.

$$\det(A) = (3)(5) - (2)(-9) = 33$$

(b) Okay, with this one we'll copy the two columns over and sketch in the diagonals to make sure we've got the idea of these down.

$$\det(B) = \left| \begin{array}{ccc|cc} 3 & 5 & 4 & 3 & 5 \\ -2 & -1 & 8 & -2 & -1 \\ -11 & 1 & 7 & -11 & 1 \end{array} \right.$$

Now, just remember to add products along the left to right diagonals and subtract products along the right to left diagonals.

$$\begin{aligned} \det(B) &= (3)(-1)(7) + (5)(8)(-11) + (4)(-2)(1) - (5)(-2)(7) - \\ &\qquad (3)(8)(1) - (4)(-1)(-11) \\ &= -467 \end{aligned}$$

(c) We'll do this one with a little less detail. We'll copy the columns but not bother to actually sketch in the diagonals this time.

$$\begin{aligned} \det(C) &= \left| \begin{array}{ccc|cc} 2 & -6 & 2 & 2 & -6 \\ 2 & -8 & 3 & 2 & -8 \\ -3 & 1 & 1 & -3 & 1 \end{array} \right. \\ &= (2)(-8)(1) + (-6)(3)(-3) + (2)(2)(1) - (-6)(2)(1) - \\ &\qquad (2)(3)(1) - (2)(-8)(-3) \\ &= 0 \end{aligned}$$

As this example has shown determinants of matrices can be positive, negative or zero.

It is again worth noting that there are no such tricks for computing determinants for matrices larger that 3×3

In the remainder of this chapter we'll take a look at some properties of determinants, two alternate methods for computing them that are not restricted by the size of the matrix as the two quick tricks we saw in this section were and an application of determinants.

Properties of Determinants

In this section we'll be taking a look at some of the basic properties of determinants and towards the end of this section we'll have a nice test for the invertibility of a matrix. In this section we'll give a fair number of theorems (and prove a few of them) as well as examples illustrating the theorems. Any proofs that are omitted are generally more involved that we want to get into in this class.

Most of the theorems in this section will not help us to actually compute determinants in general. Most of these theorems are really more about how the determinants of different matrices will relate to each other. We will take a look at a couple of theorems that will help show us how to find determinants for some special kinds of matrices, but we'll have to wait until the next two sections to start looking at how to compute determinants in general.

All of the determinants that we'll be computing in the examples in this section will be of a 2×2 or a 3×3 matrix. If you need a refresher on how to compute determinants of these kinds of matrices check out this example in the previous section. We won't actually be showing any of that work here in this section.

Let's start with the following theorem.

Theorem 1 Let *A* be an $n\times n$ matrix and *c* be a scalar then,

$$\det(cA) = c^n \det(A)$$

Proof : This is a really simply proof. From the definition of the determinant function in the previous section we know that the determinant is the sum of all the signed elementary products for the matrix. So, for *cA* we will sum signed elementary products that are of the form,

$$(ca_{1i_1})(ca_{2i_2})\cdots(ca_{ni_n}) = c^n\left(a_{1i_1}a_{2i_2}\cdots a_{ni_n}\right)$$

Recall that for scalar multiplication we multiply all the entries by *c* and so we'll have a *c* on each entry as shown above. Also, as shown, we can fact all *n* of the *c*'s out and we'll get what we've shown above. Note that $a_{1i_1}a_{2i_2}\cdots a_{ni_n}$ is the signed elementary product for *A*.

Now, if we add all the signed elementary products for cA we can factor the c^n that is on each term out of the sum and what we're left with is the sum of all the signed elementary products of A, or in other words, we're left with det(A). So, we're done.

■

Here's a quick example to verify the results of this theorem.

Example 1 For the given matrix below compute both det(A) and det(2A).

$$A = \begin{bmatrix} 4 & -2 & 5 \\ -1 & -7 & 10 \\ 0 & 1 & -3 \end{bmatrix}$$

Solution

We'll leave it to you to verify all the details of this problem. First the scalar multiple.

$$2A = \begin{bmatrix} 8 & -4 & 10 \\ -2 & -14 & 20 \\ 0 & 2 & -6 \end{bmatrix}$$

The determinants.

$$\det(A) = 45 \qquad \det(2A) = 360 = (8)(45) = 2^3 \det(A)$$

Now, let's investigate the relationship between det(A), det(B) and det(A+B). We'll start with the following example.

Example 2 Compute det(A), det(B) and det(A+B) for the following matrices.

$$A = \begin{bmatrix} 10 & -6 \\ -3 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 5 & -6 \end{bmatrix}$$

Solution

Here all the determinants.

$$\det(A) - 28 \qquad \det(B) = -16 \qquad \det(A+B) = -69$$

Notice here that for this example we have $\det(A+B) \neq \det(A) + \det(B)$. In fact this will generally be the case.

There is a very special case where we will get equality for the sum of determinants, but it doesn't happen all that often. Here is the theorem detailing this special case.

Theorem 2 Suppose that A, B, and C are all $n \times n$ matrices and that they differ by only a row, say the k^{th} row. Let's further suppose that the k^{th} row of C can be found by adding the corresponding entries from the k^{th} rows of A and B. Then in this case we will have that

$$\det(C) = \det(A) + \det(B)$$

The same result will hold if we replace the word row with column above.

Here is an example of this theorem.

Example 3 Consider the following three matrices.

$$A=\begin{bmatrix} 4 & 2 & -1 \\ 6 & 1 & 7 \\ -1 & -3 & 9 \end{bmatrix} \qquad B=\begin{bmatrix} 4 & 2 & -1 \\ -2 & -5 & 3 \\ -1 & -3 & 9 \end{bmatrix} \qquad C=\begin{bmatrix} 4 & 2 & -1 \\ 4 & -4 & 10 \\ -1 & -3 & 9 \end{bmatrix}$$

First, notice that we can write C as,

$$C=\begin{bmatrix} 4 & 2 & -1 \\ 4 & -4 & 10 \\ -1 & -3 & 9 \end{bmatrix}=\begin{bmatrix} 4 & 2 & -1 \\ 6+(-2) & 1+(-5) & 7+3 \\ -1 & -3 & 9 \end{bmatrix}$$

All three matrices differ only in the second row and the second row of C can be found by adding the corresponding entries from the second row of A and B.

The determinants of these matrices are,

$$\det(A)=15 \qquad \det(B)=-115 \qquad \det(C)=-110=15+(-115)$$

Next let's look at the relationship between the determinants of matrices and their products.

Theorem 3 If A and B are matrices of the same size then

$$\det(AB)=\det(A)\det(B)$$

This theorem can be extended out to as many matrices as we want. For instance,

$$\det(ABC)=\det(A)\det(B)\det(C)$$

Let's check out an example of this.

Example 4 For the given matrices compute det(A), det(B), and det(AB).

$$A=\begin{bmatrix} 1 & -2 & 3 \\ 2 & 7 & 4 \\ 3 & 1 & 4 \end{bmatrix} \qquad B=\begin{bmatrix} 0 & 1 & 8 \\ 4 & -1 & 1 \\ 0 & 3 & 3 \end{bmatrix}$$

Solution

Here's the product of the two matrices.

$$AB=\begin{bmatrix} -8 & 12 & 15 \\ 28 & 7 & 35 \\ 4 & 14 & 37 \end{bmatrix}$$

Here are the determinants.

$$\det(A) = -41 \qquad \det(B) = 84$$
$$\det(AB) = -3444 = (-41)(84) = \det(A)\det(B)$$

Here is a theorem relating determinants of matrices and their inverse (provided the matrix is invertible of course…).

Theorem 4 Suppose that A is an invertible matrix then,

$$\det\left(A^{-1}\right) = \frac{1}{\det(A)}$$

Proof : The proof of this theorem is a direct result of the previous theorem. Since A is invertible we know that $AA^{-1} = I$. So take the determinant of both sides and then use the previous theorem on the left side.

$$\det\left(AA^{-1}\right) = \det(A)\det\left(A^{-1}\right) = \det(I)$$

Now, all that we need is to know that $\det(I) = 1$ which you can prove using Theorem 8 below.

$$\det(A)\det\left(A^{-1}\right) = 1 \qquad \Rightarrow \qquad \det(A) = \frac{1}{\det\left(A^{-1}\right)}$$

■

Here's a quick example illustrating this.

Example 5 For the given matrix compute det(A) and $\det\left(A^{-1}\right)$.

$$A = \begin{bmatrix} 8 & -9 \\ 2 & 5 \end{bmatrix}$$

Solution

We'll leave it to you to verify that A is invertible and that its inverse is,

$$A^{-1} = \begin{bmatrix} \frac{5}{58} & \frac{9}{58} \\ -\frac{1}{29} & \frac{4}{29} \end{bmatrix}$$

Here are the determinants for both of these matrices.

$$\det(A) = 58 \qquad \det\left(A^{-1}\right) = \frac{1}{58} = \frac{1}{\det(A)}$$

The next theorem that we want to take a look at is a nice test for the invertibility of matrices.

Theorem 5 A square matrix A is invertible if and only if $\det(A) \neq 0$. A matrix that is invertible is often called **non-singular** and a matrix that is not invertible is often called **singular**.

Before doing an example of this let's talk a little bit about the phrase "if and only if" that appears in this theorem. That phrase means that this is kind of like a two way street. This theorem, because of the "if and only if" phrase, says that if we know that A is invertible then we will have $\det(A) \neq 0$. If, on the other hand, we know that $\det(A) \neq 0$ then we will also know that A is invertible.

Most theorems in this presented in these notes are not "two way streets" so to speak. They only work one way, if however, we do have a theorem that does work both ways you will always be able to identify it by the phrase "if and only if".

Now let's work an example to verify this theorem.

Example 6 Compute the determinants of the following two matrices.

$$C = \begin{bmatrix} 3 & 1 & 0 \\ -1 & 2 & 2 \\ 5 & 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 3 & 6 \\ 0 & 1 & 2 \\ -2 & 0 & 0 \end{bmatrix}$$

Solution
We determined the invertibility of both of these matrices in the section on Finding Inverses so we already know what the answers should be (at some level) for the determinants. In that section we determined that C was invertible and so by Theorem 5 we know that the det(C) should be non-zero. We also determined that B was singular (*i.e.* not invertible) and so we know by Theorem 5 that det(B) should be zero.

Here are those determinants of these two matrices.

$$\det(C) = 3 \qquad \det(B) = 0$$

Sure enough we got zero where we should have and didn't get zero where we should have.

Here is a theorem relating the determinants of a matrix and its transpose.

Theorem 6 If A is a square matrix then,

$$\det(A) = \det(A^T)$$

Here is an example that verifies the results of this theorem.

Example 7 Compute det(A) and $\det(A^T)$ for the following matrix.

$$A = \begin{bmatrix} 5 & 3 & 2 \\ -1 & -8 & -6 \\ 0 & 1 & 1 \end{bmatrix}$$

Solution
We'll leave it to you to verify that

$$\det(A) = \det(A^T) = -9$$

There are a couple special cases of matrices that we can quickly find the determinant for so let's take care of those at this point.

Theorem 7 If A is a square matrix with a row or column of all zeroes then

$$\det(A) = 0$$

and so A will be singular.

Proof : The proof here is fairly straight forward. The determinant is the sum of all the signed elementary products and each of these will have a factor from each row and a factor from each column. So, in particular it will have a factor from the row or column of all zeroes and hence will have a factor of zero making the whole product zero.

All of the products are zero and upon summing them up we will also get zero for the determinant.

■

Note that in the following example we don't need to worry about the size of the matrix now since this theorem gives us a value for the determinant. You might want to check the 2×2 and 3×3 to verify that the determinants are in fact zero. You also might want to come back and verify the other after the next section where we'll learn methods for computing determinants in general.

Example 8 Each of the following matrices are singular.

$$A = \begin{bmatrix} 3 & 9 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 0 & 1 \\ -9 & 0 & 2 \\ 4 & 0 & -3 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & 12 & 8 & 0 \\ 5 & -3 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 5 & 1 & 3 & 6 \end{bmatrix}$$

It is actually very easy to compute the determinant of any triangular (and hence any diagonal) matrix. Here is the theorem that tells us how to do that.

Theorem 8 Suppose that A is an $n\times n$ triangular matrix then,

$$\det(A) = a_{11}a_{22}\cdots a_{nn}$$

So, what this theorem tells us is that the determinant of any triangular matrix (upper or lower) or any diagonal matrix is simply the product of the entries from the matrices main diagonal.

We won't do a formal proof here. We'll just give a quick outline.

Proof Outline : Since we know that the determinant is the sum of the signed elementary products and each elementary products has a factor from each row and a factor from each column because of the triangular nature of the matrix, the only elementary product that won't have at least one zero is $a_{11}a_{22}\cdots a_{nn}$. All the others will have at least one zero in them. Hence the determinant of the matrix must be $\det(A) = a_{11}a_{22}\cdots a_{nn}$

■

Let's take the determinant of a couple of triangular matrices. You should verify the 2×2 and 3×3 matrices and after the next section come back and verify the other.

Example 9 Compute the determinant of each of the following matrices.

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & 0 \\ 2 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} 10 & 5 & 1 & 3 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & 6 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Solution
Here are these determinants.

$$\det(A) = (5)(-3)(4) = -60$$
$$\det(B) = (6)(-1) = -6$$
$$\det(C) = (10)(0)(6)(5) = 0$$

We have one final theorem to give in this section. In the Finding Inverse section we gave a theorem that listed several equivalent statements. Because of Theorem 5 above we can add a statement to that theorem so let's do that.

Here is the improved theorem.

Theorem 9 If A is an $n\times n$ matrix then the following statements are equivalent.

(a) A is invertible.
(b) The only solution to the system $A\mathbf{x} = 0$ is the trivial solution.
(c) A is row equivalent to I_n.
(d) A is expressible as a product of elementary matrices.
(e) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n\times 1$ matrix **b**.
(f) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n\times 1$ matrix **b**.
(g) $\det(A) \neq 0$

The Method of Cofactors

In this section we're going to examine one of the two methods that we're going to be looking at for computing the determinant of a general matrix. We'll also see how some of the ideas we're going to look at in this section can be used to determine the inverse of an invertible matrix.

So, before we actually give the method of cofactors we need to get a couple of definitions taken care of.

Definition 1 If A is a square matrix then the **minor of** a_{ij}, denoted by M_{ij}, is the determinant of the submatrix that results from removing the i^{th} row and j^{th} column of A.

Definition 2 If A is a square matrix then the **cofactor of** a_{ij}, denoted by C_{ij}, is the number $\left(-1\right)^{i+j} M_{ij}$.

Let's take a look at computing some minors and cofactors.

Example 1 For the following matrix compute the cofactors C_{12}, C_{24}, and C_{32}.

$$A = \begin{bmatrix} 4 & 0 & 10 & 4 \\ -1 & 2 & 3 & 9 \\ 5 & -5 & -1 & 6 \\ 3 & 7 & 1 & -2 \end{bmatrix}$$

Solution

In order to compute the cofactors we'll first need the minor associated with each cofactor. Remember that in order to compute the minor we will remove the i^{th} row and j^{th} column of A.

So, to compute M_{12} (which we'll need for C_{12}) we'll need to compute the determinate of the matrix we get by removing the 1st row and 2nd column of A. Here is that work.

$$\begin{bmatrix} 4 & 0 & 10 & 4 \\ -1 & 2 & 3 & 9 \\ 5 & -5 & -1 & 6 \\ 3 & 7 & 1 & -2 \end{bmatrix} \quad \Rightarrow \quad M_{12} = \begin{vmatrix} -1 & 3 & 9 \\ 5 & -1 & 6 \\ 3 & 1 & -2 \end{vmatrix} = 160$$

We've marked out the row and column that we eliminated and we'll leave it to you to verify the determinant computation. Now we can get the cofactor.

$$C_{12} = \left(-1\right)^{1+2} M_{12} = \left(-1\right)^{3}\left(160\right) = -160$$

Let's now move onto the second cofactor. Here is the work for the minor.

$$\begin{bmatrix} 4 & 0 & 10 & 4 \\ -1 & 2 & 3 & 9 \\ 5 & -5 & -1 & 6 \\ 3 & 7 & 1 & -2 \end{bmatrix} \quad \Rightarrow \quad M_{24} = \begin{vmatrix} 4 & 0 & 10 \\ 5 & -5 & -1 \\ 3 & 7 & 1 \end{vmatrix} = 508$$

The cofactor in this case is,

$$C_{24} = (-1)^{2+4} M_{24} = (-1)^{6} (508) = 508$$

Here is the work for the final cofactor.

$$\begin{bmatrix} 4 & 0 & 10 & 4 \\ -1 & 2 & 3 & 9 \\ 5 & -5 & -1 & 6 \\ 3 & 7 & 1 & -2 \end{bmatrix} \Rightarrow M_{32} = \begin{vmatrix} 4 & 10 & 4 \\ -1 & 3 & 9 \\ 3 & 1 & -2 \end{vmatrix} = 150$$

$$C_{32} = (-1)^{3+2} M_{32} = (-1)^{5} (150) = -150$$

Notice that the cofactor is really just $\pm M_{i\,j}$ depending upon *i* and *j*. If the subscripts of the cofactor add to an even number then we leave the minor alone (*i.e.* no "-" sign) when writing down the cofactor. Likewise, if the subscripts on the cofactor sum to an odd number then we add a "-" to the minor when writing down the cofactor.

We can use this fact to derive a table that will allow us to quickly determine whether or not we should add a "-" onto the minor or leave it alone when writing down the cofactor.

Let's start with C_{11}. In this case the subscripts sum to an even number and so we don't tack on a minus sign to the minor. Now, let's move along the first row. The next cofactor would then be C_{12} and in this case the subscripts add to an odd number and so we tack on a minus sign to the minor. For the next cofactor, C_{13}, we would leave the minor alone and for the next, C_{14}, we'd tack a minus sign on, *etc*.

As you can see from this work, if we start at the leftmost entry of the first row we have a "+" in front of the minor and then as we move across the row the signs alternate. If you think about it, this will also happen as we move down the first column. In fact, this will happen as we move across any row and down any column.

We can summarize this idea in the following "sign matrix" that will tell us if we should leave the minor along (*i.e.* tack on a "+") or change its sign (*i.e.* tack on a "-") when writing down the cofactor.

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Okay, we can now talk about how to use cofactors to compute the determinant of a general square matrix. In fact there are two ways we can used cofactors as the following theorem shows.

Theorem 1 If A is an $n \times n$ matrix.

(a) Choose any row, say row i, then,

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots a_{in}C_{in}$$

(b) Choose any column, say column j, then,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

What this theorem tells us is that if we pick any row all we need to do is go across that row and multiply each entry by its cofactor, add all these products up and we'll have the determinant for the matrix. It also says that we could do the same thing only instead of going across any row we could move down any column.

The process of moving across a row or down a column is often called a **cofactor expansion**.

Let's work some examples of this so we can see it in action.

Example 2 For the following matrix compute the determinant using the given cofactor expansions.

$$A = \begin{bmatrix} 4 & 2 & 1 \\ -2 & -6 & 3 \\ -7 & 5 & 0 \end{bmatrix}$$

(a) Expand along the first row.
(b) Expand along the third row.
(c) Expand along the second column.

Solution

First, notice that according to the theorem we should get the same result in all three parts.

(a) Here is the cofactor expansion in terms of symbols for this part.

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

Now, let's plug in for all the quantities. We will just plug in for the entries. For the cofactors we'll write down the minor and a "+1" or a "-1" depending on which sign each minor needs. We'll determine these signs by going to our "sign matrix" above starting at the first entry in the particular row/column we're expanding along and then as we move along that row or column we'll write down the appropriate sign.

Here is the work for this expansion.

$$\det(A) = (4)(+1)\begin{vmatrix} -6 & 3 \\ 5 & 0 \end{vmatrix} + (2)(-1)\begin{vmatrix} -2 & 3 \\ -7 & 0 \end{vmatrix} + (1)(+1)\begin{vmatrix} -2 & -6 \\ -7 & 5 \end{vmatrix}$$
$$= 4(-15) - 2(21) + (1)(-52)$$
$$= -154$$

We'll leave it to you to verify the 2×2 determinant computations.

(b) We'll do this one without all the explanations.

$$\det(A) = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$
$$(-7)(+1)\begin{vmatrix} 2 & 1 \\ -6 & 3 \end{vmatrix} + (5)(-1)\begin{vmatrix} 4 & 1 \\ -2 & 3 \end{vmatrix} + (0)(+1)\begin{vmatrix} 4 & 2 \\ -2 & -6 \end{vmatrix}$$
$$= -7(12) - 5(14) + (0)(-20)$$
$$= -154$$

So, the same answer as the first part which is good since that was supposed to happen.

Notice that the signs for the cofactors in this case were the same as the signs in the first case. This is because the first and third row of our "sign matrix" are identical. Also, notice that we didn't really need to compute the third cofactor since the third entry was zero. We did it here just to get one more example of a cofactor into the notes.

(c) Let's take a look at the final expansion. In this one we're going down a column and notice that from our "sign matrix" that this time we'll be starting the cofactor signs off with a "-1" unlike the first two expansions.

$$\det(A) = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$$
$$(2)(-1)\begin{vmatrix} -2 & 3 \\ -7 & 0 \end{vmatrix} + (-6)(+1)\begin{vmatrix} 4 & 1 \\ -7 & 0 \end{vmatrix} + (5)(-1)\begin{vmatrix} 4 & 1 \\ -2 & 3 \end{vmatrix}$$
$$= -2(21) - 6(7) - 5(14)$$
$$= -154$$

Again, the same as the first two as we expected.

There was another point to the previous problem apart from showing that the row or column we choose to expand along won't matter. Because we are allowed to expand along any row that means unless the problem statement forces to use a particular row or column we will get to choose the row/column to expand along.

When choosing we should choose a row/column that will reduce the amount of work we've got to do if possible. Comparing the parts of the previous example should suggest to us something we should be looking for in making this choice. In part **(b)** it was pointed out that we didn't really need to compute the third cofactor since the third entry in that row was zero.

Choosing to expand along a row/column with zeroes in it will instantly cut back on the number of cofactors that we'll need to compute. So, when allowed to choose which row/column to expand along we should look for the one with the most zeroes. In the case of the previous example that means that the quickest expansions would be either the 3rd row or the 3rd column since both of those have a zero in them and none of the other rows/columns do.

So, let's take a look at a couple more examples.

Example 3 Using a cofactor expansion compute the determinant of,

$$A = \begin{bmatrix} 5 & -2 & 2 & 7 \\ 1 & 0 & 0 & 3 \\ -3 & 1 & 5 & 0 \\ 3 & -1 & -9 & 4 \end{bmatrix}$$

Solution

Since the row or column to use for the cofactor expansion was not given in the problem statement we get to choose which one we want to use. Recalling the brief discussion after the last example we know that we want to choose the row/column with the most zeroes in it since that will mean we won't have to compute cofactors for each entry that is a zero.

So, it looks like the second row would be a good choice for the expansion since it has two zeroes in it. Here is the expansion for this row. As with the previous expansions we'll explicitly give the "+1" or "-1" for the cofactors and the minors as well so you can see where everything in the expansion is coming from.

$$\det(A) = (1)(-1)\begin{vmatrix} -2 & 2 & 7 \\ 1 & 5 & 0 \\ -1 & 9 & 4 \end{vmatrix} + (0)(+1)M_{22} + (0)(-1)M_{23} + (3)(+1)\begin{vmatrix} 5 & -2 & 2 \\ -3 & 1 & 5 \\ 3 & -1 & -9 \end{vmatrix}$$

We didn't bother to write down the minors M_{22} and M_{23} because of the zero entry.

How we choose to compute the determinants for the first and last entry is up to us at this point. We could use a cofactor expansion on each of them or we could use the technique we learned in the first section of this chapter. Either way will get the same answer and we'll leave it to you to verify these determinants.

The determinant for this matrix is,

$$\det(A) = -(-76) + 3(4) = 88$$

Example 4 Using a cofactor expansion compute the determinant of,

$$B=\begin{bmatrix} 2 & -2 & 0 & 3 & 4 \\ 4 & -1 & 0 & 1 & -1 \\ 0 & 5 & 0 & 0 & -1 \\ 3 & 2 & -3 & 4 & 3 \\ 7 & -2 & 0 & 9 & -5 \end{bmatrix}$$

Solution

This is a large matrix, but if you check out the third column we'll see that there is only one non-zero entry in that column and so that looks like a good column to do a cofactor expansion on. Here's the cofactor expansion for this matrix. Again, we explicitly added in the "+1" and "-1" and won't bother to write down the minors for the zero entries.

$$\begin{aligned} \det(B) = (0)(+1)M_{13} + (0)(-1)M_{23} + (0)(+1)M_{33} + & \\ (-3)(-1)\begin{vmatrix} 2 & -2 & 3 & 4 \\ 4 & -1 & 1 & -1 \\ 0 & 5 & 0 & -1 \\ 7 & -2 & 9 & -5 \end{vmatrix} + (0)(+1)M_{53} & \end{aligned}$$

Now, in order to complete this problem we'll need to take the determinant of a 4×4 matrix and the only way that we've got to do that is to once again do a cofactor expansion on it. In this case it looks like the third row will be the best option since it's got more zero entries than any other row or column.

This time we'll just put in the terms that come from non-zero entries. Here is the remainder of this problem. Also don't forget that there is still a coefficient of 3 in front of this determinant!

$$\begin{aligned} \det(B) &= 3\begin{vmatrix} 2 & -2 & 3 & 4 \\ 4 & -1 & 1 & -1 \\ 0 & 5 & 0 & -1 \\ 7 & -2 & 9 & -5 \end{vmatrix} \\ &= 3\left((5)(-1)\begin{vmatrix} 2 & 3 & 4 \\ 4 & 1 & -1 \\ 7 & 9 & -5 \end{vmatrix} + (-1)(-1)\begin{vmatrix} 2 & -2 & 3 \\ 4 & -1 & 1 \\ 7 & -2 & 9 \end{vmatrix} \right) \\ &= 3\big(-5(163) + (1)(41)\big) \\ &= -2322 \end{aligned}$$

This last example has shown one of the drawbacks to this method. Once the size of the matrix gets large there can be a lot of work involved in the method. Also, for anything larger than a 4×4 matrix you are almost assured of having to do cofactor expansions multiple times until the size of the matrix gets down to 3×3 and other methods can be used.

There is a way to simplify things down somewhat, but we'll need to the topic of the next section before we can show that.

Now let's move onto the final topic of this section.

It turns out that we can also use cofactors to determine the inverse of an invertible matrix. To see how this is done we'll first need a quick definition.

Definition 3 Let A be an $n \times n$ matrix and C_{ij} be the cofactor of a_{ij}. The **matrix of cofactors from *A*** is,

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

The **adjoint of *A*** is the transpose of the matrix of cofactors and is denoted by adj(A).

Example 5 Compute the adjoint of the following matrix.

$$A = \begin{bmatrix} 4 & 2 & 1 \\ -2 & -6 & 3 \\ -7 & 5 & 0 \end{bmatrix}$$

Solution

We need the cofactors for each of the entries from this matrix. This is the matrix from Example 2 and in that example we computed all the cofactors except for C_{21} and C_{23} so here are those computations.

$$C_{21} = (-1)\begin{vmatrix} 2 & 1 \\ 5 & 0 \end{vmatrix} = (-1)(-5) = 5$$

$$C_{23} = (-1)\begin{vmatrix} 4 & 2 \\ -7 & 5 \end{vmatrix} = (-1)(34) = -34$$

Here are the others from Example 2.

$$C_{11} = -15 \qquad C_{12} = -21 \qquad C_{13} = -52 \qquad C_{22} = 7$$
$$C_{31} = 12 \qquad C_{32} = -14 \qquad C_{33} = -20$$

The matrix of cofactors is then,

$$\begin{bmatrix} -15 & -21 & -52 \\ 5 & 7 & -34 \\ 12 & -14 & -20 \end{bmatrix}$$

The adjoint is then,

$$\text{adj}(A) = \begin{bmatrix} -15 & 5 & 12 \\ -21 & 7 & -14 \\ -52 & -34 & -20 \end{bmatrix}$$

We started this portion of this section off by saying that we were going to see how to use cofactors to determine the inverse of a matrix. Here is the theorem that will tell us how to do that.

Theorem 2 If A is an invertible matrix then

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$$

Example 6 Use the adjoint matrix to compute the inverse of the following matrix.

$$A = \begin{bmatrix} 4 & 2 & 1 \\ -2 & -6 & 3 \\ -7 & 5 & 0 \end{bmatrix}$$

Solution

We've done most of the work for this problem already. In Example 2 we determined that

$$\det(A) = -154$$

and in Example 5 we found the adjoint to be

$$\text{adj}(A) = \begin{bmatrix} -15 & 5 & 12 \\ -21 & 7 & -14 \\ -52 & -34 & -20 \end{bmatrix}$$

Therefore, the inverse of the matrix is,

$$A^{-1} = \frac{1}{-154}\begin{bmatrix} -15 & 5 & 12 \\ -21 & 7 & -14 \\ -52 & -34 & -20 \end{bmatrix} = \begin{bmatrix} \frac{15}{154} & -\frac{5}{154} & -\frac{6}{77} \\ \frac{3}{22} & -\frac{1}{22} & \frac{1}{11} \\ \frac{26}{77} & \frac{17}{77} & \frac{10}{77} \end{bmatrix}$$

You might want to verify this using the row reduction method we used in the previous chapter for the practice.

Using Row Reduction To Compute Determinants

In this section we'll take a look at the second method for computing determinants. The idea in this section is to use row reduction on a matrix to get it down to a row-echelon form.

Since we're computing determinants we know that the matrix, A, we're working with will be square and so the row-echelon form of the matrix will be an upper triangular matrix and we know how to quickly compute the determinant of a triangular matrix. So, since we already know how to do row reduction all we need to know before we can work some problems is how the row operations used in the row reduction process will affect the determinant.

Before proceeding we should point out that there are a set of elementary column operations that mirror the elementary row operations. We can multiply a column by a scalar, c, we can interchange two columns and we add a multiple of one column onto another column. The operations could just as easily be used as row operations and so all the theorems in this section will make note of that. We'll just be using row operations however in our examples.

Here is the theorem that tells us how row or column operations will affect the value of the determinant of a matrix.

Theorem 1 Let A be a square matrix.

(a) If B is the matrix that results from multiplying a row or column of A by a scalar, c, then $\det(B) = c\det(A)$

(b) If B is the matrix that results from interchanging two rows or two columns of A then $\det(B) = -\det(A)$

(c) If B is the matrix that results from adding a multiple of one row of A onto another row of A or adding a multiple of one column of A onto another column of A then $\det(B) = \det(A)$

Notice that the row operation that we'll be using the most in the row reduction process will not change the determinant at all. The operations that we're going to need to worry about are the first two and the second is easy enough to take care of. If we interchange two rows the determinant changes by a minus sign. We are going to have to be a little careful with the first one however. Let's check out an example of how this method works in order to see what's going on.

Example 1 Use row reduction to compute the determinant of the following matrix.

$$A = \begin{bmatrix} 4 & 12 \\ -7 & 5 \end{bmatrix}$$

Solution

There is of course no real reason to do row reduction on this matrix in order to compute the determinant. We can find it easily enough at this point. In fact let's do that so we can check the results of our work after we do row reduction on this.

$$\det(A) = (4)(5) - (-7)(12) = 104$$

Okay, now let's do with row reduction to see what we've got. We need to reduce this down to row-echelon form and while we could easily use a multiple of the third row to get a 1 in the first entry of the first row let's just divide the first row by 4 since that's the one operation we're going to need to careful with. So, let's do the first operation and see what we've got.

$$A = \begin{bmatrix} 4 & 12 \\ -7 & 5 \end{bmatrix} \xrightarrow{\frac{1}{4}R_1} \begin{bmatrix} 1 & 3 \\ -7 & 5 \end{bmatrix} = B$$

So, we called the result *B* and let's see what the determinant of this matrix is.

$$\det(B) = (1)(5) - (-7)(3) = 26 = \frac{1}{4}\det(A)$$

So, the results of the theorem are verified for this step. The next step is then to convert the -7 into a zero. Let's do that and see what we get.

$$B = \begin{bmatrix} 1 & 3 \\ -7 & 5 \end{bmatrix} \xrightarrow{R_2 + 7R_1} \begin{bmatrix} 1 & 3 \\ 0 & 26 \end{bmatrix} = C$$

According to the theorem *C* should have the same determinant as *B* and it does (you should verify this statement).

The final step is to convert the 26 into a 1.

$$C = \begin{bmatrix} 1 & 3 \\ 0 & 26 \end{bmatrix} \xrightarrow{\frac{1}{26}R_2} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = D$$

Now, we've got the following,

$$\det(D) = 1 = \frac{1}{26}\det(C)$$

Once again the theorem is verified.

Now, just how does all of this help us to find the determinant of the original matrix? We could work our way backwards from det(*D*) and figure out what det(*A*) is. However, there is a way to modify our work above that will allow us to also get the answer once we reach row-echelon form.

To see how we do this let's go back to the first operation that we did and we saw when we were done we had,

$$\det(B) = \frac{1}{4}\det(A) \qquad \text{OR} \qquad \det(A) = 4\det(B)$$

Written in another way this is,

$$\det(A) = \begin{vmatrix} 4 & 12 \\ -7 & 5 \end{vmatrix} = (4)\begin{vmatrix} 1 & 3 \\ -7 & 5 \end{vmatrix} = \det(B)$$

Notice that the determinants, when written in the "matrix" form, are pretty much what we originally wrote down when doing the row operation. Therefore, instead of writing down the row operation as we did above let's just use this "matrix" form of the determinant and write the row operation as follows.

$$\det(A) = \begin{vmatrix} 4 & 12 \\ -7 & 5 \end{vmatrix} \overset{\frac{1}{4}R_1}{=} (4)\begin{vmatrix} 1 & 3 \\ -7 & 5 \end{vmatrix}$$

In going from the matrix on the left to the matrix on the right we performed the operation $\frac{1}{4}R_1$ and in the process we changed the value of the determinant. So, since we've got an equal sign here we need to also modify the determinant of the matrix on the right so that it will remain equal to the determinant of the matrix on the left. As shown above, we can do this by multiplying the matrix on the right by the reciprocal of the scalar we used in the row operation.

Let's complete this and notice that in the second step we aren't going to change the value of the determinant since we're adding a multiple of the second row onto the first row so we'll not change the value of the determinant on the right. In the final operation we divided the second row by 26 and so we'll need to multiply the determinant on the right by 26 to persevere the equality of the determinants.

Here is the complete work for this problem using these ideas.

$$\begin{aligned} \det(A) = \begin{vmatrix} 4 & 12 \\ -7 & 5 \end{vmatrix} &\overset{\frac{1}{4}R_1}{=} (4)\begin{vmatrix} 1 & 3 \\ -7 & 5 \end{vmatrix} \\ &\overset{R_2+7R_1}{=} (4)\begin{vmatrix} 1 & 3 \\ 0 & 26 \end{vmatrix} \\ &\overset{\frac{1}{26}R_2}{=} (4)(26)\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} \end{aligned}$$

Okay, we're down to row-echelon form so let's strip out all the intermediate steps out and see what we've got.

$$\det(A) = (4)(26)\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix}$$

The matrix on the right is triangular and we know that determinants of triangular matrices are just the product of the main diagonal entries and so the determinant of A is,

$$\det(A) = (4)(26)(1)(1) = 104$$

Now, that was a lot of work to compute the determinant and in general we wouldn't use this method on a 2×2 matrix, but by doing on one here it allowed us to investigate the method in a detail without having to deal with a lot of steps.

There are a couple of issues to point out before we move into another more complicated problem. First, we didn't do any row interchanges in the above example, but the theorem tells us that will only change the sign on the determinant. So, if we do a row interchange in our work we'll just tack a minus sign onto the determinant.

Second, we took the matrix all the way down to row-echelon form, but if you stop to think about it there's really nothing special about that in this case. All we need to do is reduce the matrix to a triangular matrix and then use the fact that can quickly find the determinant of any triangular matrix.

From this point on we'll not be going all the way to row-echelon form. We'll just make sure that we reduce the matrix down to a triangular matrix and then stop and compute the determinant.

Example 2 Use row reduction to compute the determinant of the following matrix.

$$A = \begin{bmatrix} -2 & 10 & 2 \\ 1 & 0 & 7 \\ 0 & -3 & 5 \end{bmatrix}$$

Solution

We'll do this one with less explanation. Just remember that if we interchange rows tack a minus sign onto the determinant and if we multiply a row by a scalar we'll need to multiply the new determinant by the reciprocal of the scalar.

$$\det(A) = \begin{vmatrix} -2 & 10 & 2 \\ 1 & 0 & 7 \\ 0 & -3 & 5 \end{vmatrix} \overset{R_1 \leftrightarrow R_2}{=} -\begin{vmatrix} 1 & 0 & 7 \\ -2 & 10 & 2 \\ 0 & -3 & 5 \end{vmatrix}$$

$$\overset{R_2 + 2R_1}{=} -\begin{vmatrix} 1 & 0 & 7 \\ 0 & 10 & 16 \\ 0 & -3 & 5 \end{vmatrix}$$

$$\overset{\frac{1}{10}R_2}{=} -(10)\begin{vmatrix} 1 & 0 & 7 \\ 0 & 1 & \frac{8}{5} \\ 0 & -3 & 5 \end{vmatrix}$$

$$\overset{R_3 + 3R_2}{=} -(10)\begin{vmatrix} 1 & 0 & 7 \\ 0 & 1 & \frac{8}{5} \\ 0 & 0 & \frac{49}{5} \end{vmatrix}$$

Okay, we've gotten the matrix down to triangular form and so at this point we can stop and just take the determinant of that and make sure to keep the scalars that are multiplying it. Here is the final computation for this problem.

$$\det(A) = -10(1)(1)\left(\frac{49}{5}\right) = -98$$

Example 3 Use row reduction to compute the determinant of the following matrix.

$$A = \begin{bmatrix} 3 & 0 & 6 & -3 \\ 0 & 2 & 3 & 0 \\ -4 & -7 & 2 & 0 \\ 2 & 0 & 1 & 10 \end{bmatrix}$$

Solution

Okay, there's going to be some work here so let's get going on it.

$$\det(A) = \begin{vmatrix} 3 & 0 & 6 & -3 \\ 0 & 2 & 3 & 0 \\ -4 & -7 & 2 & 0 \\ 2 & 0 & 1 & 10 \end{vmatrix} \overset{\frac{1}{3}R_1}{=} (3)\begin{vmatrix} 1 & 0 & 2 & -1 \\ 0 & 2 & 3 & 0 \\ -4 & -7 & 2 & 0 \\ 2 & 0 & 1 & 10 \end{vmatrix}$$

$$\overset{\substack{R_3+4R_1 \\ R_4-2R_1}}{=} (3)\begin{vmatrix} 1 & 0 & 2 & -1 \\ 0 & 2 & 3 & 0 \\ 0 & -7 & 10 & -4 \\ 0 & 0 & -3 & 12 \end{vmatrix}$$

$$\overset{\frac{1}{2}R_2}{=} (3)(2)\begin{vmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & -7 & 10 & -4 \\ 0 & 0 & -3 & 12 \end{vmatrix}$$

$$\overset{R_3+7R_2}{=} (3)(2)\begin{vmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{41}{2} & -4 \\ 0 & 0 & -3 & 12 \end{vmatrix}$$

$$\overset{\frac{2}{41}R_3}{=} (3)(2)\left(\frac{41}{2}\right)\begin{vmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 1 & -\frac{8}{41} \\ 0 & 0 & -3 & 12 \end{vmatrix}$$

$$\overset{R_4+3R_3}{=} (3)(2)\left(\frac{41}{2}\right)\begin{vmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 1 & -\frac{8}{41} \\ 0 & 0 & 0 & \frac{468}{41} \end{vmatrix}$$

Okay, that was a lot of work, but we've gotten it into a form we can deal with. Here's the determinant.

$$\det(A) = (3)(2)\left(\frac{41}{2}\right)\left(\frac{468}{41}\right) = 1404$$

Now, as the previous example has shown us, this method can be a lot of work and its work that if we aren't paying attention it will be easy to make a mistake.

There is a method that we could have used here to significantly reduce our work and it's not even a new method. Notice that with this method at each step we have a new determinant that needs computing. We continued down until we got a triangular matrix since that would be easy for us to compute. However, there's nothing keeping us from stopping at any step and using some other method for computing the determinant. In fact, if you look at our work, after the second step we've gotten a column with a 1 in the first entry and zeroes below it. If we were in the previous section we'd just do a cofactor expansion along this column for this determinant. So, let's do that. No one ever said we couldn't mix the methods from this and the previous section in a problem.

Example 4 Use row reduction and a cofactor expansion to compute the determinant of the matrix in Example 3.

Solution.
Okay, this "new" method says to use row reduction until we get a matrix that would be easy to do a cofactor expansion on. As noted earlier that means only doing the first two steps. So, for the sake of completeness here are those two steps again.

$$\det(A) = \begin{vmatrix} 3 & 0 & 6 & -3 \\ 0 & 2 & 3 & 0 \\ -4 & -7 & 2 & 0 \\ 2 & 0 & 1 & 10 \end{vmatrix} \overset{\frac{1}{3}R_1}{=} (3)\begin{vmatrix} 1 & 0 & 2 & -1 \\ 0 & 2 & 3 & 0 \\ -4 & -7 & 2 & 0 \\ 2 & 0 & 1 & 10 \end{vmatrix}$$

$$\begin{matrix} R_3 + 4R_1 \\ R_4 - 2R_1 \\ = \end{matrix} (3)\begin{vmatrix} 1 & 0 & 2 & -1 \\ 0 & 2 & 3 & 0 \\ 0 & -7 & 10 & -4 \\ 0 & 0 & -3 & 12 \end{vmatrix}$$

At this point we'll just do a cofactor expansion along the first column.

$$\det(A) = (3)\left((1)(+1)\begin{vmatrix} 2 & 3 & 0 \\ -7 & 10 & -4 \\ 0 & -3 & 12 \end{vmatrix} + (0)C_{21} + (0)C_{31} + (0)C_{41} \right)$$

$$= 3\begin{vmatrix} 2 & 3 & 0 \\ -7 & 10 & -4 \\ 0 & -3 & 12 \end{vmatrix}$$

At this point we can use any method to compute the determinant of the new 3×3 matrix so we'll leave it to you to verify that

$$\det(A) = (3)(468) = 1404$$

There is one final idea that we need to discuss in this section before moving on.

Theorem 2 Suppose that A is a square matrix and that two of its rows are proportional or two of its columns are proportional. Then $\det(A) = 0$.

When we say that two rows or two columns are proportional that means that one of the rows(columns) is a scalar times another row(column) of the matrix.

We're not going to prove this theorem but it you think about it, it should make some sense. Let's suppose that two rows are proportional. So we know that one of the rows is a scalar multiple of another row. This means we can use the third row operation to make one of the rows all zero. From Theorem 1 above we know that both of these matrices must have the same determinant and from Theorem 7 from the Determinant Properties section we know that if a matrix has a row or column of all zeroes, then that matrix is singular, *i.e.* its determinant is zero. Therefore both matrices must have a zero determinant.

Here is a quick example showing this.

Example 5 Show that the following matrix is singular.

$$A = \begin{bmatrix} 4 & -1 & 3 \\ 2 & 5 & -1 \\ -8 & 2 & -6 \end{bmatrix}$$

Solution

We can use Theorem 2 above upon noticing that the third row is -2 times the first row. That's all we need to use this theorem.

So, technically we've answered the question. However, let's go through the steps outlined above to also show that this matrix is singular. To do this we'd do one row reduction step to get the row of all zeroes into the matrix as follows.

$$\det(A) = \begin{vmatrix} 4 & -1 & 3 \\ 2 & 5 & -1 \\ -8 & 2 & -6 \end{vmatrix} \overset{R_2 + 2R_1}{=} \begin{vmatrix} 4 & -1 & 3 \\ 2 & 5 & -1 \\ 0 & 0 & 0 \end{vmatrix}$$

We know by Theorem 1 above that these two matrices have the same determinant. Then because we see a row of all zeroes we can invoke Theorem 7 from the Determinant Properties to say that the determinant on the right must be zero, and so be singular.

Then, as we pointed out, these two matrices have the same determinant and so we've also got $\det(A) = 0$ and so A is singular.

You might want to verify that this matrix is singular by computing its determinant with one of the other methods we've looked at for the practice.

We've now looked at several methods for computing determinants and as we've seen each can be long and prone to mistakes. On top of that for some matrices one method may work better than the other. So, when faced with a determinant you'll need to look at it and determine which method to use and unless otherwise specified by the problem statement you should use the one that you find the easiest to use. Note that this may not be the method that somebody else chooses to use, but you shouldn't worry about that. You should use the method you are the most comfortable with.

Cramer's Rule

In this section we're going to come back and take one more look at solving systems of equations. In this section we're actually going to be able to get a general solution to certain systems of equations. It won't work on all systems of equations and as we'll see if the system is too large it will probably be quicker to use one of the other methods that we've got for solving systems of equations.

So, let's jump into the method.

Theorem 1 Suppose that A is an $n \times n$ invertible matrix. Then the solution to the system $A\mathbf{x} = \mathbf{b}$ is given by,

$$x_1 = \frac{\det(A_1)}{\det(A)}, \; x_2 = \frac{\det(A_2)}{\det(A)}, \; \ldots, \; x_n = \frac{\det(A_n)}{\det(A)}$$

where A_i is the matrix found by replacing the i^{th} column of A with $\mathbf{b}$.

Proof : The proof to this is actually pretty simple. First, because we know that A is invertible then we know that the inverse exists and that $\det(A) \neq 0$. We also know that the solution to the system can be given by,

$$\mathbf{x} = A^{-1}\mathbf{b}$$

From the section on cofactors we know how to define the inverse in terms of the adjoint of A. Using this gives us,

$$\mathbf{x} = \frac{1}{\det(A)} \operatorname{adj}(A)\mathbf{b} = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Recall that C_{ij} is the cofactor of a_{ij}. Also note that the subscripts on the cofactors above appear to be backwards but they are correctly placed. Recall that we get the adjoint by first forming a matrix with C_{ij} in the i^{th} row and j^{th} column and then taking the transpose to get the adjoint.

Now, multiply out the matrices to get,

$$\mathbf{x} = \frac{1}{\det(A)} \begin{bmatrix} b_1 C_{11} + b_2 C_{21} + \cdots b_n C_{n\,n} \\ b_1 C_{12} + b_2 C_{22} + \cdots b_n C_{n\,2} \\ \vdots \\ b_1 C_{1n} + b_2 C_{2n} + \cdots b_n C_{n\,n} \end{bmatrix}$$

The entry in the i^{th} row of $\mathbf{x}$, which is x_i in the solution, is

$$x_i = \frac{b_1 C_{1i} + b_2 C_{2i} + \cdots b_n C_{n\,i}}{\det(A)}$$

Next let's define,

$$A_i = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1\,i-1} & b_1 & a_{1\,i+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2\,i-1} & b_2 & a_{2\,i+1} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n\,i-1} & b_n & a_{n\,i+1} & \cdots & a_{n\,n} \end{bmatrix}$$

So, A_i is the matrix we get by replacing the i^{th} column of A with $\mathbf{b}$. Now, if we were to compute the determinate of A_i by expanding along the i^{th} column the products would be one of the b_i's times the appropriate cofactor. Notice however that since the only difference between A_i and A is the i^{th} column and so the cofactors of we get by expanding A_i along the i^{th} column will be exactly the same as the cofactors we would get by expanding A along the i^{th} column.

Therefore, the determinant of A_i is given be,

$$\det(A_i) = b_1 C_{1i} + b_2 C_{2i} + \cdots b_n C_{n\,i}$$

where $C_{k\,i}$ is the cofactor of $a_{k\,i}$ from the matrix A. Note however that this is exactly the numerator of x_i and so we have,

$$x_i = \frac{\det(A_i)}{\det(A)}$$

as we wanted to prove.

■

Let's work a quick example to illustrate the method.

Example 1 Use Cramer's Rule to determine the solution to the following system of equations.

$$\begin{aligned} 3x_1 - x_2 + 5x_3 &= -2 \\ -4x_1 + x_2 + 7x_3 &= 10 \\ 2x_1 + 4x_2 - x_3 &= 3 \end{aligned}$$

Solution

First let's put the system into matrix form and verify that the coefficient matrix is invertible.

$$\underbrace{\begin{bmatrix} 3 & -1 & 5 \\ -4 & 1 & 7 \\ 2 & 4 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} -2 \\ 10 \\ 3 \end{bmatrix}}_{\mathbf{b}}$$

$$\det(A) = -187 \neq 0$$

So, the coefficient matrix is invertible and Cramer's Rule can be used on the system. We'll also need det(A) in a bit so it's good that we now have it. Let's now write down the formulas for the solution to this system.

$$x_1 = \frac{\det(A_1)}{\det(A)} \qquad x_2 = \frac{\det(A_2)}{\det(A)} \qquad x_3 = \frac{\det(A_3)}{\det(A)}$$

where A_1 is the matrix formed by replacing the 1st column of A with **b**, A_2 is the matrix formed by replacing the 2nd column of A with **b**, and A_3 is the matrix formed by replacing the 3rd column of A with **b**.

We'll leave it to you to verify the following determinants.

$$\det(A_1) = \begin{vmatrix} -2 & -1 & 5 \\ 10 & 1 & 7 \\ 3 & 4 & -1 \end{vmatrix} = 212$$

$$\det(A_2) = \begin{vmatrix} 3 & -2 & 5 \\ -4 & 10 & 7 \\ 2 & 3 & -1 \end{vmatrix} = -273$$

$$\det(A_3) = \begin{vmatrix} 3 & -1 & -2 \\ -4 & 1 & 10 \\ 2 & 4 & 3 \end{vmatrix} = -107$$

The solution to the system is then,

$$x_1 = -\frac{212}{187} \qquad x_2 = \frac{273}{187} \qquad x_3 = \frac{107}{187}$$

Now, the solution to this system had some somewhat messy solutions and that would have made the row reduction method prone to mistake. However, since this solution required us to compute 4 determinants as you can see if your system gets too large this would be a very time consuming method to use. For example a system with 5 equations and 5 unknowns would require us to compute 6 5×5 determinants. At that point, regardless of how messy the final answers are there is a good chance that the row reduction method would be easier.

Euclidean n-Space

Introduction

In this chapter we are going to start looking at the idea of a vector and the ultimate goal of this chapter will be to define something called Euclidean *n*-space. In this chapter we'll be looking at some very specific examples of vectors so we can build up some of the ideas that surround them. We will reserve general vectors for the next chapter.

We will also be taking a quick look at the topic of linear transformations. Linear transformations are a very important idea in the study of Linear Algebra.

Here is a listing of the topics in this chapter.

Vectors – In this section we'll introduce vectors in 2-space and 3-space as well as some of the important ideas about them.

Dot Product & Cross Product – Here we'll look at the dot product and the cross product, two important products for vectors. We'll also take a look at an application of the dot product.

Euclidean *n*-Space – We'll introduce the idea of Euclidean *n*-space in this section and extend many of the ideas of the previous two sections.

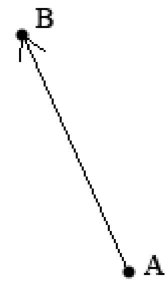
Linear Transformations – In this section we'll introduce the topic of linear transformations and look at many of their properties.

Examples of Linear Transformations – We'll take a look at quite a few examples of linear transformations in this section.

Vectors

In this section we're going to start taking a look at vectors in 2-space (normal two dimensional space) and 3-space (normal three dimensional space). Later in this chapter we'll be expanding the ideas here to *n*-space and we'll be looking at a much more general definition of a vector in the next chapter. However, if we start in 2-space and 3-space we'll be able to use a geometric interpretation that may help understand some of the concepts we're going to be looking at.

So, let's start off with defining a vector in 2-space or 3-space. A vector can be represented geometrically by a directed line segment that starts at a point *A*, called the **initial point**, and ends at a point *B*, called the **terminal point**. Below is an example of a vector in 2-space.



Vectors are typically denoted with a boldface lower case letter. For instance we could represent the vector above by **v**, **w**, **a**, or **b**, *etc*. Also when we've explicitly given the initial and terminal points we will often represent the vector as,

$$\mathbf{v} = \overrightarrow{AB}$$

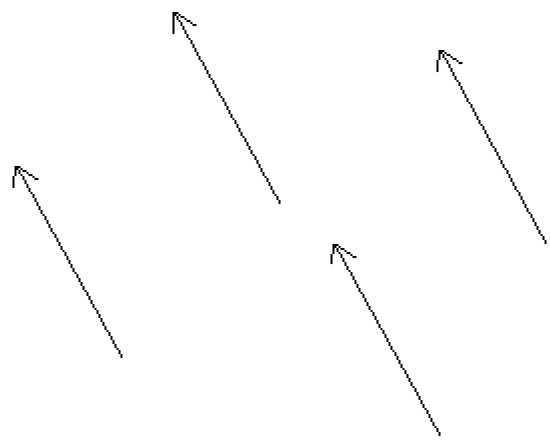
where the positioning of the upper case letters is important. The *A* is the initial point and so is listed first while the terminal point, *B*, is listed second.

As we can see in the figure of the vector shown above a vector imparts two pieces of information. A vector will have a direction and a magnitude (the length of the directed line segment). Two vectors with the same magnitude but different directions are different vectors and likewise two vectors with the same direction but different magnitude are different.

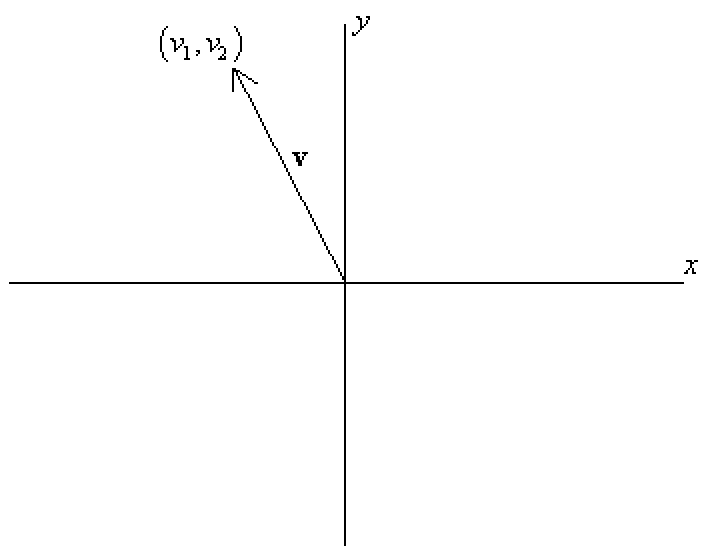
Vectors with the same direction and same magnitude are called **equivalent** and even though they may have different initial and terminal points we think of them as equal and so if **v** and **u** are two equivalent vectors we will write,

$$\mathbf{v} = \mathbf{u}$$

To illustrate this idea all of the vectors in the image below (all in 2-space) are equivalent since they have the same direction and magnitude.



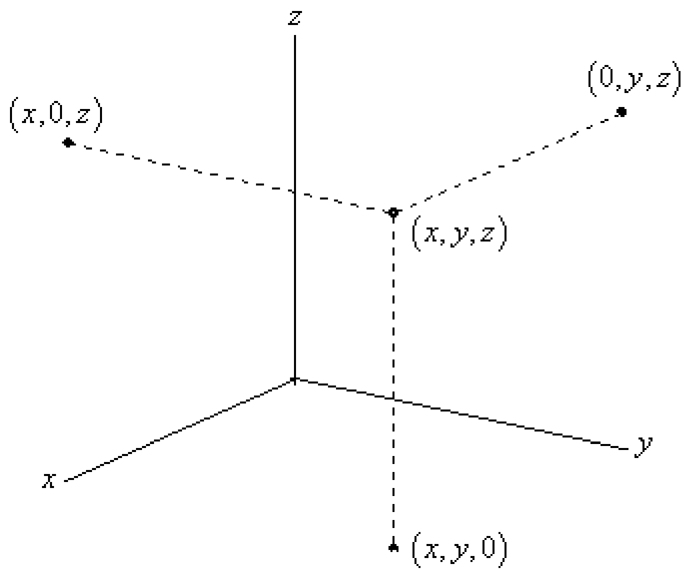
It is often difficult to really visualize a vector without a frame of reference and so we will often introduce a coordinate system to the picture. For example, in 2-space, suppose that **v** is any vector whose initial point is at the origin of the rectangular coordinate system and its terminal point is at the coordinates $\left(v_1, v_2\right)$ as shown below.



In these cases we call the coordinates of the terminal point the **components of v** and write,

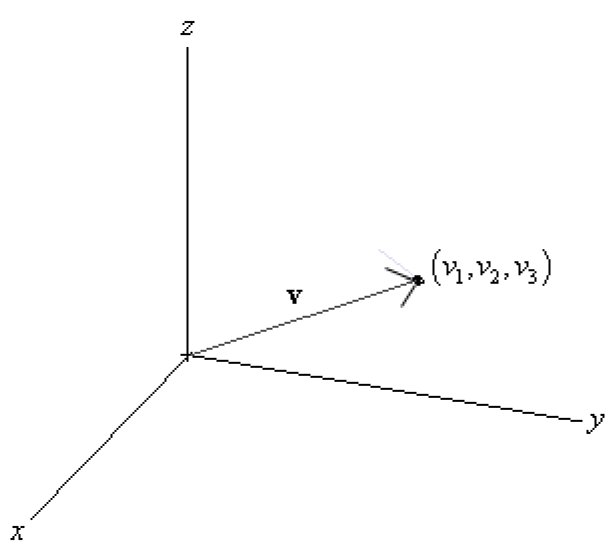
$$\mathbf{v} = \left(v_1, v_2\right)$$

We can do a similar thing for vectors in 3-space. Before we get into that however, let's make sure that you're familiar with all the concepts we might run across in dealing with 3-space. Below is a point in 3-space.



Just as a point in 2-space is described by a pair (x, y) we describe a point in 3-space by a triple (x, y, z). Next if we take each pair of coordinate axes and look at the plane they form we call these the coordinate planes and denote them as ***xy*-plane**, ***yz*-plane**, and ***xz*-plane** respectively. Also note that if we take the general point and move it straight into one of the coordinate planes we get a new point where one of the coordinates is zero. For instance in the *xy*-plane we have the point $(x, y, 0)$, *etc*.

Just as in 2-space, suppose that we've got a vector **v** whose initial point is the origin of the coordinate system and whose terminal point is given by (v_1, v_2, v_3) as shown below,



Just as in 2-space we call (v_1, v_2, v_3) the components of **v** and write,

$$\mathbf{v} = (v_1, v_2, v_3)$$

Before proceeding any further we should briefly talk about the notation we're using because it can be confusing sometimes. We are using the notation (v_1, v_2, v_3) to represent both a point in 3-space and a vector in 3-space as shown in the figure above. This is something you'll need to get used to. In this class (v_1, v_2, v_3) can be either a point or a vector and we'll need to be careful and pay attention to the context of the problem, although in many problems it won't really matter. We'll be able to use it as a point or a vector as we need to. The same comment could be made for points/vectors in 2-space.

Now, let's get back to the discussion at hand and notice that the component form of the vector is really telling how to get from the initial point of the vector to the terminal point of the vector. For example, lets suppose that $\mathbf{v} = (v_1, v_2)$ is a vector in 2-space with initial point $A = (x_1, y_1)$. The first component of the vector, v_1, is the amount we have to move to the right (if v_1 is positive) or to the left (if v_1 is negative). The second component tells us how much to move up or down depending on the sign of v_2. The terminal point of **v** is then given by,

$$B = (x_1 + v_1, y_1 + v_2)$$

Likewise if $\mathbf{v} = (v_1, v_2, v_3)$ is a vector in 2-space with initial point $A = (x_1, y_1, z_1)$ the terminal point is given by,

$$B = (x_1 + v_1, y_1 + v_2, z_1 + v_3)$$

Notice as well that if the initial point is the origin then the final point will be $B = (v_1, v_2, v_3)$ and we once again see that (v_1, v_2, v_3) can represent both a point and a vector.

This can all be turned around as well. Let's suppose that we've got two points in 2-space, $A = (x_1, y_1)$ and $B = (x_2, y_2)$. Then the vector with initial point *A* and terminal point *B* is given by,

$$\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1)$$

Note that the order of the points is important. The components are found by subtracting the coordinates of the initial point from the coordinates of the terminal point. If we turned this around and wanted the vector with initial point *B* and terminal point *A* we'd have,

$$\overrightarrow{BA} = (x_1 - x_2, y_1 - y_2)$$

Of course we can also do this in 3-space. Suppose that we want the vector that has an initial point of $A = (x_1, y_1, z_1)$ and a terminal point of $B = (x_2, y_2, z_2)$. This vector is given by,

$$\overrightarrow{AB} = \left(x_2 - x_1, y_2 - y_1, z_2 - z_1\right)$$

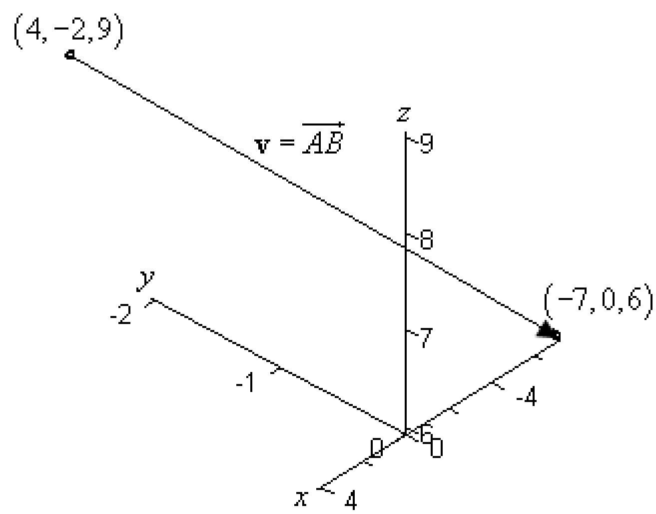
Let's see an example of this.

Example 1 Find the vector that starts at $A = (4,-2,9)$ and ends at $B = (-7,0,6)$.

Solution
There really isn't much to do here other than use the formula above.

$$\mathbf{v} = \overrightarrow{AB} = \left(-7-4, 0-(-2), 6-9\right) = (-11, 2, -3)$$

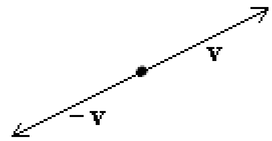
Here is a sketch showing the points and the vector.



Okay, it's now time to move into arithmetic of vectors. For each operation we'll look at both a geometric and a component interpretation. The geometric interpretation will help with understanding just what the operation is doing and the component interpretation will help us to actually do the operation.

There are two quick topics that we first need to address in vector arithmetic. The first is the **zero vector**. The zero vector, denoted by **0**, is a vector with no length. Because the zero vector has no length it is hard to talk about its direction so by convention we say that the zero vector can have any direction that we need for it to have in a given problem.

The next quick topic to discuss is that of **negative** of a vector. If **v** is a vector then the negative of the vector, denoted by **–v**, is defined to be the vector with the same length as **v** but has the opposite direction as **v** as shown below.

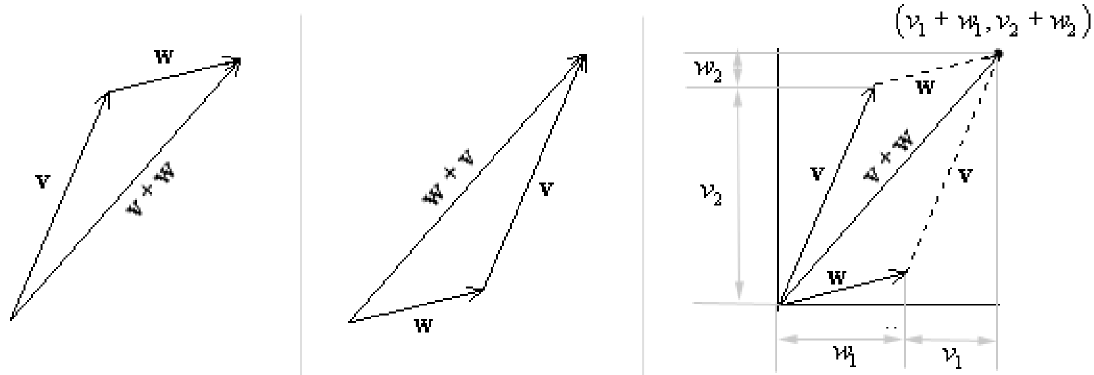


We'll see how to compute the negative vector in a bit. Also note that sometimes the negative is called the **additive inverse** of the vector **v**.

Okay let's start off the arithmetic with addition.

Definition 1 Suppose that **v** and **w** are two vectors then to find the **sum** of the two vectors, denoted $\mathbf{v}+\mathbf{w}$, we position **w** so that its initial point coincides with the terminal point of **v**. The new vector whose initial point is the initial point of **v** and whose terminal point is the terminal point of **w** will be the sum of the two vectors, or $\mathbf{v}+\mathbf{w}$.

Below are three sketches of what we've got here with addition of vectors in 2-space. In terms of components we have $\mathbf{v}=(v_1,v_2)$ and $\mathbf{w}=(w_1,w_2)$.



The sketch on the left matches the definition above. We first sketch in **v** and the sketch **w** starting where **v** left off. The resultant vector is then the sum. In the middle we have the sketch for $\mathbf{w}+\mathbf{v}$ and as we can see we get exactly the same resultant vector. From this we can see that we will have,

$$\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$$

The sketch on the right merges the first two sketches into one and also adds in the components for each of the vectors. It's a little "busy", but you can see that the coordinates of the sum are (v_1+w_1, v_2+w_2). Therefore, for the vectors in 2-space we can compute the sum of two vectors using the following formula.

$$\mathbf{v}+\mathbf{w}=(v_1+w_1, v_2+w_2)$$

Likewise, if we have two vectors in 3-space, say $\mathbf{v}=(v_1,v_2,v_3)$ and $\mathbf{w}=(w_1,w_2,w_3)$, then we'll have,

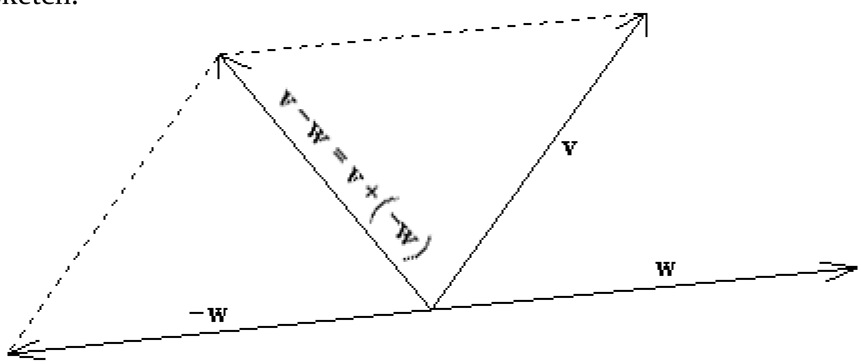
$$\mathbf{v}+\mathbf{w}=(v_1+w_1, v_2+w_2, v_3+w_3)$$

Now that we've got addition and the negative of a vector out of the way we can do subtraction.

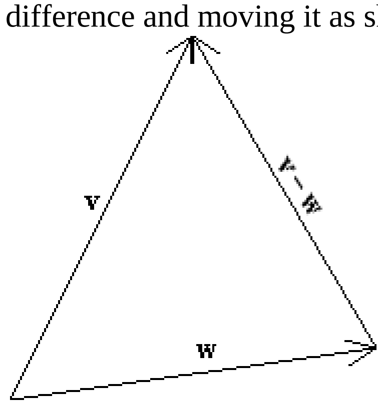
Definition 2 Suppose that we have two vectors **v** and **w** then the difference of **w** from **v**, denoted by $\mathbf{v}-\mathbf{w}$ is defined to be,

$$\mathbf{v}-\mathbf{w}=\mathbf{v}+\left(-\mathbf{w}\right)$$

If we make a sketch, in 2-space, for the summation form of the difference we the following sketch.



Now, while this sketch shows us what the vector for the difference is as a summation we generally like to have a sketch that relates to the two original vectors and not one of the vectors and the negative of the other. We can do this by recalling that any two vectors are equal if the have the same magnitude and direction. Upon recalling this we can pick up the vector representing the difference and moving it as show below.



Finally, if we were to go back to the original sketch add in components for the vectors we will see that in 2-space we can compute the difference as follows,

$$\mathbf{v}-\mathbf{w}=\left(v_1-w_1,v_2-w_2\right)$$

and if the vectors are in 3-space the difference is,

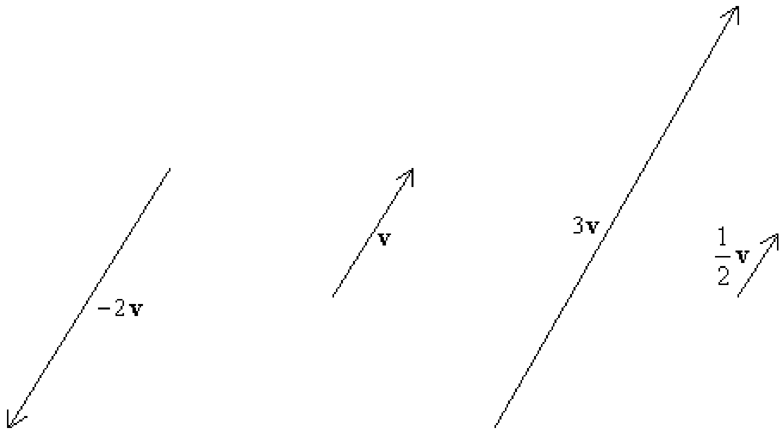
$$\mathbf{v}-\mathbf{w}=\left(v_1-w_1,v_2-w_2,v_3-w_3\right)$$

Note that both addition and subtraction will extend naturally to more than two vectors.

The final arithmetic operation that we want to take a look at is scalar multiples.

Definition 3 Suppose that **v** is a vector and c is a non-zero scalar (*i.e.* c is a number) then the **scalar multiple**, $c\mathbf{v}$, is the vector whose length is $|c|$ times the length of **v** and is in the direction of **v** if c is positive and in the opposite direction of **v** is c is negative.

Here is a sketch of some scalar multiples of a vector **v**.



Note that we can see from this that scalar multiples are parallel. In fact it can be shown that if **v** and **w** are two parallel vectors then there is a non-zero scalar c such that $\mathbf{v} = c\mathbf{w}$, or in other words the two vectors will be scalar multiples of each other.

It can also be shown that if **v** is a vector in either 2-space or 3-space then the scalar multiple can be computed as follows,

$$c\mathbf{v} = (cv_1, cv_2) \qquad \text{OR} \qquad c\mathbf{v} = (cv_1, cv_2, cv_3)$$

At this point we can give a formula for the negative of a vector. Let's examine the scalar multiple, $(-1)\mathbf{v}$. This is a vector whose length is the same as **v** since $|-1| = 1$ and is in the opposite direction of **v** since the scalar is negative. Hence this vector represents the negative of **v**. In 3-space this gives,

$$-\mathbf{v} = (-1)\mathbf{v} = (-v_1, -v_2, -v_3)$$

and in 2-space we'll have,

$$-\mathbf{v} = (-1)\mathbf{v} = (-v_1, -v_2)$$

Before we move on to an example let's get some properties of vector arithmetic written down.

Theorem 1 If **u**, **v**, and **w** are vectors in 2-space or 3-space and c and k are scalars then,

(a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(b) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

(c) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(d) $\mathbf{u}-\mathbf{u}=\mathbf{u}+(-\mathbf{u})=\mathbf{0}$
(e) $1\mathbf{u}=\mathbf{u}$
(f) $(ck)\mathbf{u}=c(k\mathbf{u})=k(c\mathbf{u})$
(g) $(c+k)\mathbf{u}=c\mathbf{u}+k\mathbf{u}$
(h) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

The proof of all these comes directly from the component definition of the operations and so are left to you to verify.

At this point we should probably do a couple of examples of vector arithmetic to say that we've done that.

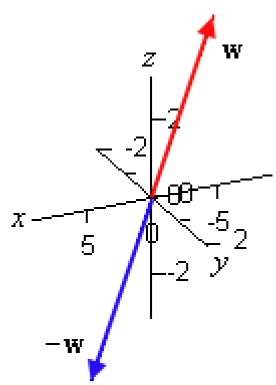
Example 2 Given the following vectors compute the indicated quantity.

$$\mathbf{a}=(4,-6) \qquad \mathbf{b}=(-3,-7) \qquad \mathbf{c}=(-1,5)$$
$$\mathbf{u}=(1,-2,6) \qquad \mathbf{v}=(0,4,-1) \qquad \mathbf{w}=(9,2,-3)$$

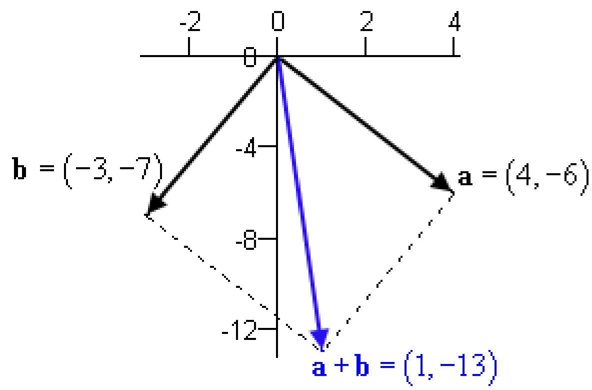
(a) $-\mathbf{w}$
(b) $\mathbf{a}+\mathbf{b}$
(c) $\mathbf{a}-\mathbf{c}$
(d) $\mathbf{a}-3\mathbf{b}+10\mathbf{c}$
(e) $4\mathbf{u}+\mathbf{v}-2\mathbf{w}$

Solution
There really isn't too much to these other than to compute the scalar multiples and the do the addition and/or subtraction. For the first three we'll include sketches so you can visualize what's going on with each operation.

(a)
$$-\mathbf{w}=(-9,-2,3)$$
Here is a sketch of this vector as well as **w**.



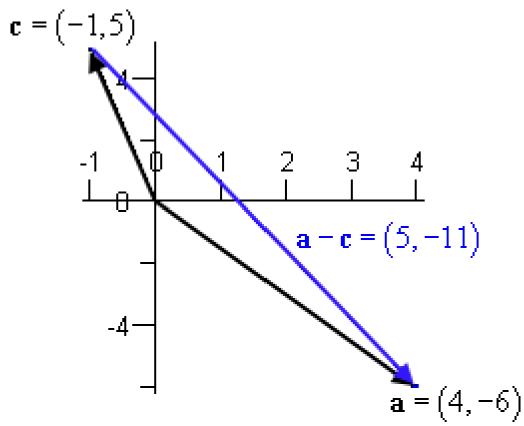
(b)
$$\mathbf{a}+\mathbf{b}=(4+(-3),-6+(-7))=(1,-13)$$
Here is a sketch of **a** and **b** as well as the sum.



(c)

$$\mathbf{a}-\mathbf{c}=\left(4-\left(-1\right),-6-5\right)=\left(5,-11\right)$$

Here is a sketch of **a** and **c** as well as the difference



(d)

$$\mathbf{a}-3\mathbf{b}+10\mathbf{c}=\left(4,-6\right)-\left(-9,-21\right)+\left(-10+90\right)=\left(3,105\right)$$

(e)

$$4\mathbf{u}+\mathbf{v}-2\mathbf{w}=\left(4,-8,24\right)+\left(0,4,-1\right)-\left(18,4,-6\right)=\left(-14,-8,29\right)$$

There is one final topic that we need to discus in this section. We are often interested in the length or magnitude of a vector so we've got a name and notation to use when we're talking about the magnitude of a vector.

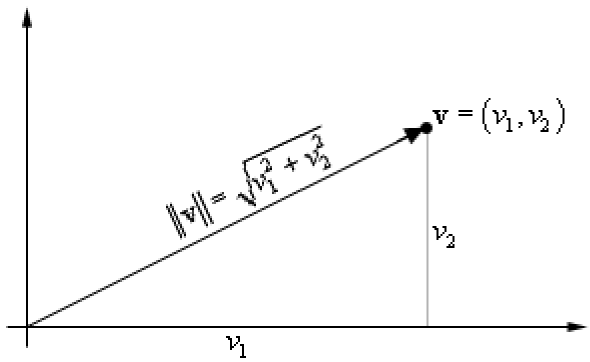
Definition 4 If **v** is a vector then the magnitude of the vector is called the **norm** of the vector and denoted by $\left\|\mathbf{v}\right\|$. Furthermore, if **v** is a vector in 2-space then,

$$\left\|\mathbf{v}\right\|=\sqrt{v_1^2+v_2^2}$$

and if **v** is in 3-space we have,

$$\left\|\mathbf{v}\right\|=\sqrt{v_1^2+v_2^2+v_3^2}$$

In the 2-space case the formula is fairly easy to see from a geometric perspective. Let's suppose that we have $\mathbf{v} = (v_1, v_2)$ and we want to find the magnitude (or length) of this vector. Let's consider the following sketch of the vector.



Since we know that the components of **v** are also the coordinates of the terminal point of the vector when its initial point is the origin (as it is here) we know then the lengths of the sides of a right triangle as shown. Then using the Pythagorean Theorem we can find the length of the hypotenuse, but that is also the length of the vector. A similar argument can be done on the 3-space version.

From above we know that $c\mathbf{v}$ is a scalar multiple of **v** and that its length is $|c|$ times the length of **v** and so we have,

$$\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$$

We can also get this from the definition of the norm. Here is the 3-space case, the 2-space argument is identical.

$$\|c\mathbf{v}\| = \sqrt{(cv_1)^2 + (cv_2)^2 + (cv_3)^2} = \sqrt{c^2\left(v_1^2 + v_2^2 + v_3^2\right)} = |c|\sqrt{v_1^2 + v_2^2 + v_3^2} = |c|\,\|\mathbf{v}\|$$

There is one norm that we'll be particularly interested in on occasion. Suppose **v** is a vector in 2-space or 3-space. We call **v** a **unit vector** if $\|\mathbf{v}\| = 1$.

Let's compute a couple of norms.

Example 3 Compute the norms of the given vectors.

(a) $\mathbf{v} = (-5, 3, 9)$

(b) $\mathbf{j} = (0, 1, 0)$

(c) $\mathbf{w} = (3, -4)$ and $\frac{1}{5}\mathbf{w}$

Solution

Not much to do with these other than to use the formula.

(a) $\|\mathbf{v}\| = \sqrt{(-5)^2 + 3^2 + 9^2} = \sqrt{115}$

(b) $\|\mathbf{j}\| = \sqrt{0^2 + 1^2 + 0^2} = \sqrt{1} = 1$, so **j** is a unit vector!

(c) Okay with this one we've got two norms to compute. Here is the first one.

$$\|\mathbf{w}\| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$$

To get the second we'll first need,

$$\frac{1}{5}\mathbf{w} = \left(\frac{3}{5}, -\frac{4}{5}\right)$$

and here is the norm using the fact that $\|c\mathbf{v}\| = |c|\;\|\mathbf{v}\|$.

$$\left\|\frac{1}{5}\mathbf{w}\right\| = \frac{1}{5}\|\mathbf{w}\| = \left(\frac{1}{5}\right)(5) = 1$$

As a check let's also compute this using the formula for the norm.

$$\left\|\frac{1}{5}\mathbf{w}\right\| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = 1$$

Both methods get the same answer as they should. Notice as well that **w** is not a unit vector but $\frac{1}{5}\mathbf{w}$ is a unit vector.

We now need to take a look at a couple facts about the norm of a vector.

Theorem 2 Given a vector **v** in 2-space or 3-space then $\|\mathbf{v}\| \geq 0$. Also, $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

Proof : The proof of the first part of this comes directly from the definition of the norm. The norm is defined to be a square root and by convention the value of a square root is always greater than or equal to zero and so a norm will always be greater than or equal to zero.

Now, for the second part, recall that when we say "if and only if" in a theorem statement we're saying that this is kind of a two way street. This statement is saying that if $\|\mathbf{v}\| = 0$ then we must also have $\mathbf{v} = \mathbf{0}$ and in the reverse it's also saying that if $\mathbf{v} = \mathbf{0}$ then we must also have $\|\mathbf{v}\| = 0$. To prove this we need to make each assumption and then prove that this will imply the other portion of the statement.

We're only going to show the proof for the case where **v** is in 2-space. The proof for in 3-space is identical. So, assume that $\mathbf{v} = (v_1, v_2)$ and let's start the proof by assuming that $\|\mathbf{v}\| = 0$. Plugging into the formula for the norm gives,

$$0 = \sqrt{v_1^2 + v_2^2} \qquad \Rightarrow \qquad v_1^2 + v_2^2 = 0$$

As shown, the only way we'll get zero out of a square root is if the quantity under the radical is zero. Now at this point we've got a sum of squares equaling zero. The only way this will happen is if the individual terms are zero. So, this means that,

$$v_1 = 0 \quad \& \quad v_2 = 0 \qquad \Rightarrow \qquad \mathbf{v} = (0,0) = \mathbf{0}$$

So, if $\|\mathbf{v}\| = 0$ we must have $\mathbf{v} = \mathbf{0}$.

Next, let's assume that $\mathbf{v} = \mathbf{0}$. In this case simply plug the components into the formula for the norm and a quick computation will show that $\|\mathbf{v}\| = 0$ and so we're done.

■

Theorem 3 Given a non-zero vector **v** in 2-space or 3-space define a new vector $\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$, then **u** is a unit vector.

Proof : This is a really simple proof, just notice that **u** is a scalar multiple of **v** and take the norm of **u**.

$$\|\mathbf{u}\| = \left\| \frac{1}{\|\mathbf{v}\|}\mathbf{v} \right\|$$

Now we know that $\|\mathbf{v}\| > 0$ because norms are always greater than or equal to zero, but will only be zero if we have the zero vector. In this case we've explicitly assumed that we don't have the zero vector and so we now the norm will be strictly positive and this will allow us to drop the absolute value bars on the norm when we do the computation.

We can now do the following,

$$\|\mathbf{u}\| = \left\| \frac{1}{\|\mathbf{v}\|}\mathbf{v} \right\| = \left| \frac{1}{\|\mathbf{v}\|} \right| \|\mathbf{v}\| = \frac{1}{\|\mathbf{v}\|} \|\mathbf{v}\| = 1$$

So, **u** is a unit vector.

■

This theorem tells us that we can always turn a non-zero vector into a unit vector simply be dividing by the norm. Note as well that because all we're doing to compute this new unit vector is scalar multiplication by a positive number this new unit vector will point in the same direction as the original vector.

Example 4 Given $\mathbf{v} = (3,-1,-2)$ find a unit vector that,

(a) points in the same direction as **v**

(b) points in the opposite direction as **v**

Solution

(a) Now, as pointed out after the proof of the previous theorem, the unit vector computed in the theorem will point in the same direction as **v** so all we need to do is compute the norm of **v** and then use the theorem to find a unit vector that will point in the same direction as **v**.

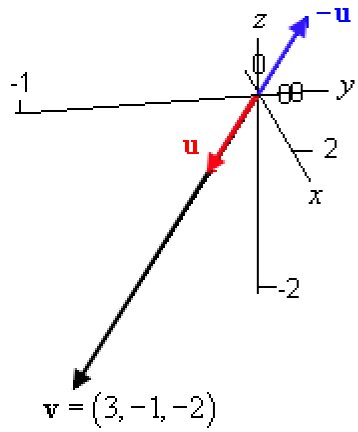
$$\|\mathbf{v}\| = \sqrt{3^2 + (-1)^2 + (-2)^2} = \sqrt{14}$$

$$\mathbf{u} = \frac{1}{\sqrt{14}}(3,-1,-2) = \left(\frac{3}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}\right)$$

(b) We've done most of the work for this one. Since **u** is a unit vector that points in the same direction as **v** then its negative will be a unit vector that points in the opposite directions as **v**. So, here is the negative of **u**.

$$-\mathbf{u} = \left(-\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}\right)$$

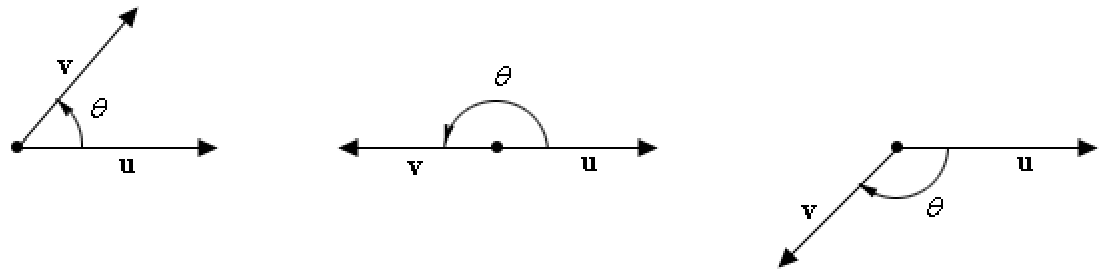
Finally, here is a sketch of all three of these vectors.



Dot Product & Cross Product

In this section we're going to be taking a look at two special products of vectors, the dot product and the cross product. However, before we look at either on of them we need to get a quick definition out of the way.

Suppose that **u** and **v** are two vectors in 2-space or 3-space that are placed so that their initial points are the same. Then the **angle between u and v** is angle θ that is formed by **u** and **v** such that $0 \le \theta \le \pi$. Below are some examples of angles between vectors.



Notice that there are always two angles that are formed by the two vectors and the one that we will always chose is the one that satisfies $0 \le \theta \le \pi$. We'll be using this angle with both products.

So, let's get started by taking a look at the dot product. Of the two products we'll be looking at in this section this is the one we're going to run across most often in later sections. We'll start with the definition.

Definition 1 If **u** and **v** are two vectors in 2-space or 3-space and θ is the angle between them then the **dot product**, denoted by $\mathbf{u}\bullet\mathbf{v}$ is defined as,

$$\mathbf{u}\bullet\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$

Note that the dot product is sometimes called the **scalar product** or the **Euclidean inner product**. Let's see a quick example or two of the dot product.

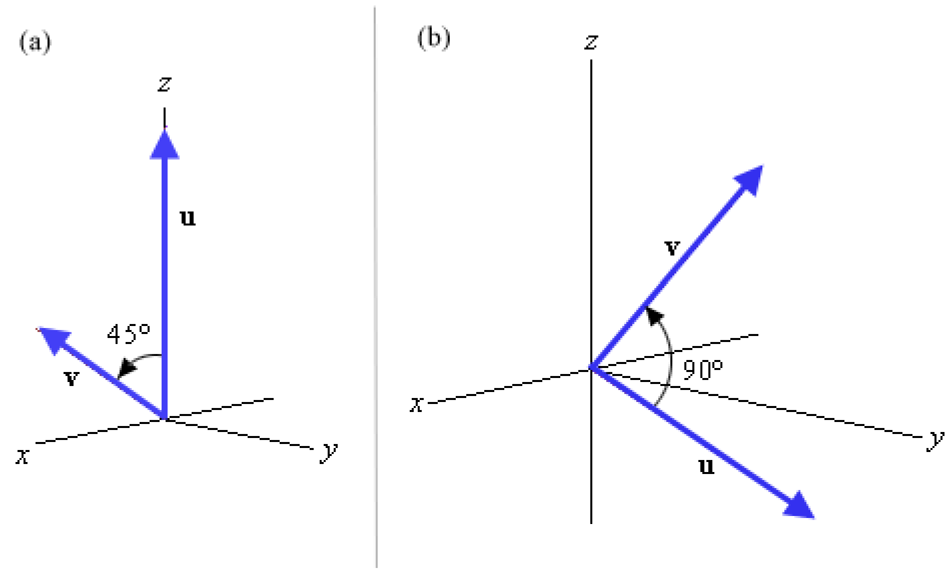
Example 1 Compute the dot product for the following pairs of vectors.

(a) $\mathbf{u} = (0,0,3)$ and $\mathbf{v} = (2,0,2)$ which makes the angle between them $45°$.

(b) $\mathbf{u} = (0,2,-1)$ and $\mathbf{v} = (-1,1,2)$ which makes the angle between them $90°$.

Solution

For reference purposes here is a sketch of the two sets of vectors.



(a) There really isn't too much to do here with this problem.

$$\|\mathbf{u}\| = \sqrt{0+0+9} = 3 \qquad\qquad \|\mathbf{v}\| = \sqrt{4+0+4} = \sqrt{8} = 2\sqrt{2}$$

$$\mathbf{u}\bullet\mathbf{v} = (3)\left(2\sqrt{2}\right)\cos(45) = 6\sqrt{2}\left(\frac{\sqrt{2}}{2}\right) = 6$$

(b) Nor is there a lot of work to do here.

$$\|\mathbf{u}\| = \sqrt{0+4+1} = \sqrt{5} \qquad \|\mathbf{v}\| = \sqrt{1+1+4} = \sqrt{6}$$
$$\mathbf{u}\bullet\mathbf{v} = \left(\sqrt{5}\right)\left(\sqrt{6}\right)\cos\left(90\right) = \sqrt{30}\left(0\right) = 0$$

Now, there should be a question in everyone's mind at this point. Just how did we arrive at those angles above? They are the correct angles, but just how did we get them? That is the problem with this definition of the dot product. If you don't have the angles between two vectors you can't easily compute the dot product and sometimes finding the correct angles is not the easiest thing to do.

Fortunately, there is another formula that we can use to compute the formula that relies only on the components of the vectors and not the angle between them.

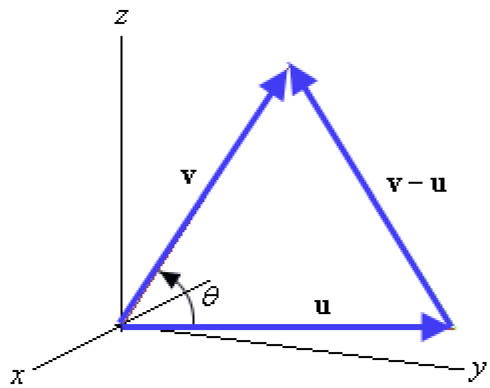
Theorem 1 Suppose that $\mathbf{u} = \left(u_1, u_2, u_3\right)$ and $\mathbf{v} = \left(v_1, v_2, v_3\right)$ are two vectors in 3-space then,

$$\mathbf{u}\bullet\mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Likewise, if $\mathbf{u} = \left(u_1, u_2\right)$ and $\mathbf{v} = \left(v_1, v_2\right)$ are two vectors in 2-space then,

$$\mathbf{u}\bullet\mathbf{v} = u_1 v_1 + u_2 v_2$$

Proof : We'll just prove the 3-space version of this theorem. The 2-space version has a similar proof. Let's start out with the following figure.



So, these three vectors form a triangle and the lengths of each side is $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and $\|\mathbf{v}-\mathbf{u}\|$. Now, from the Law of Cosines we know that,

$$\|\mathbf{v}-\mathbf{u}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 - 2\|\mathbf{v}\|\,\|\mathbf{u}\|\cos\theta$$

Now, plug in the definition of the dot product and solve for $\mathbf{u}\bullet\mathbf{v}$.

$$\|\mathbf{v}-\mathbf{u}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 - 2\left(\mathbf{u}\bullet\mathbf{v}\right)$$

$$\mathbf{u}\bullet\mathbf{v} = \frac{1}{2}\left(\|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 - \|\mathbf{v}-\mathbf{u}\|^2\right) \qquad \textbf{(1)}$$

Next, we know that $\mathbf{v}-\mathbf{u}=\left(v_1-u_1,v_2-u_2,v_3-u_3\right)$ and so we can compute $\left\|\mathbf{v}-\mathbf{u}\right\|^2$. Note as well that because of the square on the norm we won't have a square root. We'll also do all of the multiplications.

$$\begin{aligned}\left\|\mathbf{v}-\mathbf{u}\right\|^2&=\left(v_1-u_1\right)^2+\left(v_2-u_2\right)^2+\left(v_3-u_3\right)^2\\&=v_1^2-2v_1u_1+u_1^2+v_2^2-2v_2u_2+u_2^2+v_3^2-2v_3u_3+u_3^2\\&=v_1^2+v_2^2+v_3^2+u_1^2+u_2^2+u_3^2-2\left(v_1u_1+v_2u_2+v_3u_3\right)\end{aligned}$$

The first three terms of this are nothing more than the formula for $\left\|\mathbf{v}\right\|^2$ and the next three terms are the formula for $\left\|\mathbf{u}\right\|^2$. So, let's plug this into (1).

$$\begin{aligned}\mathbf{u}\bullet\mathbf{v}&=\frac{1}{2}\left(\left\|\mathbf{v}\right\|^2+\left\|\mathbf{u}\right\|^2-\left(\left\|\mathbf{v}\right\|^2+\left\|\mathbf{u}\right\|^2-2\left(v_1u_1+v_2u_2+v_3u_3\right)\right)\right)\\&=\frac{1}{2}\left(2\left(v_1u_1+v_2u_2+v_3u_3\right)\right)\\&=v_1u_1+v_2u_2+v_3u_3\end{aligned}$$

And we're done with the proof.

■

Before we work an example using this new (easier to use) formula let's notice that if we rewrite the definition of the dot product as follows,

$$\cos\theta=\frac{\mathbf{u}\bullet\mathbf{v}}{\left\|\mathbf{u}\right\|\,\left\|\mathbf{v}\right\|},\qquad 0\le\theta\le\pi$$

we now have a very easy way to determine the angle between any two vectors. In fact this is how we got the angles between the vectors in the first example!

Example 2 Determine the angle between the following pairs of vectors.

(a) $\mathbf{a}=\left(9,-2\right)\quad\mathbf{b}=\left(4,18\right)$

(b) $\mathbf{u}=\left(3,-1,6\right)\quad\mathbf{v}=\left(4,2,0\right)$

Solution

(a) Here are all the important quantities for this problem.

$$\left\|\mathbf{a}\right\|=\sqrt{85}\qquad\left\|\mathbf{b}\right\|=\sqrt{340}\qquad\mathbf{a}\bullet\mathbf{b}=\left(9\right)\left(4\right)+\left(-2\right)\left(18\right)=0$$

The angle is then,

$$\cos\theta=\frac{0}{\sqrt{85}\,\sqrt{340}}=0\qquad\Rightarrow\qquad\theta=90^\circ$$

(b) The important quantities for this part are,

$$\left\|\mathbf{u}\right\|=\sqrt{46}\qquad\left\|\mathbf{v}\right\|=\sqrt{20}\qquad\mathbf{u}\bullet\mathbf{v}=\left(3\right)\left(4\right)+\left(-1\right)\left(2\right)+\left(6\right)\left(0\right)=10$$

The angle is then,

$$\cos\theta=\frac{10}{\sqrt{46}\,\sqrt{20}}=0.3296902\qquad\Rightarrow\qquad\theta=70.75^\circ$$

Note that we did need to use a calculator to get this result.

Twice now we've seen two vectors whose dot product is zero and in both cases we've seen that the angle between them was $90°$ and so the two vectors in question each time where perpendicular. Perpendicular vectors are called **orthogonal** and as we'll see on occasion we often want to know if two vectors are orthogonal. The following theorem will give us a nice check for this.

Theorem 2 Two non-zero vectors, **u** and **v**, are orthogonal if and only if $\mathbf{u}\bullet\mathbf{v} = 0$.

Proof :
First suppose that **u** and **v** are orthogonal. This means that the angle between them is $90°$ and so from the definition of the dot product we have,

$$\mathbf{u}\bullet\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos(90) = \|\mathbf{u}\|\,\|\mathbf{v}\|(0) = 0$$

and so we have $\mathbf{u}\bullet\mathbf{v} = 0$.

Next suppose that $\mathbf{u}\bullet\mathbf{v} = 0$, then from the definition of the dot product we have,

$$0 = \mathbf{u}\bullet\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta \qquad \Rightarrow \qquad \cos\theta = 0 \qquad \Rightarrow \qquad \theta = 90°$$

and so the two vectors are orthogonal.

Note that we used the fact that the two vectors are non-zero, and hence would have non-zero magnitudes, in determining that we must have $\cos\theta = 0$.

■

If we take the convention that the zero vector is orthogonal to any other vector we can say that for any two vectors **u** and **v** they will be orthogonal provided $\mathbf{u}\bullet\mathbf{v} = 0$. Using this convention means we don't need to worry about whether or not we have zero vectors.

Here are some nice properties about the dot product.

Theorem 3 Suppose that **u, v**, and **w** are three vectors that are all in 2-space or all in 3-space and that *c* is a scalar. Then,

(a) $\mathbf{v}\bullet\mathbf{v} = \|\mathbf{v}\|^2$ (this implies that $\|\mathbf{v}\| = (\mathbf{v}\bullet\mathbf{v})^{\frac{1}{2}}$)

(b) $\mathbf{u}\bullet\mathbf{v} = \mathbf{v}\bullet\mathbf{u}$

(c) $\mathbf{u}\bullet(\mathbf{v}+\mathbf{w}) = \mathbf{u}\bullet\mathbf{v} + \mathbf{u}\bullet\mathbf{w}$

(d) $c(\mathbf{u}\bullet\mathbf{v}) = (c\mathbf{u})\bullet\mathbf{v} = \mathbf{u}\bullet(c\mathbf{v})$

(e) $\mathbf{v}\bullet\mathbf{v} > 0$ if $\mathbf{v} \neq \mathbf{0}$

(f) $\mathbf{v}\bullet\mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$

We'll prove the first couple and leave the rest to you to prove since the follow pretty much from either the definition of the dot product or the formula from Theorem 2. The proof of the last one is nearly identical to the proof of Theorem 2 in the previous section.

Proof :
(a) The angle between **v** and **v** is 0 since they are the same vector and so by the definition of the dot product we've got.

$$\mathbf{v}\bullet\mathbf{v} = \|\mathbf{v}\|\,\|\mathbf{v}\|\cos(0) = \|\mathbf{v}\|^2$$

To get the second part just take the square root of both sides.

(b) This proof is going to seem tricky but it's really not that bad. Let's just look at the 3-space case. So, $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ and the dot product $\mathbf{u}\bullet\mathbf{v}$ is

$$\mathbf{u}\bullet\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$$

We can also compute $\mathbf{v}\bullet\mathbf{u}$ as follows,

$$\mathbf{v}\bullet\mathbf{u} = v_1u_1 + v_2u_2 + v_3u_3$$

However, since $u_1v_1 = v_1u_1$, *etc*. (they are just real numbers after all) these are identical and so we've got $\mathbf{u}\bullet\mathbf{v} = \mathbf{v}\bullet\mathbf{u}$.

■

Example 3 Given $\mathbf{u} = (5, -2)$, $\mathbf{v} = (0, 7)$ and $\mathbf{w} = (4, 10)$ compute the following.

(a) $\mathbf{u}\bullet\mathbf{u}$ and $\|\mathbf{u}\|^2$

(b) $\mathbf{u}\bullet\mathbf{w}$

(c) $(-2\mathbf{u})\bullet\mathbf{v}$ and $\mathbf{u}\bullet(-2\mathbf{v})$

Solution
(a) Okay, in this one we'll be verifying part **(a)** of the previous theorem. Note as well that because the norm is squared we'll not need to have the square root in the computation. Here are the computations for this part.

$$\mathbf{u}\bullet\mathbf{u} = (5)(5) + (-2)(-2) = 25 + 4 = 29$$

$$\|\mathbf{u}\|^2 = 5^2 + (-2)^2 = 29$$

So, as the theorem suggested we do have $\mathbf{u}\bullet\mathbf{u} = \|\mathbf{u}\|^2$.

(b) Here's the dot product for this part.

$$\mathbf{u}\bullet\mathbf{w} = (5)(4) + (-2)(10) = 0$$

So, it looks like **u** and **w** are orthogonal.

(c) In this part we'll be verifying part **(d)** of the previous theorem. Here are the computations for this part.

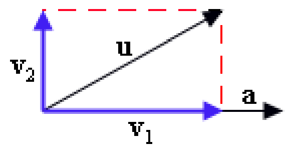
$$-2\mathbf{u} = (-10, 4) \qquad\qquad -2\mathbf{v} = (0, -14)$$

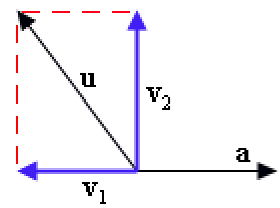
$$(-2\mathbf{u})\bullet\mathbf{v} = (-10)(0) + (4)(7) = 28$$

$$\mathbf{u}\bullet(-2\mathbf{v}) = (5)(0) + (-2)(-14) = 28$$

Again, we got the result that we should expect .

We now need to take a look at a very important application of the dot product. Let's suppose that **u** and **a** are two vectors in 2-space or 3-space and let's suppose that they are positioned so that their initial points are the same. What we want to do is "decompose" the vector **u** into two components. One, which we'll denote $\mathbf{v}_1$ for now, will be parallel to the vector **a** and the other, denoted $\mathbf{v}_2$ for now, will be orthogonal to **a**. See the image below to see some examples of kind of decomposition.





From these figures we can see how to actually construct the two pieces of our decomposition. Starting at **u** we drop a line straight down until it intersects **a** (or the line defined by **a** as in the second case). The parallel vector $\mathbf{v}_1$ is then the vector that starts at the initial point of **u** and end there the perpendicular line intersects **a**. Finding $\mathbf{v}_2$ is actually really simple provided we first have $\mathbf{v}_1$. From the image we can see that we have,

$$\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{u} \qquad \Rightarrow \qquad \mathbf{v}_2 = \mathbf{u} - \mathbf{v}_1$$

We now need to get some terminology and notation out of the way. The parallel vector, $\mathbf{v}_1$, is called the **orthogonal projection of u on a** and is denoted by $\mathrm{proj}_{\mathbf{a}}\,\mathbf{u}$. Note that sometimes $\mathrm{proj}_{\mathbf{a}}\,\mathbf{u}$ is called the **vector component of u along a**. The orthogonal vector, $\mathbf{v}_2$, is called the **vector component of u orthogonal to a**.

The following theorem gives us formulas for computing both of these vectors.

Theorem 4 Suppose that **u** and $\mathbf{a} \neq \mathbf{0}$ are both vectors in 2-space or 3-space then,

$$\mathrm{proj}_{\mathbf{a}}\,\mathbf{u} = \frac{\mathbf{u}\bullet\mathbf{a}}{\left\|\mathbf{a}\right\|^2}\mathbf{a}$$

and the vector component of **u** orthogonal to **a** is given by,

$$\mathbf{u} - \mathrm{proj}_{\mathbf{a}}\,\mathbf{u} = \mathbf{u} - \frac{\mathbf{u}\bullet\mathbf{a}}{\left\|\mathbf{a}\right\|^2}\mathbf{a}$$

Proof : First let $\mathbf{v}_1 = \mathrm{proj}_{\mathbf{a}}\,\mathbf{u}$ then $\mathbf{u} - \mathrm{proj}_{\mathbf{a}}\,\mathbf{u}$ will be the vector component of **u** orthogonal to **a** and so all we need to do is show the formula for $\mathbf{v}_1$ is what we claimed it to be.

To do this let's first note that since $\mathbf{v}_1$ is parallel to **a** then it must be a scalar multiple of **a** since we know from last section that parallel vectors are scalar multiples of each other. Therefore there is a scalar c such that $\mathbf{v}_1 = c\,\mathbf{a}$. Now, let's start with the following,

$$\mathbf{u} = \mathbf{v}_1 + \mathbf{v}_2 = c\,\mathbf{a} + \mathbf{v}_2$$

Next take the dot product of both sides with **a** and distribute the dot product through the parenthesis.

$$\begin{aligned}\mathbf{u}\bullet\mathbf{a} &= \left(c\,\mathbf{a} + \mathbf{v}_2\right)\bullet\mathbf{a} \\ &= c\,\mathbf{a}\bullet\mathbf{a} + \mathbf{v}_2\bullet\mathbf{a}\end{aligned}$$

Now, $\mathbf{a}\bullet\mathbf{a} = \left\|\mathbf{a}\right\|^2$ and $\mathbf{v}_2\bullet\mathbf{a} = 0$ because $\mathbf{v}_2$ is orthogonal to **a**. Therefore this reduces to,

$$\mathbf{u}\bullet\mathbf{a} = c\left\|\mathbf{a}\right\|^2 \qquad \Rightarrow \qquad c = \frac{\mathbf{u}\bullet\mathbf{a}}{\left\|\mathbf{a}\right\|^2}$$

and so we get,

$$\mathbf{v}_1 = \operatorname{proj}_{\mathbf{a}} \mathbf{u} = \frac{\mathbf{u}\bullet\mathbf{a}}{\left\|\mathbf{a}\right\|^2}\mathbf{a}$$

■

We can also take a quick norm of $\operatorname{proj}_{\mathbf{a}} \mathbf{u}$ to get a nice formula for the magnitude of the orthogonal projection of **u** on **a**.

$$\left\|\operatorname{proj}_{\mathbf{a}} \mathbf{u}\right\| = \left\|\frac{\mathbf{u}\bullet\mathbf{a}}{\left\|\mathbf{a}\right\|^2}\mathbf{a}\right\| = \left|\frac{\mathbf{u}\bullet\mathbf{a}}{\left\|\mathbf{a}\right\|^2}\right| \left\|\mathbf{a}\right\| = \frac{\left|\mathbf{u}\bullet\mathbf{a}\right|}{\left\|\mathbf{a}\right\|}$$

Let's work a quick example or two of orthogonal projections.

Example 4 Compute the orthogonal projection of **u** on **a** and the vector component of **u** orthogonal to **a** for each of the following.

(a) $\mathbf{u} = \left(-3,1\right) \quad \mathbf{a} = \left(7,2\right)$

(b) $\mathbf{u} = \left(4,0,-1\right) \quad \mathbf{a} = \left(3,1,-5\right)$

Solution

There really isn't much to do here other than to plug into the formulas so we'll leave it to you to verify the details.

(a) First,

$$\mathbf{u}\bullet\mathbf{a} = -19 \qquad \left\|\mathbf{a}\right\|^2 = 53$$

Now the orthogonal projection of **u** on **a** is,

$$\operatorname{proj}_{\mathbf{a}} \mathbf{u} = \frac{-19}{53}\left(7,2\right) = \left(-\frac{133}{53}, -\frac{38}{53}\right)$$

and the vector component of **u** orthogonal to **a** is,

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\,\mathbf{u} = (-3,1) - \left(-\frac{133}{53}, -\frac{38}{53}\right) = \left(-\frac{26}{53}, \frac{91}{53}\right)$$

(b) First,

$$\mathbf{u}\bullet\mathbf{a} = 17 \qquad\qquad \|\mathbf{a}\|^2 = 35$$

Now the orthogonal projection of **u** on **a** is,

$$\text{proj}_{\mathbf{a}}\,\mathbf{u} = \frac{17}{35}(3,1,-5) = \left(\frac{51}{35}, \frac{17}{35}, -\frac{17}{7}\right)$$

and the vector component of **u** orthogonal to **a** is,

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\,\mathbf{u} = (4,0,-1) - \left(\frac{51}{35}, \frac{17}{35}, -\frac{17}{7}\right) = \left(\frac{89}{35}, -\frac{17}{35}, \frac{10}{7}\right)$$

We need to be very careful with the notation $\text{proj}_{\mathbf{a}}\,\mathbf{u}$. In this notation we are looking for the orthogonal projection of **u** (the second vector listed) on **a** (the vector that is subscripted). Let's do a quick example illustrating this.

Example 5 Given $\mathbf{u} = (4,-5)$ and $\mathbf{a} = (1,-1)$ compute,

(a) $\text{proj}_{\mathbf{a}}\,\mathbf{u}$

(b) $\text{proj}_{\mathbf{u}}\,\mathbf{a}$

Solution

(a) In this case we are looking for the component of **u** that is parallel to **a** and so the orthogonal projection is given by,

$$\text{proj}_{\mathbf{a}}\,\mathbf{u} = \frac{\mathbf{u}\bullet\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$$

so let's get all the quantities that we need.

$$\mathbf{u}\bullet\mathbf{a} = (4)(1) + (-5)(-1) = 9 \qquad\qquad \|\mathbf{a}\|^2 = (1)^2 + (-1)^2 = 2$$

The projection is then,

$$\text{proj}_{\mathbf{a}}\,\mathbf{u} = \frac{9}{2}(1,-1) = \left(\frac{9}{2}, -\frac{9}{2}\right)$$

(b) Here we are looking for the component of **a** that is parallel to **u** and so the orthogonal projection is given by,

$$\text{proj}_{\mathbf{u}}\,\mathbf{a} = \frac{\mathbf{a}\bullet\mathbf{u}}{\|\mathbf{u}\|^2}\mathbf{u}$$

so let's get the quantities that we need for this part.

$$\mathbf{a}\bullet\mathbf{u} = \mathbf{u}\bullet\mathbf{a} = 9 \qquad\qquad \|\mathbf{u}\|^2 = (4)^2 + (-5)^2 = 31$$

The projection is then,

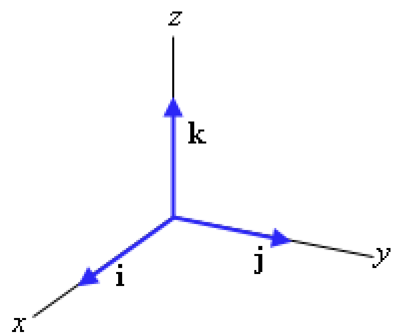
$$\text{proj}_{\mathbf{u}}\,\mathbf{a} = \frac{9}{31}(4,-5) = \left(\frac{36}{31}, -\frac{45}{31}\right)$$

As this example has shown we need to pay attention to the placement of the two vectors in the projection notation. Each part above was asking for something different and as shown we did in fact get different answers so be careful.

It's now time to move into the second vector product that we're going to look at in this section. However before we do that we need to introduce the idea of the **standard unit vectors** or **standard basis vectors** for 3-space. These vectors are defined as follows,

$$\mathbf{i} = (1,0,0) \qquad\qquad \mathbf{j} = (0,1,0) \qquad\qquad \mathbf{k} = (0,0,1)$$

Each of these have a magnitude of 1 and so are unit vectors. Also note that each one lies along the coordinate axes of 3-space and point in the positive direction as shown below.



Notice that any vector in 3-space, say $\mathbf{u} = (u_1, u_2, u_3)$, can be written in terms of these three vectors as follows,

$$\begin{aligned} \mathbf{u} &= (u_1, u_2, u_3) \\ &= (u_1, 0, 0) + (0, u_2, 0) + (0, 0, u_3) \\ &= u_1(1,0,0) + u_2(0,1,0) + u_3(0,0,1) \\ &= u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \end{aligned}$$

So, for example we can do the following,

$$(-10,4,3) = -10\mathbf{i} + 4\mathbf{j} + 3\mathbf{k} \qquad\qquad (-1,0,2) = -\mathbf{i} + 2\mathbf{k}$$

Also note that if we define $\mathbf{i} = (1,0)$ and $\mathbf{j} = (0,1)$ these two vectors are the **standard basis vectors** for 2-space and any vector in 2-space, say $\mathbf{u} = (u_1, u_2)$ can be written as,

$$\mathbf{u} = (u_1, u_2) = u_1\mathbf{i} + u_2\mathbf{j}$$

We're not going to need the 2-space version of things here, but it was worth pointing out that there was a 2-space version since we'll need that down the road.

Okay we are now ready to look at the cross product. The first thing that we need to point out here is that, unlike the dot product, this is only valid in 3-space. There are three different ways of defining it depending on how you want to do it. The following definition gives all three definitions.

Definition 2 If **u** and **v** are two vectors in 3-space then the **cross product**, denoted by $\mathbf{u}\times\mathbf{v}$ and is defined in one of three ways.

(a) $\mathbf{u}\times\mathbf{v}=\left(u_2v_3-u_3v_2,u_3v_1-u_1v_3,u_1v_2-u_2v_1\right)$ - Vector Notation.

(b) $\mathbf{u}\times\mathbf{v}=\left(\begin{vmatrix}u_2 & u_3\\ v_2 & v_3\end{vmatrix},-\begin{vmatrix}u_1 & u_3\\ v_1 & v_3\end{vmatrix},\begin{vmatrix}u_1 & u_2\\ v_1 & v_2\end{vmatrix}\right)$ - Using 2×2 determinants

(c) $\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}$ - Using 3×3 determinants

Note that all three of these definitions are equivalent as you can check be computing the determinants in the second and third definition and verifying that you get the same formula as in the first definition.

Notice that the cross product of two vectors is a new vector unlike the dot product which gives a scalar. Make sure to keep these two products straight.

Let's take a quick look at an example of a cross product.

Example 6 Compute $\mathbf{u}\times\mathbf{v}$ for $\mathbf{u}=\left(4,-9,1\right)$ and $\mathbf{v}=\left(3,-2,7\right)$.

Solution
You can use either of the three definitions above to compute this cross product. We'll use the third one. If you don't remember how to compute determinants you might want to go back and check out the first section of the Determinants chapter. In that section you'll find the formulas for computing determinants of both 2×2 and 3×3 matrices.

$$\mathbf{u}\times\mathbf{v}=\left|\begin{array}{ccc|cc}\mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j}\\ 4 & -9 & 1 & 4 & -9\\ 3 & -2 & 7 & 3 & -2\end{array}\right.$$

$$=-63\mathbf{i}+3\mathbf{j}-8\mathbf{k}-28\mathbf{j}+2\mathbf{i}+27\mathbf{k}=-61\mathbf{i}-25\mathbf{j}+19\mathbf{k}$$

When we're using this definition of the cross product we'll always get the answer in terms of the standard basis vectors. However, we can always go back to the form we're used to. Doing this gives,

$$\mathbf{u}\times\mathbf{v}=\left(-61,-25,19\right)$$

Here is a theorem listing the main properties of the cross product.

Theorem 5 Suppose **u, v**, and **w** are vectors in 3-space and *c* is any scalar then

(a) $\mathbf{u}\times\mathbf{v}=-\left(\mathbf{v}\times\mathbf{u}\right)$

(b) $\mathbf{u}\times\left(\mathbf{v}+\mathbf{w}\right)=\left(\mathbf{u}\times\mathbf{v}\right)+\left(\mathbf{u}\times\mathbf{w}\right)$

(c) $(\mathbf{u}+\mathbf{v})\times\mathbf{w}=(\mathbf{u}\times\mathbf{w})+(\mathbf{v}\times\mathbf{w})$

(d) $c(\mathbf{u}\times\mathbf{v})=(c\,\mathbf{u})\times\mathbf{v}=\mathbf{u}\times(c\,\mathbf{v})$

(e) $\mathbf{u}\times\mathbf{0}=\mathbf{0}\times\mathbf{u}=\mathbf{0}$

(f) $\mathbf{u}\times\mathbf{u}=\mathbf{0}$

The proof of all these properties come directly from the definition of the cross product and so are left to you to verify.

There are also quite a few properties that relate the dot product and the cross product. Here is a theorem giving those properties.

Theorem 6 Suppose **u, v**, and **w** are vectors in 3-space then,

(a) $\mathbf{u}\bullet(\mathbf{u}\times\mathbf{v})=0$

(b) $\mathbf{v}\bullet(\mathbf{u}\times\mathbf{v})=0$

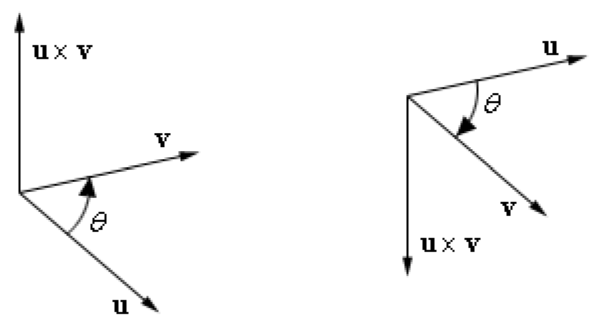
(c) $\|\mathbf{u}\times\mathbf{v}\|^2=\|\mathbf{u}\|^2\,\|\mathbf{v}\|^2-(\mathbf{u}\bullet\mathbf{v})^2$ - This is called **Lagrange's Identity**

(d) $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})=(\mathbf{u}\bullet\mathbf{w})\mathbf{v}-(\mathbf{u}\bullet\mathbf{v})\mathbf{w}$

(e) $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}=(\mathbf{u}\bullet\mathbf{w})\mathbf{v}-(\mathbf{v}\bullet\mathbf{w})\mathbf{u}$

The proof of all these properties come directly from the definition of the cross product and the dot product and so are left to you to verify.

The first two properties deserve some closer inspection. That they are saying is that given two vectors **u** and **v** in 3-space then the cross product $\mathbf{u}\times\mathbf{v}$ is orthogonal to both **u** and **v**. The image below shows this idea.



As this figure shows there are two directions in which the cross product to be orthogonal to **u** and **v** and there is a nice way to determine which it will be. Take your hand and cup your fingers show that they point in the direction of rotation that is shown in the figures (*i.e.* rotate **u** until it lies on top of **v**) and hold your thumb out. Your thumb will point in the direction of the cross product.

Notice that part **(a)** of Theorem 5 above also gives this same result. If we flip the order in which we take the cross product (which is really what we did in the figure above when we interchanged the letters) we get $\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})$. In other words, in one order we get a cross product that points in one direction and if we flip the order we get a new cross product that points in the opposite direction as the first one.

Let's work a couple more cross products to verify some of the properties listed above and so we can say we've got a couple more examples in the notes.

Example 7 Given $\mathbf{u} = (3,-1,4)$ and $\mathbf{v} = (2,0,1)$ compute each of the following.

(a) $\mathbf{u}\times\mathbf{v}$ and $\mathbf{v}\times\mathbf{u}$
(b) $\mathbf{u}\times\mathbf{u}$
(c) $\mathbf{u}\bullet(\mathbf{u}\times\mathbf{v})$ and $\mathbf{v}\bullet(\mathbf{u}\times\mathbf{v})$

Solution
In the solutions to these problems we will be using the third definition above and we'll be setting up the determinant. We will not be showing the determinant computation however, if you need a reminder on how to take determinants go back to the first section in the Determinant chapter for a refresher.

(a) Let's compute $\mathbf{u}\times\mathbf{v}$ first.

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 4 \\ 2 & 0 & 1 \end{vmatrix} = -\mathbf{i} + 5\mathbf{j} + 2\mathbf{k} = (-1,5,2)$$

Remember that we'll get the answers here in terms of the standard basis vectors and these can always be put back into the standard vector notation that we've been using to this point as we did above.

Now let's compute $\mathbf{v}\times\mathbf{u}$.

$$\mathbf{v}\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 1 \\ 3 & -1 & 4 \end{vmatrix} = \mathbf{i} - 5\mathbf{j} - 2\mathbf{k} = (1,-5,-2)$$

So, as part **(a)** of Theorem 5 suggested we got $\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})$.

(b) Not much to do here other than do the cross product and note that part **(f)** of Theorem 5 implies that we should get $\mathbf{u}\times\mathbf{u} = \mathbf{0}$.

$$\mathbf{u}\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 4 \\ 3 & -1 & 4 \end{vmatrix} = (0,0,0)$$

So, sure enough we got **0**.

(c) We've already got $\mathbf{u}\times\mathbf{v}$ computed so we just need to do a couple of dot products and according to Theorem 6 both **u** and **v** are orthogonal to $\mathbf{u}\times\mathbf{v}$ and so we should get zero out of both of these.

$$\mathbf{u} \bullet (\mathbf{u} \times \mathbf{v}) = (3)(-1) + (-1)(5) + (4)(2) = 0$$
$$\mathbf{v} \bullet (\mathbf{u} \times \mathbf{v}) = (2)(-1) + (0)(5) + (1)(2) = 0$$

And we did get zero as expected.

We'll give one theorem on cross products relating the magnitude of the cross product to the magnitudes of the two vectors we're taking the cross product of.

Theorem 7 Suppose that **u** and **v** are vectors in 3-space and let θ be the angle between them then,

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin\theta$$

Let's take a look at one final example here.

Example 8 Given $\mathbf{u} = (1,-1,0)$ and $\mathbf{v} = (0,-2,0)$ verify the results of Theorem 7.

Solution

Let's get the cross product and the norms taken care of first.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 0 & -2 & 0 \end{vmatrix} = (0,0,-2) \qquad \|\mathbf{u} \times \mathbf{v}\| = \sqrt{0+0+4} = 2$$

$$\|\mathbf{u}\| = \sqrt{1+1+0} = \sqrt{2} \qquad \|\mathbf{v}\| = \sqrt{0+4+0} = 2$$

Now, in order to verify Theorem 7 we'll need the angle between the two vectors and we can use the definition of the dot product above to find this. We'll first need the dot product.

$$\mathbf{u} \bullet \mathbf{v} = 2 \Rightarrow \qquad \cos\theta = \frac{2}{(\sqrt{2})(2)} = \frac{1}{\sqrt{2}} \qquad \Rightarrow \qquad \theta = 45°$$

All that's left is to check the formula.

$$\|\mathbf{u}\| \, \|\mathbf{v}\| \sin\theta = (\sqrt{2})(2)\sin(45°) = (\sqrt{2})(2)\left(\frac{\sqrt{2}}{2}\right) = 2 = \|\mathbf{u} \times \mathbf{v}\|$$

So, the theorem is verified.

Euclidean n-Space

In the first two sections of this chapter we looked at vectors in 2-space and 3-space. You probably noticed that with the exception of the cross product (which is only defined in

3-space) all of the formulas that we had for vectors in 3-space were natural extensions of the 2-space formulas.

In this section we're going to extend things out to a much more general setting. We won't be able to visualize things in a geometric setting as we did in the previous two sections but things will extend out nicely. In fact, that was why we started in 2-space and 3-space. We wanted to start out in a setting where we could visualize some of what was going on before we generalized things into a setting where visualization was a very difficult thing to do.

So, let's get things started off with the following definition.

Definition 1 Given a positive integer n an **ordered *n*-tuple** is a sequence of n real numbers denoted by $\left(a_1, a_2, \ldots, a_n\right)$. The complete set of all ordered n-tuples is called ***n*-space** and is denoted by $\mathbb{R}^n$

In the previous sections we were looking at $\mathbb{R}^2$ (what we were calling 2-space) and $\mathbb{R}^3$ (what we were calling 3-space). Also the more standard terms for 2-tuples and 3-tuples are **ordered pair** and **ordered triplet** and that's the terms we'll be using from this point on.

Also, as we pointed out in the previous sections an ordered pair, $\left(a_1, a_2\right)$, or an ordered triplet, $\left(a_1, a_2, a_3\right)$, can be thought of as either a point or a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ respectively. In general an ordered n-tuple, $\left(a_1, a_2, \ldots, a_n\right)$, can also be thought of as a "point" or a vector in $\mathbb{R}^n$. Again, we can't really visualize a point or a vector in $\mathbb{R}^n$, but we will think of them as points or vectors in $\mathbb{R}^n$ anyway and try not to worry too much about the fact that we can't really visualize them.

Next, we need to get the standard arithmetic definitions out of the way and all of these are going to be natural extensions of the arithmetic we saw in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Definition 2 Suppose $\mathbf{u} = \left(u_1, u_2, \ldots, u_n\right)$ and $\mathbf{v} = \left(v_1, v_2, \ldots, v_n\right)$ are two vectors in $\mathbb{R}^n$.

(a) We say that **u** and **v** are equal if,

$$u_1 = v_1 \quad u_2 = v_2 \quad \cdots \quad u_n = v_n$$

(b) The sum of **u** and **v** is defined to be,

$$\mathbf{u} + \mathbf{v} = \left(u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n\right)$$

(c) The negative (or additive inverse) of **u** is defined to be,

$$-\mathbf{u} = \left(-u_1, -u_2, \ldots, -u_n\right)$$

(d) The difference of two vectors is defined to be,

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + \left(-\mathbf{v}\right) = \left(u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n\right)$$

(e) If c is any scalar then the scalar multiple of **u** is defined to be,

$$c\mathbf{u} = (cu_1, cu_2, \ldots, cu_n)$$

(f) The **zero vector** in $\mathbb{R}^n$ is denoted by $\mathbf{0}$ and is defined to be,

$$\mathbf{0} = (0, 0, \ldots, 0)$$

So, these really are just the natural extensions of the arithmetic we were doing in the previous two sections.

The basic properties of arithmetic are still valid in $\mathbb{R}^n$ so let's also give those so that we can say that we've done that.

Theorem 1 Suppose $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ are vectors in $\mathbb{R}^n$ and c and k are scalars then,

(a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(b) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

(c) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(d) $\mathbf{u} - \mathbf{u} = \mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

(e) $1\mathbf{u} = \mathbf{u}$

(f) $(ck)\mathbf{u} = c(k\mathbf{u}) = k(c\mathbf{u})$

(g) $(c + k)\mathbf{u} = c\mathbf{u} + k\mathbf{u}$

(h) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

The proof of all of these come directly from the definitions above and so won't be given here.

We now need to extend the dot product we saw in the previous section to $\mathbb{R}^n$ and we'll be giving it a new name as well.

Definition 3 Suppose $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are two vectors in $\mathbb{R}^n$ then the **Euclidean inner product** denoted by $\mathbf{u} \bullet \mathbf{v}$ is defined to be

$$\mathbf{u} \bullet \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

So, we can see that it's the same notation and is a natural extension to the dot product that we looked at in the previous section, we're just going to call it something different now. In fact, this is probably the more correct name for it and we should instead say that we've renamed this to the dot product when we were working exclusively in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Note that when we add in addition, scalar multiplication and the Euclidean inner product to $\mathbb{R}^n$ we will often call this **Euclidean *n*-space**.

We also have natural extensions of the properties of the dot product that we saw in the previous section.

Theorem 2 Suppose $\mathbf{u}=(u_1,u_2,\ldots,u_n)$, $\mathbf{v}=(v_1,v_2,\ldots,v_n)$, and $\mathbf{w}=(w_1,w_2,\ldots,w_n)$ are vectors in $\mathbb{R}^n$ and let c be a scalar then,

(a) $\mathbf{u}\bullet\mathbf{v}=\mathbf{v}\bullet\mathbf{u}$

(b) $(\mathbf{u}+\mathbf{v})\bullet\mathbf{w}=\mathbf{u}\bullet\mathbf{w}+\mathbf{v}\bullet\mathbf{w}$

(c) $c(\mathbf{u}\bullet\mathbf{v})=(c\,\mathbf{u})\bullet\mathbf{v}=\mathbf{u}\bullet(c\,\mathbf{v})$

(d) $\mathbf{u}\bullet\mathbf{u}\geq 0$

(e) $\mathbf{u}\bullet\mathbf{u}=0$ if and only if $\mathbf{u}=\mathbf{0}$.

The proof of this theorem falls directly from the definition of the Euclidean inner product and are extensions of proofs given in the previous section and so aren't given here.

The final extension to the work of the previous sections that we need to do is to give the definition of the norm for vectors in $\mathbb{R}^n$ and we'll use this to define distance in $\mathbb{R}^n$.

Definition 4 Suppose $\mathbf{u}=(u_1,u_2,\ldots,u_n)$ is a vector in $\mathbb{R}^n$ then the **Euclidean norm** is,

$$\|\mathbf{u}\|=(\mathbf{u}\bullet\mathbf{u})^{\frac{1}{2}}=\sqrt{u_1^2+u_2^2+\cdots+u_n^2}$$

Definition 5 Suppose $\mathbf{u}=(u_1,u_2,\ldots,u_n)$ and $\mathbf{v}=(v_1,v_2,\ldots,v_n)$ are two points in $\mathbb{R}^n$ then the **Euclidean distance** between them is defined to be,

$$d(\mathbf{u},\mathbf{v})=\|\mathbf{u}-\mathbf{v}\|=\sqrt{(u_1-v_1)^2+(u_2-v_2)^2+\cdots+(u_n-v_n)^2}$$

Notice in this definition that we called **u** and **v** points and then used them as vectors in the norm. This comes back to the idea that an n-tuple can be thought of as both a point and a vector and so will often be used interchangeably where needed.

Okay, we're not going to be working many examples in this section since all of this is an extension to previous work, but we should work one or two just to make sure we're all on the same page.

Example 1 Given $\mathbf{u}=(9,3,-4,0,1)$ and $\mathbf{v}=(0,-3,2,-1,7)$ compute

(a) $\mathbf{u}-4\mathbf{v}$

(b) $\mathbf{v}\bullet\mathbf{u}$

(c) $\mathbf{u}\bullet\mathbf{u}$

(d) $\|\mathbf{u}\|$

(e) $d(\mathbf{u},\mathbf{v})$

Solution

There really isn't much to do here other than use the appropriate definition.

(a)

$$\mathbf{u}-4\mathbf{v}=(9,3,-4,0,1)-4(0,-3,2,-1,7)$$
$$=(9,3,-4,0,1)-(0,-12,8,-4,28)$$
$$=(9,15,-12,4,-27)$$

(b)

$$\mathbf{v}\bullet\mathbf{u}=(0)(9)+(-3)(3)+(2)(-4)+(-1)(0)+(7)(1)=-10$$

(c)

$$\mathbf{u}\bullet\mathbf{u}=9^2+3^2+(-4)^2+0^2+1^2=107$$

(d)

$$\|\mathbf{u}\|=\sqrt{9^2+3^2+(-4)^2+0^2+1^2}=\sqrt{107}$$

(e)

$$d(\mathbf{u},\mathbf{v})=\sqrt{(9-0)^2+(3-(-3))^2+(-4-2)^2+(0-(-1))^2+(1-7)^2}=\sqrt{190}$$

Just as we saw in the section on vectors if we have $\|\mathbf{u}\|=1$ then we will call $\mathbf{u}$ a **unit vector** and so the vector $\mathbf{u}$ from the previous set of examples is not a unit vector

Now that we've gotten both the inner product and the norm taken care of we can give the following theorem.

Theorem 3 Suppose $\mathbf{u}$ and $\mathbf{v}$ are two vectors in $\mathbb{R}^n$ and θ is the angle between them. Then,

$$\mathbf{u}\bullet\mathbf{v}=\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$

Of course since we are in $\mathbb{R}^n$ it is hard to visualize just what the angle between the two vectors is, but provided we can find it we can use this theorem. Also note that this was the definition of the dot product that we gave in the previous section and like that section this theorem is most useful for actually determining the angle between two vectors.

The proof of this theorem is identical to the proof of Theorem 1 in the previous section and so isn't given here.

The next theorem is very important and has many uses in the study of vectors. In fact we'll need it in the proof of at least one theorem in these notes. The following theorem is called the **Cauchy-Schwarz Inequality**.

Theorem 4 Suppose $\mathbf{u}$ and $\mathbf{v}$ are two vectors in $\mathbb{R}^n$ then

$$|\mathbf{u}\bullet\mathbf{v}|\le\|\mathbf{u}\|\,\|\mathbf{v}\|$$

Proof : This proof is surprisingly simple. We'll start with the result of the previous theorem and take the absolve value of both sides.

$$|\mathbf{u}\bullet\mathbf{v}|=\|\mathbf{u}\|\,\|\mathbf{v}\|\,|\cos\theta|$$

However, we know that $\left|\cos\theta\right| \le 1$ and so we get our result by using this fact.

$$\left|\mathbf{u}\bullet\mathbf{v}\right| = \left\|\mathbf{u}\right\|\,\left\|\mathbf{v}\right\|\left|\cos\theta\right| \le \left\|\mathbf{u}\right\|\,\left\|\mathbf{v}\right\|(1) = \left\|\mathbf{u}\right\|\,\left\|\mathbf{v}\right\|$$

■

Here are some nice properties of the Euclidean norm.

Theorem 5 Suppose **u** and **v** are two vectors in $\mathbb{R}^n$ and that c is a scalar then,

- **(a)** $\left\|\mathbf{u}\right\| \ge 0$
- **(b)** $\left\|\mathbf{u}\right\| = 0$ if and only if **u**=**0**.
- **(c)** $\left\|c\,\mathbf{u}\right\| = \left|c\right|\,\left\|\mathbf{u}\right\|$
- **(d)** $\left\|\mathbf{u}+\mathbf{v}\right\| \le \left\|\mathbf{u}\right\| + \left\|\mathbf{v}\right\|$ - Usually called the **Triangle Inequality**

The proof of the first two part is a direct consequence of the definition of the Euclidean norm and so won't be given here.

Proof :

(c) We'll just run through the definition of the norm on this one.

$$\begin{aligned}\left\|c\,\mathbf{u}\right\| &= \sqrt{(cu_1)^2 + (cu_2)^2 + \cdots + (cu_n)^2} \\ &= \sqrt{c^2\left(u_1^2 + u_2^2 + \cdots + u_n^2\right)} \\ &= \left|c\right|\sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} \\ &= \left|c\right|\,\left\|\mathbf{u}\right\|\end{aligned}$$

(d) The proof of this one isn't too bad once you see the steps you need to take. We'll start with the following.

$$\left\|\mathbf{u}+\mathbf{v}\right\|^2 = (\mathbf{u}+\mathbf{v})\bullet(\mathbf{u}+\mathbf{v})$$

So, we're starting with the definition of the norm and squaring both sides to get rid of the square root on the right side. Next, we'll use the properties of the Euclidean inner product to simplify this.

$$\begin{aligned}\left\|\mathbf{u}+\mathbf{v}\right\|^2 &= \mathbf{u}\bullet(\mathbf{u}+\mathbf{v}) + \mathbf{v}\bullet(\mathbf{u}+\mathbf{v}) \\ &= \mathbf{u}\bullet\mathbf{u} + \mathbf{u}\bullet\mathbf{v} + \mathbf{v}\bullet\mathbf{u} + \mathbf{v}\bullet\mathbf{u} \\ &= \mathbf{u}\bullet\mathbf{u} + 2(\mathbf{u}\bullet\mathbf{v}) + \mathbf{v}\bullet\mathbf{u}\end{aligned}$$

Now, notice that we can convert the first and third terms into norms so we'll do that. Also, $\mathbf{u}\bullet\mathbf{v}$ is a number and so we know that if we take the absolute value of this we'll have $\mathbf{u}\bullet\mathbf{v} \le \left|\mathbf{u}\bullet\mathbf{v}\right|$. Using this and converting the first and third terms to norms gives,

$$\begin{aligned}\left\|\mathbf{u}+\mathbf{v}\right\|^2 &= \left\|\mathbf{u}\right\|^2 + 2(\mathbf{u}\bullet\mathbf{v}) + \left\|\mathbf{v}\right\|^2 \\ &\le \left\|\mathbf{u}\right\|^2 + 2\left|\mathbf{u}\bullet\mathbf{v}\right| + \left\|\mathbf{v}\right\|^2\end{aligned}$$

We can now use the Cauchy-Schwarz inequality on the second term to get,

$$\|\mathbf{u}+\mathbf{v}\|^2 \le \|\mathbf{u}\|^2 + 2\|\mathbf{u}\|\ \|\mathbf{v}\| + \|\mathbf{v}\|^2$$

We're almost done. Let's notice that the left side can now be rewritten as,

$$\|\mathbf{u}+\mathbf{v}\|^2 \le \left(\|\mathbf{u}\| + \|\mathbf{v}\|\right)^2$$

Finally, take the square root of both sides.

$$\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$$

■

Example 2 Given $\mathbf{u} = (-2,3,1,-1)$ and $\mathbf{v} = (7,1,-4,-2)$ verify the Cauchy-Schwarz inequality and the Triangle Inequality.

Solution
Let's first verify the Cauchy-Schwarz inequality. To do this we need to following quantities.

$$\mathbf{u}\bullet\mathbf{v} = -14+3-4+2 = -13$$

$$\|\mathbf{u}\| = \sqrt{4+9+1+1} = \sqrt{15} \qquad \|\mathbf{v}\| = \sqrt{49+1+16+4} = \sqrt{70}$$

Now, verify the Cauchy-Schwarz inequality.

$$|\mathbf{u}\bullet\mathbf{v}| = |-13| = 13 \le 32.4037 = \sqrt{15}\ \sqrt{70} = \|\mathbf{u}\|\ \|\mathbf{v}\|$$

Sure enough the Cauchy-Schwarz inequality holds true.

To verify the Triangle inequality all we need is,

$$\mathbf{u}+\mathbf{v} = (5,4,-3,-3) \qquad \|\mathbf{u}+\mathbf{v}\| = \sqrt{25+16+9+9} = \sqrt{59}$$

Now verify the Triangle Inequality.

$$\|\mathbf{u}+\mathbf{v}\| = \sqrt{59} = 7.6811 \le 12.2396 = \sqrt{15} + \sqrt{70} = \|\mathbf{u}\| + \|\mathbf{v}\|$$

So, the Triangle Inequality is also verified for this problem.

Here are some nice properties pertaining to the Euclidean distance.

Theorem 6 Suppose **u**, **v**, and **w** are vectors in $\mathbb{R}^n$ then,

(a) $d(\mathbf{u},\mathbf{v}) \ge 0$

(b) $d(\mathbf{u},\mathbf{v}) = 0$ if and only if **u**=**v**.

(c) $d(\mathbf{u},\mathbf{v}) = d(\mathbf{v},\mathbf{u})$

(d) $d(\mathbf{u},\mathbf{v}) \le d(\mathbf{u},\mathbf{w}) + d(\mathbf{w},\mathbf{v})$ - Usually called the **Triangle Inequality**

The proof of the first two parts is a direct consequence of the previous theorem and the proof of the third part is a direct consequence of the definition of distance and won't be proven here.

Proof (d) : Let's start off with the definition of distance.

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|$$

Now, add in and subtract out **w** as follows,

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{w}+\mathbf{w}-\mathbf{v}\| = \|(\mathbf{u}-\mathbf{w})+(\mathbf{w}-\mathbf{v})\|$$

Next use the Triangle Inequality for norms on this.

$$d(\mathbf{u},\mathbf{v}) \le \|\mathbf{u}-\mathbf{w}\| + \|\mathbf{w}-\mathbf{v}\|$$

Finally, just reuse the definition of distance again.

$$d(\mathbf{u},\mathbf{v}) \le d(\mathbf{u},\mathbf{w}) + d(\mathbf{w},\mathbf{v})$$

■

We have one final topic that needs to be generalized into Euclidean *n*-space.

Definition 6 Suppose **u** and **v** are two vectors in $\mathbb{R}^n$. We say that **u** and **v** are **orthogonal** if $\mathbf{u}\bullet\mathbf{v} = 0$.

So, this definition of orthogonality is identical to the definition that we saw when we were dealing with $\mathbb{R}^2$ and $\mathbb{R}^3$.

Here's is the **Pythagorean Theorem in** $\mathbb{R}^n$.

Theorem 7 Suppose **u** and **v** are two orthogonal vectors in $\mathbb{R}^n$ then,

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Proof : The proof of this theorem is fairly simple. From the proof of the triangle inequality for norms we have the following statement.

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2(\mathbf{u}\bullet\mathbf{v}) + \|\mathbf{v}\|^2$$

However, because **u** and **v** are orthogonal we have $\mathbf{u}\bullet\mathbf{v} = 0$ and so we get,

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

■

Example 3 Show that $\mathbf{u} = (3,0,1,0,4,-1)$ and $\mathbf{v} = (-2,5,0,2,-3,-18)$ are orthogonal and verify that the Pythagorean Theorem holds.

Solution

Showing that these two vectors is easy enough.

$$\mathbf{u}\bullet\mathbf{v} = (3)(-2)+(0)(5)+(1)(0)+(0)(2)+(4)(-3)+(-1)(-18) = 0$$

So, the Pythagorean Theorem should hold, but let's verify that. Here's the sum

$$\mathbf{u}+\mathbf{v}=(1,5,1,2,1,-19)$$

and here's the square of the norms.

$$\|\mathbf{u}+\mathbf{v}\|^2=1^2+5^2+1^2+2^2+1^2+(-19)^2=393$$
$$\|\mathbf{u}\|^2=3^2+0^2+1^2+0^2+4^2+(-1)^2=27$$
$$\|\mathbf{v}\|^2=(-2)^2+5^2+0^2+2^2+(-3)^2+(-18)^2=366$$

A quick computation then confirms that $\|\mathbf{u}+\mathbf{v}\|^2=\|\mathbf{u}\|^2+\|\mathbf{v}\|^2$.

We've got one more theorem that gives a relationship between the Euclidean inner product and the norm. This may seem like a silly theorem, but we'll actually need this theorem towards the end of the next chapter.

Theorem 8 If **u** and **v** are two vectors in $\mathbb{R}^n$ then,

$$\mathbf{u}\bullet\mathbf{v}=\frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2-\frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2$$

Proof : The proof here is surprisingly simple. First, start with,

$$\|\mathbf{u}+\mathbf{v}\|^2=\|\mathbf{u}\|^2+2(\mathbf{u}\bullet\mathbf{v})+\|\mathbf{v}\|^2$$
$$\|\mathbf{u}-\mathbf{v}\|^2=\|\mathbf{u}\|^2-2(\mathbf{u}\bullet\mathbf{v})+\|\mathbf{v}\|^2$$

The first of these we've seen a couple of times already and the second is derived in the same manner that the first was and so you should verify that formula.

Now subtract the second from the first to get,

$$4(\mathbf{u}\bullet\mathbf{v})=\|\mathbf{u}+\mathbf{v}\|^2-\|\mathbf{u}-\mathbf{v}\|^2$$

Finally, divide by 4 and we get the result we were after.

$$\mathbf{u}\bullet\mathbf{v}=\frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2-\frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2$$

■

In the previous section we saw the three standard basis vectors for $\mathbb{R}^3$, **i**, **j**, and **k**. This idea can also be extended out to $\mathbb{R}^n$. In $\mathbb{R}^n$ we will define the **standard basis vectors** or **standard unit vectors** to be,

$$\mathbf{e}_1=(1,0,0,\ldots,0)\qquad \mathbf{e}_2=(0,1,0,\ldots,0)\qquad \cdots \qquad \mathbf{e}_n=(0,0,0,\ldots,1)$$

and just as we saw in that section we can write any vector $\mathbf{u}=(u_1,u_2,\ldots,u_n)$ in terms of these standard basis vectors as follows,

$$\begin{aligned}\mathbf{u}&=u_1(1,0,0,\ldots 0)+u_2(0,1,0,\ldots 0)+\cdots+u_n(0,0,0,\ldots 1)\\&=u_1\mathbf{e}_1+u_2\mathbf{e}_2+\cdots+u_n\mathbf{e}_n\end{aligned}$$

Note that in $\mathbb{R}^3$ we have $\mathbf{e}_1 = \mathbf{i}$, $\mathbf{e}_2 = \mathbf{j}$ and $\mathbf{e}_3 = \mathbf{k}$.

Now that we've gotten the general vector in Euclidean *n*-space taken care of we need to go back and remember some of the work that we did in the first chapter. It is often convenient to write the vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ as either a row matrix or a column matrix as follows,

$$\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \qquad\qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

In this notation we can use matrix addition and scalar multiplication for matrices to show that we'll get the same results as if we'd done vector addition and scalar multiplication for vectors on the original vectors.

So, why do we do this? We'll let's use the column matrix notation for the two vectors **u** and **v**.

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad\qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Now compute the following matrix product.

$$\mathbf{v}^T\mathbf{u} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} u_1v_1 + u_2v_2 + \cdots + u_nv_n \end{bmatrix} = \begin{bmatrix} \mathbf{u}\bullet\mathbf{v} \end{bmatrix} = \mathbf{u}\bullet\mathbf{v}$$

So, we can think of the Euclidean inner product can be thought of as a matrix multiplication using,

$$\mathbf{u}\bullet\mathbf{v} = \mathbf{v}^T\mathbf{u}$$

provided we consider **u** and **v** as column vectors.

The natural question this is just why is this important? We'll let's consider the following scenario. Suppose that **u** and **v** are two vectors in $\mathbb{R}^n$ and that *A* is an $n \times n$ matrix. Now consider the following inner product and write it as a matrix multiplication.

$$(A\mathbf{u})\bullet\mathbf{v} = \mathbf{v}^T(A\mathbf{u})$$

Now, rearrange the order of the multiplication and recall one of the properties of transposes.

$$(A\mathbf{u})\bullet\mathbf{v} = (\mathbf{v}^T A)\mathbf{u} = (A^T\mathbf{v})^T\,\mathbf{u}$$

Don't forget that we switch the order on the matrices when we move the transpose out of the parenthesis. Finally, this last matrix product can be rewritten as an inner product.

$$(A\mathbf{u})\bullet\mathbf{v} = \mathbf{u}\bullet\left(A^T\mathbf{v}\right)$$

This tells us that if we've got an inner product and the first vector (or column matrix) is multiplied by a matrix then we can move that matrix to the second vector (or column matrix) if we simply take its transpose.

A similar argument can also show that,

$$\mathbf{u}\bullet(A\mathbf{v}) = \left(A^T\mathbf{u}\right)\bullet\mathbf{v}$$

Linear Transformations

In this section we're going to take a look at a special kind of function that arises very naturally in the study of Linear Algebra and has many applications in fields outside of mathematics such as physics and engineering. This section is devoted mostly to the basic definitions and facts associated with this special kind of function. We will be looking at a couple of examples, but we'll reserve most of the examples for the next section.

Now, the first thing that we need to do is take a step back and make sure that we're all familiar with some of the basics of functions in general. A function, *f*, is a rule (usually defined by an equation) that takes each element of the set *A* (called the **domain**) and associates it with exactly one element of a set *B* (called the **codomain**). The notation that we'll be using to denote our function is

$$f : A \to B$$

When we see this notation we know that we're going to be dealing with a function that takes elements from the set *A* and associates them with elements from the set *B*. Note as well that it is completely possible that not every element of the set *B* will be associated with an element from *A*. The subset of all elements from *B* that are associated with elements from *A* is called the **range**.

In this section we're going to be looking at functions of the form,

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

In other words, we're going to be looking at functions that take elements/points/vectors from $\mathbb{R}^n$ and associate them with elements/points/vectors from $\mathbb{R}^m$. These kinds of functions are called **transformations** and we say that *f* **maps** $\mathbb{R}^n$ into $\mathbb{R}^m$. On an element basis we will also say that *f* **maps** the element **u** from $\mathbb{R}^n$ to the element **v** from $\mathbb{R}^m$.

So, just what do transformations look like? Consider the following scenario. Suppose that we have *m* functions of the following form,

$$\begin{aligned} w_1 &= f_1(x_1, x_2, \ldots, x_n) \\ w_2 &= f_2(x_1, x_2, \ldots, x_n) \\ &\vdots \qquad \vdots \\ w_m &= f_m(x_1, x_2, \ldots, x_n) \end{aligned}$$

Each of these functions takes a point in $\mathbb{R}^n$, namely $(x_1, x_2, \ldots, x_n)$, and maps it to the number w_i. We can now define a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ as follows,

$$T(x_1, x_2, \ldots, x_n) = (w_1, w_2, \ldots, w_m)$$

In this way we associate with each point $(x_1, x_2, \ldots, x_n)$ from $\mathbb{R}^n$ a point $(w_1, w_2, \ldots, w_m)$ from $\mathbb{R}^m$ and we have a transformation.

Let's take a look at a couple of transformations.

Example 1 Given

$$w_1 = 3x_1 - 4x_2 \qquad w_2 = x_1 + 2x_2 \qquad w_3 = 6x_1 - x_2 \qquad w_4 = 10x_2$$

define $T : \mathbb{R}^2 \to \mathbb{R}^4$ as,

$$T(x_1, x_2) = (w_1, w_2, w_3, w_4) \qquad \text{OR}$$

$$T(x_1, x_2) = (3x_1 - 4x_2,\ x_1 + 2x_2,\ 6x_1 - x_2,\ 10x_2)$$

Note that the second form is more convenient since we don't actually have to define any of the *w*'s in that way and is how we will define most of our transformations.

We evaluate this just as we evaluate the functions that we're used to working with. Namely, pick a point from $\mathbb{R}^2$ and plug into the transformation and we'll get a point out of the function that is in $\mathbb{R}^4$. For example,

$$T(-5, 2) = (-23, -1, -32, 20)$$

Example 2 Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ as $T(x_1, x_2, x_3) = (4x_2^2 + x_3^2,\ x_1^2 - x_2x_3)$. A sample evaluation of this transformation is,

$$T(3, -1, 6) = (40,\ 15)$$

Now, in this section we're going to be looking at a special kind of transformation called a linear transformation. Here is the definition of a linear transformation.

Definition 1 A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a **linear transformation** if for all **u** and **v** in $\mathbb{R}^n$ and all scalars *c* we have,

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \qquad T(c\mathbf{u}) = cT(\mathbf{u})$$

We looked at two transformations above and only one of them is linear. Let's take a look at each one and see what we've got.

Example 3 Determine if the transformation from Example 2 is linear or not.

Solution
Okay, if this is going to be linear then it must satisfy both of the conditions from the definition. In other words, both of the following will need to be true.

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= T(u_1+v_1,\, u_2+v_2,\, u_3+v_3) \\ &= T(u_1,\, u_2,\, u_3)+T(v_1,\, v_2,\, v_3) \\ &= T(\mathbf{u})+T(\mathbf{v}) \end{aligned}$$

$$T(c\,\mathbf{u}) = T(cu_1, cu_2, cu_3) = cT(u_1, u_2, u_3) = cT(\mathbf{u})$$

In this case let's take a look at the second condition.

$$\begin{aligned} T(c\,\mathbf{u}) &= T(cu_1, cu_2, cu_3) \\ &= \left(4c^2u_2^2 + c^2u_3^2,\; c^2u_1^2 - c^2u_2u_3\right) \\ &= c^2\left(4u_2^2 + u_3^2,\; u_1^2 - u_2u_3\right) \\ &= c^2T(\mathbf{u}) \neq cT(\mathbf{u}) \end{aligned}$$

The second condition is not satisfied and so this is not a linear transformation. You might want to verify that in this case the first is also not satisfied. It's not too bad, but the work does get a little messy.

Example 4 Determine if the transformation in Example 1 is linear or not.

Solution
To do this one we're going to need to rewrite things just a little. The transformation is defined as $T(x_1, x_2) = (w_1, w_2, w_3, w_4)$ where,

$$\begin{aligned} w_1 &= 3x_1 - 4x_2 \\ w_2 &= x_1 + 2x_2 \\ w_3 &= 6x_1 - x_2 \\ w_4 &= 10x_2 \end{aligned}$$

Now, each of the components are given by a system of linear (hhmm, makes one instantly wonder if the transformation is also linear…) equations and we saw in the first chapter that we can always write a system of linear equations in matrix form. Let's do that for this system.

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & 2 \\ 6 & -1 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{w} = A\mathbf{x}$$

Now, notice that if we plug in any column matrix **x** and do the matrix multiplication we'll get a new column matrix out, **w**. Let's pick a column matrix **x** totally at random and see what we get.

$$\begin{bmatrix} -23 \\ -1 \\ -32 \\ 20 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & 2 \\ 6 & -1 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

Of course, we didn't pick **x** completely at random. Notice that **x** we choose was the column matrix representation of the point from $\mathbb{R}^2$ that we used in Example 1 to show a sample evaluation of the transformation. Just as importantly notice that the result, **w**, is the matrix representation of the point from $\mathbb{R}^4$ that we got out of the evaluation.

In fact, this will always be the case for this transformation. So, in some way the evaluation $T(\mathbf{x})$ is the same as the matrix multiplication $A\mathbf{x}$ and so we can write the transformation as

$$T(\mathbf{x}) = A\mathbf{x}$$

Notice that we're kind of mixing and matching notation here. On the left **x** represents a point in $\mathbb{R}^2$ and on the right it is a 2×1 matrix. However, this really isn't a problem since they both can be used to represent a point in $\mathbb{R}^2$. We will have to get used to this notation however as we'll be using it quite regularly.

Okay, just what where we after here. We wanted to determine if this transformation is linear or not. With this new way of writing the transformation this is actually really simple. We'll just make use of some very nice facts that we know about matrix multiplication. Here is the work for this problem

$$T(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$$
$$T(c\,\mathbf{u}) = A(c\,\mathbf{u}) = c\,A\mathbf{u} = cT(\mathbf{u})$$

So, both conditions of the definition are met and so this transformation is a linear transformation.

There are a couple of things to note here. First, we couldn't write the transformation from Example 2 as a matrix multiplication because at least one of the equations (okay both in this case) for the components in the result were not linear.

Second, when all the equations that give the components of the result are linear then the transformation will be linear. If at least one of the equations are not linear then the transformation will not be linear either.

Now, we need to investigate the idea that we used in the previous example in more detail. There are two issues that we want to take a look at.

First, we saw that, at least in some cases, matrix multiplication can be thought of as a linear transformation. As the following theorem shows, this is in fact always the case.

Theorem 1 If A is an $m \times n$ matrix then its **induced transformation**, $T_A : \mathbb{R}^n \to \mathbb{R}^m$, defined as,

$$T_A(\mathbf{x}) = A\mathbf{x}$$

is a linear transformation.

Proof : The proof here is really simple and in fact we pretty much saw it last example.

$$T_A(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T_A(\mathbf{u}) + T_A(\mathbf{v})$$
$$T_A(c\mathbf{u}) = A(c\mathbf{u}) = cA\mathbf{u} = cT_A(\mathbf{u})$$

So, the induced function, T_A, satisfies both the conditions in the definition of a linear transformation and so it is a linear transformation.

■

So, any time we do matrix multiplication we can also think of the operation as evaluating a linear transformation.

The other thing that we saw in Example 4 is that we were able, in that case, to write a linear transformation as a matrix multiplication. Again, it turns out that every linear transformation can be written as a matrix multiplication.

Theorem 2 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, then there is an $m \times n$ matrix such that $T = T_A$ (recall that T_A is the transformation induced by A).

The matrix A is called the **matrix induced by *T*** and is sometimes denoted as $A = [T]$.

Proof : First let,

$$\mathbf{e}_1 = (1,0,0,\ldots,0) \qquad \mathbf{e}_2 = (0,1,0,\ldots,0) \qquad \cdots \qquad \mathbf{e}_n = (0,0,0,\ldots,1)$$

be the standard basis vectors for $\mathbb{R}^n$ and define A to be the $m \times n$ matrix whose i^{th} column is $T(\mathbf{e}_i)$. In other words, A is given by,

$$A = \left[T(\mathbf{e}_1) \mid T(\mathbf{e}_2) \mid \quad \cdots \quad \mid T(\mathbf{e}_n) \right]$$

Next let **x** be any vector from $\mathbb{R}^n$. We know that we can write **x** in terms of the standard basis vectors as follows,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

In order to prove this theorem we're going to need to show that for any **x** (which we've got a nice general one above) we will have $T(\mathbf{x}) = T_A(\mathbf{x})$. So, let's start off and plug **x** into *T* using the general form as written out above.

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n)$$

Now, we know that *T* is a linear transformation and so we can break this up at each of the "+"'s as follows,

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1) + T(x_2\mathbf{e}_2) + \cdots + T(x_n\mathbf{e}_n)$$

Next, each of the x_i's are scalars and again because *T* is a linear transformation we can write this as,

$$T(\mathbf{x}) = x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) + \cdots + x_nT(\mathbf{e}_n)$$

Next, let's notice that this is nothing more than the following matrix multiplication.

$$T(\mathbf{x}) = \left[\, T(\mathbf{e}_1) \;\middle|\; T(\mathbf{e}_2) \;\middle|\; \cdots \;\middle|\; T(\mathbf{e}_n) \,\right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

But the first matrix nothing more than *A* and the second is just **x** and we when we define *A* as we did above we will get,

$$T(\mathbf{x}) = A\mathbf{x} = T_A(\mathbf{x})$$

and so we've proven what we needed to.

■

In this proof we used the standard basis vectors to define the matrix *A*. As we will see in a later chapter there are other choices of vectors that we could use here and these will produce a different induced matrix, *A*, and we do need to remember that. However, when we use the standard basis vectors to define *A*, as we're going to in this chapter, then we don't actually need to evaluate *T* at each of the basis vectors as we did in the proof. All we need to do is what we did in Example 4, write down the coefficient matrix for the system of equations that we get by writing out each of the components as individual equations.

Okay, we've done a lot of work in this section and we haven't really done any examples so we should probably do a couple of them. Note that we are saving most of the examples for the next section, so don't expect a lot here. We're just going to do a couple so we can say we've done a couple.

Example 5 The **zero transformation** is the transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ that maps every vector $\mathbf{x}$ in $\mathbb{R}^n$ to the zero vector in $\mathbb{R}^m$, *i.e.* $T(\mathbf{x}) = \mathbf{0}$. The matrix induced by this transformation is the $m \times n$ zero matrix, **0** since,

$$T(\mathbf{x}) = T_0(\mathbf{x}) = \mathbf{0}\mathbf{x} = \mathbf{0}$$

To make it clear we're using the zero transformation we usually denote it by $T_0(\mathbf{x})$.

Example 6 The **identity transformation** is the transformation $T:\mathbb{R}^n \to \mathbb{R}^n$ (yes they are both $\mathbb{R}^n$) that maps every $\mathbf{x}$ to itself, *i.e.* $T(\mathbf{x}) = \mathbf{x}$. The matrix induced by this transformation is the $n \times n$ identity matrix, I_n since,

$$T(\mathbf{x}) = T_I(\mathbf{x}) = I_n\mathbf{x} = \mathbf{x}$$

We'll usually denote the identity transformation as $T_I(\mathbf{x})$ to make it clear we're working with it.

So, the two examples above are standard examples and we did need them taken care of. However, they aren't really very illustrative for seeing how to construct the matrix induced by the transformation. To see how this is done, let's take a look at some reflections in $\mathbb{R}^2$. We'll look at reflections in $\mathbb{R}^3$ in the next section.

Example 7 Determine the matrix induced by the following reflections.

(a) Reflection about the *x*-axis.

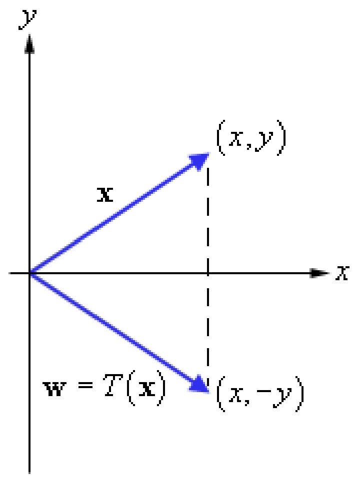
(b) Reflection about the *y*-axis.

(c) Reflection about the line *y*=*x*.

Solution

Note that all of these will be linear transformations of the form $T:\mathbb{R}^2 \to \mathbb{R}^2$.

(a) Let's start off with a sketch of what we're looking for here.



So, from this sketch we can see that the components of the for the translation (*i.e.* the equations that will map **x** into **w**) are,

$$w_1 = x$$
$$w_2 = -y$$

Remember that w_1 will be the first component of the transformed point and w_2 will be the second component of the transformed point.

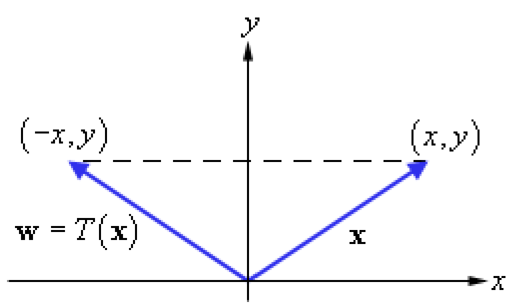
Now, just as we did in Example 4 we can write down the matrix form of this system.

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

So, it looks like the matrix induced by this reflection is,

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(b) We'll do this one a little quicker. Here's a sketch and the equations for this reflection.

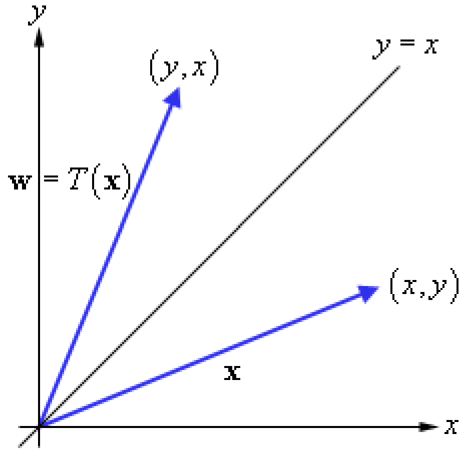


$$w_1 = -x$$
$$w_2 = y$$

The matrix induced by this reflection is,

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

(c) Here's the sketch and equations for this reflection.



$$w_1 = y$$
$$w_2 = x$$

The matrix induced by this reflection is,

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Hopefully, from these examples you're starting to get a feel for how we arrive at the induced matrix for a linear transformation. We'll be seeing more of these in the next section, but for now we need to move on to some more ideas about linear transformations.

Let's suppose that we have two linear transformations induced by the matrices A and B, $T_A : \mathbb{R}^n \to \mathbb{R}^k$ and $T_B : \mathbb{R}^k \to \mathbb{R}^m$. If we take any $\mathbf{x}$ out of $\mathbb{R}^n$ T_A will map $\mathbf{x}$ into $\mathbb{R}^k$. In other words, $T_A(\mathbf{x})$ will be in $\mathbb{R}^k$ and notice that we can then apply T_B to this and its image will be in $\mathbb{R}^m$. In summary, if we take $\mathbf{x}$ out of $\mathbb{R}^n$ and first apply T_A to $\mathbf{x}$ and then apply T_B to the result we will have a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

This process is called **composition** of transformations and is denoted as

$$(T_B \circ T_A)(\mathbf{x}) = T_B(T_A(\mathbf{x}))$$

Note that the order here is important. The first transformation to be applied is on the right and the second is on the left.

Now, because both of our original transformations were linear we can do the following,

$$(T_B \circ T_A)(\mathbf{x}) = T_B(T_A(\mathbf{x})) = T_B(A\mathbf{x}) = (BA)\mathbf{x}$$

and so the composition $T_B \circ T_A$ is the same as multiplication by BA. This means that the composition will be a linear transformation provided the two original transformations were also linear.

Note as well that we can do composition with as many transformations as we want provided all the spaces correctly match up. For instance with three transformations we require the following three transformations,

$$T_A : \mathbb{R}^n \to \mathbb{R}^k \qquad T_B : \mathbb{R}^k \to \mathbb{R}^p \qquad T_C : \mathbb{R}^p \to \mathbb{R}^m$$

and in this case the composition would be,

$$\left(T_C \circ T_B \circ T_A\right)(\mathbf{x}) = T_C\left(T_B\left(T_A(\mathbf{x})\right)\right) = (CBA)\mathbf{x}$$

Let's take a look at a couple of examples.

Example 8 Determine the matrix inducted by the composition of reflection about the *y*-axis followed by reflection about the *x*-axis.

Solution
First, notice that reflection about the *y*-axis should change the sign on the *x* coordinate and following this by a reflection about the *x*-axis should change the sign on the *y* coordinate.

The two transformations here are,

$$T_A : \mathbb{R}^2 \to \mathbb{R}^2 \qquad A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{reflection about } y\text{-axis}$$

$$T_B : \mathbb{R}^2 \to \mathbb{R}^2 \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{reflection about } x\text{-axis}$$

The matrix induced by the composition is then,

$$T_B \circ T_A = BA = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Let's take a quick look at what this does to a point. Given $\mathbf{x}$ in $\mathbb{R}^2$ we have,

$$\left(T_B \circ T_A\right)(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

This is what we expected to get. This is often called reflection about the origin.

Example 9 Determine the matrix inducted by the composition of reflection about the *y*-axis followed by another reflection about the *y*-axis.

Solution
In this case if we reflect about the *y*-axis twice we should end right back where we started.

The two transformations in this case are,

$$T_A : \mathbb{R}^2 \to \mathbb{R}^2 \qquad A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{reflection about } y\text{-axis}$$

$$T_B : \mathbb{R}^2 \to \mathbb{R}^2 \qquad B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{reflection about } y\text{-axis}$$

The induced matrix is,

$$T_B \circ T_A = BA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

So, the composition of these two transformations yields the identity transformation. So,

$$\left(T_B \circ T_A\right)(\mathbf{x}) = T_I(\mathbf{x}) = \mathbf{x}$$

and the composition will not change the original **x** as we guessed.

Examples of Linear Transformations

This section is going to be mostly devoted to giving the induced matrices for a variety of standard linear transformations. We will be working exclusively with linear transformations of the form $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ and for the most part we'll be providing equations and sketches of the transformations in $\mathbb{R}^2$ but we'll just be providing equations for the $\mathbb{R}^3$ cases.

Let's start this section out with two of the transformations we looked at in the previous section just so we can say we've got all the main examples here in one section.

Zero Transformation

In this case very vector **x** is mapped to the zero vector and so the transformation is,

$$T(\mathbf{x}) = T_0(\mathbf{x})$$

and the induced matrix is the zero matrix, **0**.

Identity Transformation

The identity transformation will map every vector **x** to itself. The transformation is,

$$T(\mathbf{x}) = T_I(\mathbf{x})$$

and so the induced matrix is the identity matrix.

Reflections

We saw a variety of reflections in $\mathbb{R}^2$ in the previous section so we'll give those again here again along with some reflections in $\mathbb{R}^3$ so we can say that we've got all of them written down in this section.

Reflection	Equations	Induced Matrix

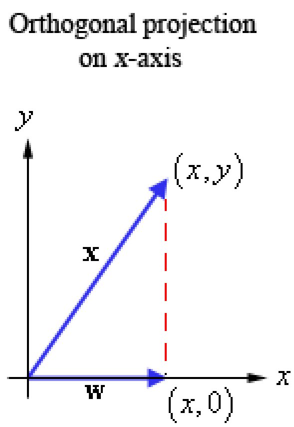
Reflection about x-axis in $\mathbb{R}^2$	$w_1 = x$ $w_2 = -y$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection about y-axis in $\mathbb{R}^2$	$w_1 = -x$ $w_2 = y$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection about line $x = y$ in $\mathbb{R}^2$	$w_1 = y$ $w_2 = x$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Reflection about origin in $\mathbb{R}^2$	$w_1 = -x$ $w_2 = -y$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection about xy-plane in $\mathbb{R}^3$	$w_1 = x$ $w_2 = y$ $w_3 = -z$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
Reflection about yz-plane in $\mathbb{R}^3$	$w_1 = -x$ $w_2 = y$ $w_3 = z$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Reflection about xz-plane in $\mathbb{R}^3$	$w_1 = x$ $w_2 = -y$ $w_3 = z$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

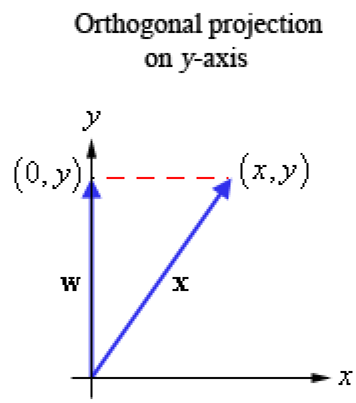
Note that in the $\mathbb{R}^3$ when we say we're reflecting about a given plane, say the *xy*-plane, all we're doing is moving from above the plane to below the plane (or visa-versa of course) and this means simply changing the sign of the other variable, z in the case of the *xy*-plane.

Orthogonal Projections

We first saw orthogonal projections in the section on the dot product. In that section we looked at projections only in the $\mathbb{R}^2$, but as we'll see eventually they can be done in any setting. Here we are going to look at some special orthogonal projections.

Let's start with the orthogonal projections in $\mathbb{R}^2$. There are two of them that we want to look at. Here is a quick sketch of both of these.





So, we project **x** onto the *x*-axis or *y*-axis depending upon which we're after. Of course we also have a variety of projections in $\mathbb{R}^3$ as well. We could project onto one of the three axes or we could project onto one of the three coordinate planes.

Here are the orthogonal projections we're going to look at in this section, their equations and their induced matrix.

Orthogonal Projection	**Equations**	**Induced Matrix**
Projection on *x*-axis in $\mathbb{R}^2$	$w_1 = x$ $w_2 = 0$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Projection on *y*-axis in $\mathbb{R}^2$	$w_1 = 0$ $w_2 = y$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
Projection on *x*-axis in $\mathbb{R}^3$	$w_1 = x$ $w_2 = 0$ $w_3 = 0$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Projection on *y*-axis in $\mathbb{R}^3$	$w_1 = 0$ $w_2 = y$ $w_3 = 0$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Projection on *z*-axis in $\mathbb{R}^3$	$w_1 = 0$ $w_2 = 0$ $w_3 = z$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Projection on *xy*-plane in $\mathbb{R}^3$	$w_1 = x$ $w_2 = y$ $w_3 = 0$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Projection on *yz*-plane in $\mathbb{R}^3$	$w_1 = 0$ $w_2 = y$ $w_3 = z$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Projection on *xz*-plane in $\mathbb{R}^3$	$w_1 = x$ $w_2 = 0$ $w_3 = z$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

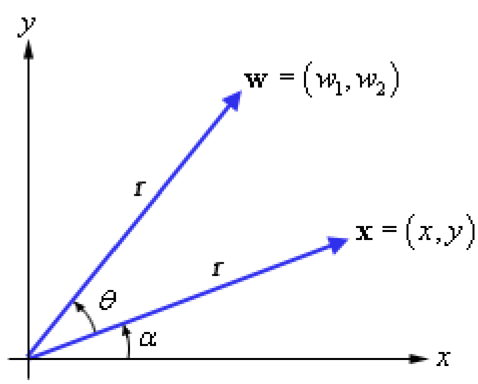
Contractions & Dilations

These transformations are really just fancy names for scalar multiplication, $\mathbf{w} = c\,\mathbf{x}$, where c is a nonnegative scalar. The transformation is called a **contraction** if $0 \le c \le 1$ and a **dilation** if $c \ge 1$. The induced matrix is identical for both a contraction and a dilation and so we'll not give separate equations or induced matrices for both.

Contraction/Dilation	Equations	Induced Matrix
Contraction/Dilation in $\mathbb{R}^2$	$w_1 = cx$ $w_2 = cy$	$\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$
Contraction/Dilation in $\mathbb{R}^3$	$w_1 = cx$ $w_2 = cy$ $w_3 = cz$	$\begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}$

Rotations

We'll start this discussion in $\mathbb{R}^2$. We're going to start with a vector $\mathbf{x}$ and we want to rotate that vector through an angle θ in the counter-clockwise manner as shown below.



Unlike the previous transformation where we could just write down the equations we'll need to do a little derivation work here. First, from our basic knowledge of trigonometry we know that

$$x = r\cos\alpha \qquad\qquad y = r\sin\alpha$$

and we also know that

$$w_1 = r\cos(\alpha + \theta) \qquad\qquad w_2 = r\sin(\alpha + \theta)$$

Now, through a trig formula we can write the equations for $\mathbf{w}$ as follows,

$$w_1 = (r\cos\alpha)\cos\theta - (r\sin\alpha)\sin\theta$$
$$w_2 = (r\cos\alpha)\sin\theta + (r\sin\alpha)\cos\theta$$

Notice that the formulas for *x* and *y* both show up in these formulas so substituting in for those gives,

$$\begin{aligned} w_1 &= x\cos\theta - y\sin\theta \\ w_2 &= x\sin\theta + y\cos\theta \end{aligned}$$

Finally, since θ is a fixed angle $\cos\theta$ and $\sin\theta$ are fixed constants and so there are our equations and the induced matrix is,

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

In $\mathbb{R}^3$ we also have rotations but the derivations are a little trickier. The three that we'll be giving here are counter-clockwise rotation about the three positive coordinate axes.

Here is a table giving all the rotational equation and induced matrices.

Rotation	**Equations**	**Induced Matrix**
Counter-clockwise rotation through an angle θ in $\mathbb{R}^2$	$w_1 = x\cos\theta - y\sin\theta$ $w_2 = x\sin\theta + y\cos\theta$	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
Counter-clockwise rotation trough an angle of θ about the positive *x*-axis in $\mathbb{R}^3$	$w_1 = x$ $w_1 = y\cos\theta - z\sin\theta$ $w_2 = y\sin\theta + z\cos\theta$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
Counter-clockwise rotation trough an angle of θ about the positive *y*-axis in $\mathbb{R}^3$	$w_1 = x\cos\theta + z\sin\theta$ $w_1 = y$ $w_2 = z\cos\theta - x\sin\theta$	$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
Counter-clockwise rotation trough an angle of θ about the positive *z*-axis in $\mathbb{R}^3$	$w_1 = x\cos\theta - y\sin\theta$ $w_1 = x\sin\theta + y\cos\theta$ $w_2 = z$	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Okay, we've given quite a few general formulas here, but we haven't worked any examples with numbers in them so let's do that.

Example 1 Determine the new point after applying the transformation to the given point. Use the induced matrix associated with each transformation to find the new point.

(a) $\mathbf{x} = (2, -4, 1)$ reflected about the *xz*-plane.

(b) $\mathbf{x} = (10, 7, -9)$ projected on the *x*-axis.

(c) $\mathbf{x} = (10, 7, -9)$ projected on the *yz*-plane.

Solution

So, it would be easier to just do all of these directly rather than using the induced matrix, but at least this way we can verify that the induced matrix gives the correct value.

(a) Here's the multiplication for this one.

$$\mathbf{w}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}=\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

So, the point $\mathbf{x}=(2,-4,1)$ maps to $\mathbf{w}=(2,4,1)$ under this transformation.

(b) The multiplication for this problem is,

$$\mathbf{w}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 10 \\ 7 \\ -9 \end{bmatrix}=\begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$

The projection here is $\mathbf{w}=(10,0,0)$

(c) The multiplication for the final transformation in this set is,

$$\mathbf{w}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 10 \\ 7 \\ -9 \end{bmatrix}=\begin{bmatrix} 0 \\ 7 \\ -9 \end{bmatrix}$$

The projection here is $\mathbf{w}=(0,7,-9)$.

Let's take a look at a couple of rotations.

Example 2 Determine the new point after applying the transformation to the given point. Use the induced matrix associated with each transformation to find the new point.

(a) $\mathbf{x}=(2,-6)$ rotated $30°$ in the counter-clockwise direction.

(b) $\mathbf{x}=(0,5,1)$ rotated $90°$ in the counter-clockwise direction about the *y*-axis.

(c) $\mathbf{x}=(-3,4,-2)$ rotated $25°$ in the counter-clockwise direction about the *z*-axis.

Solution

There isn't much to these other than plugging into the appropriate induced matrix and then doing the multiplication.

(a) Here is the work for this rotation.

$$\mathbf{w}=\begin{bmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{bmatrix}\begin{bmatrix} 2 \\ -6 \end{bmatrix}=\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 2 \\ -6 \end{bmatrix}=\begin{bmatrix} 3+\sqrt{3} \\ 1-3\sqrt{3} \end{bmatrix}$$

The new point after this rotation is then, $\mathbf{w}=\left(3+\sqrt{3},\, 1-3\sqrt{3}\right)$.

(b) The matrix multiplication for this rotation is,

$$\mathbf{w}=\begin{bmatrix}\cos 90 & 0 & \sin 90\\ 0 & 1 & 0\\ -\sin 90 & 0 & \cos 90\end{bmatrix}\begin{bmatrix}0\\5\\1\end{bmatrix}=\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ -1 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\5\\1\end{bmatrix}=\begin{bmatrix}1\\5\\0\end{bmatrix}$$

The point after this rotation becomes $\mathbf{w}=(1,5,0)$. Note that we could have predicted this one. The original point was in the *yz*-plane (because the *x* component is zero) and a $90°$ counter-clockwise rotation about the *y*-axis would put the new point in the *xy*-plane with the *z* component becoming the *x* component and that is exactly what we got.

(c) Here's the work for this part and notice that the angle is not one of the "standard" trig angles and so the answers will be in decimals.

$$\begin{aligned}\mathbf{w}&=\begin{bmatrix}\cos 25 & -\sin 25 & 0\\ \sin 25 & \cos 25 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-3\\4\\-2\end{bmatrix}\\ &=\begin{bmatrix}0.9063 & -0.4226 & 0\\ 0.4226 & 0.9063 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-3\\4\\-2\end{bmatrix}\\ &=\begin{bmatrix}-4.4093\\2.3574\\-2\end{bmatrix}\end{aligned}$$

The new point under this rotation is then $\mathbf{w}=(-4.4093,2.3574,-2)$.

Finally, let's take a look at some compositions of transformations.

Example 3 Determine the new point after applying the transformation to the given point. Use the induced matrix associated with each transformation to find the new point.

(a) Dilate $\mathbf{x}=(4,-1,-3)$ by 2 (*i.e.* 2**x**) and then project on the *y*-axis.

(b) Project $\mathbf{x}=(4,-1,-3)$ on the *y*-axis and then dilate by 2.

(c) Project $\mathbf{x}=(4,2)$ on the *x*-axis and the rotate by $45°$ counter-clockwise.

(d) Rotate $\mathbf{x}=(4,2)$ $45°$ counter-clockwise and then project on the *x*-axis.

Solution

Notice that the first two are the same translations just done in the opposite order and the same is true for the last two. Do you expect to get the same result from each composition regardless of the order the transformations are done?

Recall as well that in compositions we can get the induced matrix by multiplying the induced matrices from each transformation from the right to left in the order they are applied. For instance the induced matrix for the composition $T_B \circ T_A$ is *BA* where T_A is the first transformation applied to the point.

(a) The induced matrix for this composition is,

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Project on } y\text{-axis}} \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{\text{Dilate by 2}} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Composition}}$$

The matrix multiplication for the new point is then,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

The new point is then $\mathbf{w} = (0, -2, 0)$.

(b) In this case the induced matrix is,

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{\text{Dilate by 2}} \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Project on } y\text{-axis}} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Composition}}$$

So, in this case the induced matrix for the composition is the same as the previous part. Therefore, the new point is also the same, $\mathbf{w} = (0, -2, 0)$.

(c) Here is the induced matrix for this composition.

$$\underbrace{\begin{bmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{bmatrix}}_{\text{Rotate by } 45^\circ} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{\substack{\text{Project on} \\ x\text{-axis}}} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

The matrix multiplication for new point after applying this composition is,

$$\mathbf{w} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} \\ 2\sqrt{2} \end{bmatrix}$$

The new point is then, $\mathbf{w} = \left(2\sqrt{2},\ 2\sqrt{2}\right)$

(d) The induced matrix for the final composition is,

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{\substack{\text{Project on} \\ x\text{-axis}}} \underbrace{\begin{bmatrix} \cos 45 & -\sin 45 \\ \sin 45 & \cos 45 \end{bmatrix}}_{\text{Rotate by } 45^\circ} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 0 \end{bmatrix}$$

Note that this is different from the induced matrix from **(c)** and so we should expect the new point to also be different. The fact that the induced matrix is different shouldn't be too surprising given that matrix multiplication is not a commutative operation.

The matrix multiplication for the new point is,

$$\mathbf{w} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$

The new point is then, $\mathbf{w} = \left(\sqrt{2},\ 0\right)$ and as we expected it was not the same as that from part **(c)**.

So, as this example has shown us transformation composition is not necessarily commutative and so we shouldn't expect that to happen in most cases.

Vector Spaces

Introduction

In the previous chapter we looked at vectors in Euclidean *n*-space and while in $\mathbb{R}^2$ and $\mathbb{R}^3$ we thought of vectors as directed line segments. A vector however, is a much more general concept and it doesn't necessarily have to represent a directed line segment in $\mathbb{R}^2$ or $\mathbb{R}^3$. Nor does a vector have to represent the vectors we looked at in $\mathbb{R}^n$. As we'll soon see a vector can be a matrix or a function and that's only a couple of possibilities for vectors. With all that said a good many of our examples will be examples from $\mathbb{R}^n$ since that is a setting that most people are familiar with and/or can visualize. We will however try to include the occasional example that does not lie in $\mathbb{R}^n$.

The main idea of study in this chapter is that of a vector space. A vector space is nothing more than a collection of vectors (whatever those now are…) that satisfies a set of axioms. Once we get the general definition of a vector and a vector space out of the way we'll look at many of the important ideas that come with vector spaces. Towards the end of the chapter we'll take a quick look at inner product spaces.

Here is a listing of the topics in this chapter.

Change of Basis – In this section we will see how to change the set of basis vectors for a vector space.

Fundamental Subspaces – Here we will take a look at some of the fundamental subspaces of a matrix, including the row space, column space and null space.

Inner Product Spaces – We will be looking at a special kind of vector spaces in this section as well as define the inner product.

Orthonormal Basis – In this section we will develop and use the Gram-Schmidt process for constructing an orthogonal/orthonormal basis for an inner product space.

Least Squares – In this section we'll take a look at an application of some of the ideas that we will be discussing in this chapter.

***QR*-Decomposition** – Here we will take a look at the *QR*-Decomposition for a matrix and how it can be used in the least squares process.

Orthogonal Matrices – We will take a look at a special kind of matrix, the orthogonal matrix, in this section.

Vector Spaces

As noted in the introduction to this chapter vectors do not have to represent directed line segments in space. When we first start looking at many of the concepts of a vector space we usually start with the directed line segment idea and their natural extension to vectors in $\mathbb{R}^n$ because it is something that most people can visualize and get their hands on. So, the first thing that we need to do in this chapter is to define just what a vector space is and just what vectors really are.

However, before we actually do that we should point out that because most people can visualize directed line segments most of our examples in these notes will revolve around vectors in $\mathbb{R}^n$. We will try to always include an example or two with vectors that aren't in $\mathbb{R}^n$ just to make sure that we don't forget that vectors are more general objects, but the reality is that most of the examples will be in $\mathbb{R}^n$.

So, with all that out of the way let's go ahead and get the definition of a vector and a vector space out of the way.

Definition 1 Let V be a set on which addition and scalar multiplication are defined (this means that if $\mathbf{u}$ and $\mathbf{v}$ are objects in V and c is a scalar then we've defined $\mathbf{u}+\mathbf{v}$ and $c\mathbf{u}$ in some way). If the following axioms are true for all objects $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and all scalars c and k then V is called a **vector space** and the objects in V are called **vectors**.

(a) $\mathbf{u}+\mathbf{v}$ is in V – This is called **closed under addition**.
(b) $c\mathbf{u}$ is in V – This is called **closed under scalar multiplication**.
(c) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
(d) $\mathbf{u}+\left(\mathbf{v}+\mathbf{w}\right)=\left(\mathbf{u}+\mathbf{v}\right)+\mathbf{w}$
(e) There is a special object in V, denoted $\mathbf{0}$ and called the **zero vector**, such that for all $\mathbf{u}$ in V we have $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$.
(f) For every $\mathbf{u}$ in V there is another object in V, denoted $-\mathbf{u}$ and called the **negative** of $\mathbf{u}$, such that $\mathbf{u}-\mathbf{u}=\mathbf{u}+\left(-\mathbf{u}\right)=\mathbf{0}$.
(g) $c\left(\mathbf{u}+\mathbf{v}\right)=c\mathbf{u}+c\mathbf{v}$
(h) $\left(c+k\right)\mathbf{u}=c\mathbf{u}+k\mathbf{u}$
(i) $c\left(k\mathbf{u}\right)=\left(ck\right)\mathbf{u}$
(j) $1\mathbf{u}=\mathbf{u}$

We should make a couple of comments about these axioms at this point. First, do not get too locked into the "standard" ways of defining addition and scalar multiplication. For the most part we will be doing addition and scalar multiplication in a fairly standard way, but there will be the occasional example where we won't. In order for something to be a vector space it simply must have an addition and scalar multiplication that meets the above axioms and it doesn't matter how strange the addition of scalar multiplication might be.

Next, the first two axioms may seem a little strange at first glance. It might seem like these two will be trivially true for any definition of addition or scalar multiplication, however, we will see at least one example in this section of a set that is not closed under a particular scalar multiplication.

Finally, with the exception of the first two these axioms should all seem familiar to you. All of these axioms were in one of the theorems from the discussion on vectors and/or Euclidean n-space in the previous chapter. However, in this case they aren't properties, they are axioms. What that means is that they aren't to be proven. Axioms are simply the rules under which we're going to operate when we work with vector spaces. Given a definition of addition and scalar multiplication we'll simply need to verify that the above axioms are satisfied by our definitions.

We should also make a quick comment about the scalars that we'll be using here. To this point, and in all the examples we'll be looking at in the future, the scalars are real numbers. However, they don't have to be real numbers. They could be complex numbers. When we restrict the scalars to real numbers we generally call the vector space a **real vector space** and when we allow the scalars to be complex numbers we generally call the vector space a **complex vector space**. We will be working exclusively with real vector spaces and from this point on when we saw vector space it is to be understood that we mean a real vector space.

We should now look at some examples of vector spaces and at least a couple of examples of sets that aren't vector spaces. Some of these will be fairly standard vector spaces while others may seem a little strange at first but are fairly important to other areas of mathematics.

Example 1 If n is any positive integer then the set $V = \mathbb{R}^n$ with the standard addition and scalar multiplication as defined in the Euclidean *n*-space section is a vector space.

Technically we should show that the axioms are all met here, however that was done in Theorem 1 from the Euclidean *n*-space section and so we won't do that for this example.

Note that from this point on when we refer to the standard vector addition and standard vector scalar multiplication we are referring to that we defined in the Euclidean *n*-space section.

Example 2 The set $V = \mathbb{R}^2$ with the standard vector addition and scalar multiplication defined as,

$$c(u_1, u_2) = (u_1, cu_2)$$

is NOT a vector space.

Showing that something is not a vector space can be tricky because it's completely possible that only one of the axioms fails. In this case because we're dealing with the standard addition all the axioms involving the addition of objects from V (**a, c, d, e,** and **f**) will be valid.

Also, in this case of all the axioms involving the scalar multiplication (**b, g, h, i,** and **j**), only **(h)** is not valid. We'll show this in a bit, but the point needs to be made here that only one of the axioms will fail in this case and that is enough for this set under this definition of addition and multiplication to not be a vector space.

First we should at least show that the set meets axiom **(b)** and this is easy enough to show, in that we can see that the result of the scalar multiplication is again a point in $\mathbb{R}^2$ and so the set is closed under scalar multiplication. Again, do not get used to this happening. We will see at least one example later in this section of a set that is not closed under scalar multiplication as we'll define it there.

Now, to show that **(h)** is not valid we'll need to compute both sides of the equality and show that they aren't equal.

$$(c+k)\mathbf{u} = (c+k)(u_1, u_2) = (u_1, (c+k)u_2) = (u_1, cu_2 + ku_2)$$

$$c\mathbf{u} + k\mathbf{u} = c(u_1, u_2) + k(u_1, u_2) = (u_1, cu_2) + (u_1, ku_2) = (2u_1, cu_2 + ku_2)$$

So, we can see that $(c+k)\mathbf{u} \neq c\mathbf{u} + k\mathbf{u}$ because the first components are not the same. This means that axiom **(h)** is not valid for this definition of scalar multiplication.

We'll not verify that the remaining scalar multiplication axioms are valid for this definition of scalar multiplication. We'll leave those to you. All you need to do is compute both sides of the equal sign and show that you get the same thing on each side.

Example 3 The set $V = \mathbb{R}^3$ with the standard vector addition and scalar multiplication defined as,

$$c(u_1, u_2, u_3) = (0, 0, cu_3)$$

is NOT a vector space.

Again, there is a single axiom that fails in this case. We'll leave it to you to verify that the others hold. In this case it is the last axiom, **(j)**, that fails as the following work shows.

$$1\mathbf{u} = 1(u_1, u_2, u_3) = (0, 0, (1)u_3) = (0, 0, u_3) \neq (u_1, u_2, u_3) = \mathbf{u}$$

Example 4 The set $V = \mathbb{R}^2$ with the standard scalar multiplication and addition defined as,

$$(u_1, u_2) + (v_1 + v_2) = (u_1 + 2v_1,\ u_2 + v_2)$$

Is NOT a vector space.

To see that this is not a vector space let's take a look at the axiom **(c)**.

$$\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1 + v_2) = (u_1 + 2v_1,\ u_2 + v_2)$$

$$\mathbf{v} + \mathbf{u} = (v_1 + v_2) + (u_1, u_2) = (v_1 + 2u_1,\ v_2 + u_2)$$

So, because only the first component of the second point listed gets multiplied by 2 we can see that $\mathbf{u} + \mathbf{v} \neq \mathbf{v} + \mathbf{u}$ and so this is not a vector space.

You should go through the other axioms and determine if they are valid or not for the practice.

So, we've now seen three examples of sets of the form $V = \mathbb{R}^n$ that are NOT vector spaces so hopefully it is clear that there are sets out there that aren't vector spaces. In each case we had to change the definition of scalar multiplication or addition to make the set fail to be a vector space. However, don't read too much into that. It is possible for a set under the standard scalar multiplication and addition to fail to be a vector space as we'll see in a bit. Likewise, it's possible for a set of this form to have a non-standard scalar multiplication and/or addition and still be a vector space.

In fact, let's take a look at the following example. This is probably going to be the only example that we're going to go through and do in excruciating detail in this section. We're doing this for two reasons. First, you really should see all the detail that needs to go into actually showing that a set along with a definition of addition and scalar multiplication is a vector space. Second, our definitions are NOT going to be standard here and it would be easy to get confused with the details if you had to go through them on your own.

Example 5 Suppose that the set *V* is the set of positive real numbers (*i.e.* $x > 0$) with addition and scalar multiplication defined as follows,

$$x + y = xy \qquad\qquad cx = x^c$$

This set under this addition and scalar multiplication is a vector space.

First notice that we're taking *V* to be only a portion of $\mathbb{R}$. If we took it to be all of $\mathbb{R}$ we would not have a vector space. Next, do not get excited about the definitions of "addition" and "scalar multiplication" here. Even though they are not they are not addition and scalar multiplication as we think of them we are still going to call them the addition and scalar multiplication operations for this vector space.

Okay, let's go through each of the axioms and verify that they are valid.

First let's take a look at the closure axioms, **(a)** and **(b)**. Since by *x* and *y* are positive numbers their product *xy* is a positive real number and so the *V* is closed under addition. Since *x* is positive then for any c x^c is a positive real number and so *V* is closed under scalar multiplication.

Next we'll verify **(c)**. We'll do this one with some detail pointing out how we do each step. First assume that *x* and *y* are any two elements of *V* (*i.e.* they are two positive real numbers).

Multiplication of real numbers is commutative

$$x + y = xy = yx = y + x$$

By Definition

We'll now verify **(d)**. Again, we'll make it clear how we're going about each step with this one. Assume that *x*, *y*, and *z* are any three elements of *V*.

$$\begin{aligned} x+(y+z) &= x+(yz) && \text{By Definition of Addition} \\ &= x(yz) && \\ &= (xy)z && \text{Multiplication of real numbers is associative} \\ &= (xy)+z && \text{By Definition of Addition} \\ &= (x+y)+z && \end{aligned}$$

Next we need to find the zero vector, **0**, and we need to be careful here. We use **0** to denote the zero vector but it does NOT have to be the number zero. In fact in this case it can't be zero if for no other reason than the fact that the number zero isn't in the set *V* ! We need to find an element that is in *V* so that under our definition of addition we have,

$$x + \mathbf{0} = \mathbf{0} + x = x$$

It looks like we should define the "zero vector" in this case as : **0**=1. In other words the zero vector for this set will be the number 1! Let's see how that works and remember that our "addition" here is really multiplication and remember to substitute the number 1 in for **0**. If *x* is any element of *V*,

$$x+\mathbf{0}=x\cdot 1=x \qquad \& \qquad \mathbf{0}+x=1\cdot x=x$$

Sure enough that does what we want it to do.

We next need to define the negative, $-x$, for each element *x* that is in *V*. As with the zero vector to not confuse $-x$ with "minus *x*", this is just the notation we use to denote the negative of *x*. In our case we need an element of *V* (so it can't be minus *x* since that isn't in *V*) such that

$$x+\left(-x\right)=\mathbf{0}$$

and remember that **0**=1 in our case!

Given an *x* in *V* we know that *x* is strictly positive and so $\frac{1}{x}$ is defined (since *x* isn't zero) and is positive (since *x* is positive) and therefore $\frac{1}{x}$ is in *V*. Also, under our definition of addition and the zero vector we have,

$$x+\left(-x\right)=x\cdot\frac{1}{x}=1=\mathbf{0}$$

Therefore, for the set *V* the negative of *x* is $-x=\frac{1}{x}$.

So, at this point we've taken care of the closure and addition axioms we not just need to deal with the axioms relating to scalar multiplication.

We'll start with **(g)**. We'll do this one in some detail so you can see what we're doing at each step. If *x* and *y* are any two elements of *V* and *c* is any scalar then,

$$\begin{aligned} c\left(x+y\right) &= c\left(xy\right) && \longleftarrow \text{Definition of Addition} \\ &= \left(xy\right)^c && \longleftarrow \text{Definition of Scalar Multiplication} \\ &= x^c y^c && \longleftarrow \text{Property of Real Numbers and Exponents} \\ &= x^c + y^c && \longleftarrow \text{Definition of Addition} \\ &= cx + cy && \longleftarrow \text{Definition of Scalar Multiplication} \end{aligned}$$

So, it looks like we've verified **(g)**.

Let's now verify **(h)**. If *x* is any element of *V* and *c* and *k* are any two scalars then,

$$
\begin{aligned}
(c+k)x &= x^{c+k} && \longleftarrow \text{Definition of Scalar Multiplication} \\
&= x^c x^k && \longleftarrow \text{Property of Real Numbers and Exponents} \\
&= x^c + x^k && \longleftarrow \text{Definition of Addition} \\
&= cx + kx && \longleftarrow \text{Definition of Scalar Multiplication}
\end{aligned}
$$

So, this axiom is verified. Now, let's verify **(i)**. If x is any element of V and c and k are any two scalars then,

$$
\begin{aligned}
c(kx) &= c\left(x^k\right) && \longleftarrow \text{Definition of Scalar Multiplication} \\
&= \left(x^k\right)^c && \longleftarrow \text{Definition of Scalar Multiplication} \\
&= x^{ck} && \longleftarrow \text{Property of Real Numbers and Exponents} \\
&= (ck)x && \longleftarrow \text{Definition of Scalar Multiplication}
\end{aligned}
$$

We've got the final axiom to go here and that's a fairly simple one to verify.

$$1x = x^1 = x$$

Just remember that $1x$ is the notation for scalar multiplication and NOT multiplication of x by the number 1.

Okay, that was a lot of work and we're not going to be showing that much work in the remainder of the examples that are vector spaces. We'll leave that up to you to check most of the axioms now that you've seen one done completely out. For those examples that aren't a vector space we'll show the details on at least one of the axioms that fails. For these examples you should check the other axioms to see if they are valid or fail.

Example 6 Let the set V be the points on a line through the origin in $\mathbb{R}^2$ with the standard addition and scalar multiplication. Then V is a vector space.

First, let's think about just what V is. The set V is all the points that are on some line through the origin in $\mathbb{R}^2$. So, we know that the line must have the equation,

$$ax + by = 0$$

for some a and some b, at least one not zero. Also note that a and b are fixed constants and aren't allowed to change. In other words we are always on the same line. Now, a point (x_1, y_1) will be on the line, and hence in V, provided it satisfies the equation above.

We'll show that V is closed under addition and scalar multiplication and leave it to you to verify the remaining axioms. Let's first show that V is closed under addition. To do this we'll need the sum of two random points from V, say $\mathbf{u} = (x_1, y_1)$ and $\mathbf{v} = (x_2, y_2)$, and we'll need to show that $\mathbf{u} + \mathbf{v} = (x_1 + x_2, y_1 + y_2)$ is also in V. This amounts to showing that this point satisfies the equation of the line and that's fairly simple to do, just plug the coordinates into the equation and verify we get zero.

$$a\left(x_1+x_2\right)+b\left(y_1+y_2\right)=\left(ax_1+by_1\right)+\left(ax_2+by_2\right)=0+0=0$$

So, after some rearranging and using the fact that both **u** and **v** were both in *V* (and so satisfied the equation of the line) we see that the sum also satisfied the line and so is in *V*. We've now shown that *V* is closed under addition.

To show that *V* is closed under scalar multiplication we'll need to show that for any **u** from *V* and any scalar, *c*, the $c\mathbf{u}=\left(cx_1,cy_1\right)$ is also in *V*. This is done pretty much as we did closed under addition.

$$a\left(cx_1\right)+b\left(cy_1\right)=c\left(ax_1+by_1\right)=c\left(0\right)=0$$

So, $c\mathbf{u}$ is on the line and hence in *V*. *V* is therefore closed under scalar multiplication.

Again we'll leave it to you to verify the remaining axioms. Note however, that because we're working with the standard addition that the zero vector and negative are the standard zero vector and negative that we're used to dealing with,

$$\mathbf{0}=\left(0,0\right) \qquad\qquad -\mathbf{u}=-\left(x_1,y_1\right)=\left(-x_1,-y_1\right)$$

Note that we can extend this example to a line through the origin in $\mathbb{R}^n$ and still have a vector space. Showing that this set is a vector space can be a little difficult if you don't know the equation of a line in $\mathbb{R}^n$ however, as many of you probably don't, and so we won't show the work here.

Example 7 Let the set *V* be the points on a line that does NOT go through the origin in $\mathbb{R}^2$ with the standard addition and scalar multiplication. Then *V* is not a vector space.

In this case the equation of the line will be,

$$ax+by=c$$

for fixed constants *a*, *b*, and *c* where at least one of *a* and *b* is non-zero and *c* is not zero. This set is not closed under addition or scalar multiplication. Here is the work showing that it's not closed under addition. Let $\mathbf{u}=\left(x_1,y_1\right)$ and $\mathbf{v}=\left(x_2,y_2\right)$ be any two points from *V* (and so they satisfy the equation above). Then,

$$a\left(x_1+x_2\right)+b\left(y_1+y_2\right)=\left(ax_1+by_1\right)+\left(ax_2+by_2\right)=c+c=2c\neq c$$

So the sum, $\mathbf{u}+\mathbf{v}=\left(x_1+x_2,y_1+y_2\right)$, does not satisfy the equation and hence is not in *V* and so *V* is not closed under addition.

We'll leave it to you to verify that this particular *V* is not closed under scalar multiplication.

Also, note that since we are working on a set of points from $\mathbb{R}^2$ with the standard addition then the zero vector must be $\mathbf{0} = (0,0)$, but because this doesn't satisfy the equation it is not in *V* and so axiom **(e)** is also not satisfied. In order for *V* to be a vector space it must contain the zero vector **0**!

You should go through the remaining axioms and see if there are any others that fail.

Before moving on we should note that prior to this example all the sets that have not been vector spaces we've not been operating under the standard addition and/or scalar multiplication. In this example we've now seen that for some sets under the standard addition and scalar multiplication will not be vector spaces either.

Example 8 Let the set *V* be the points on a plane through the origin in $\mathbb{R}^3$ with the standard addition and scalar multiplication. Then *V* is a vector space.

The equation of a plane through the in $\mathbb{R}^3$ is,

$$ax + by + cz = 0$$

where *a*, *b*, and *c* are fixed constants and at least one is not zero.

Given the equation you can (hopefully) see that this will work in pretty much the same manner as the Example 6 and so we won't show any work here.

Okay, we've seen quite a few examples to this point, but they've all involved sets that were some or all of $\mathbb{R}^n$ and so we now need to see a couple of examples of vector spaces whose elements (and hence the "vectors" of the set) are not points in $\mathbb{R}^n$.

Example 9 Let *n* and *m* be fixed numbers and let M_{nm} represent the set of all $n \times m$ matrices. Also let addition and scalar multiplication on M_{nm} be the standard matrix addition and standard matrix scalar multiplication. Then M_{nm} is a vector space.

If we let *c* be any scalar and let the "vectors" **u** and **v** represent any two $n \times m$ matrices (*i.e.* they are both objects in M_{nm}) then we know from our work in the first chapter that the sum, $\mathbf{u} + \mathbf{v}$, and the scalar multiple, *c***u**, are also $n \times m$ matrices and hence are in M_{nm}. So M_{nm} is closed under addition and scalar multiplication.

Next, if we define the zero vector, **0**, to be the $n \times m$ zero matrix and if the "vector" **u** is some $n \times m$, *A*, we can define the negative, $-\mathbf{u}$, to be the matrix -*A* then the properties of matrix arithmetic will show that the remainder of the axioms are valid.

Therefore, M_{nm} is a vector space.

Note that this example now gives us a whole host of new vector spaces. For instance, the set of 2×2 matrices, M_{22}, is a vector space and the set of all 5×9 matrices, M_{59}, is a vector space, *etc*.

Also, the "vectors" in this vector space are really matrices!

Here's another important example that may appear to be even stranger yet.

Example 10 Let $F\left[a,b\right]$ be the set of all real valued functions that are defined on the interval $\left[a,b\right]$. Then given any two "vectors", $\mathbf{f} = f\left(x\right)$ and $\mathbf{g} = g\left(x\right)$, from $F\left[a,b\right]$ and any scalar *c* define addition and scalar multiplication as,

$$\left(\mathbf{f}+\mathbf{g}\right)\left(x\right) = f\left(x\right)+g\left(x\right) \qquad\qquad \left(c\mathbf{f}\right)\left(x\right) = cf\left(x\right)$$

Under these operations $F\left[a,b\right]$ is a vector space.

By assumption both **f** and **g** are real valued and defined on $\left[a,b\right]$. Then, for both addition and scalar multiplication we just going to plug *x* into both $f\left(x\right)$ and/or $g\left(x\right)$ and both of these are defined and so the sum or the product with a scalar will also be defined and so this space is closed under addition and scalar multiplication.

The "zero vector", **0**, for $F\left[a,b\right]$ is the zero function, *i.e.* the function that is zero for all *x*, and the negative of the "vector" **f** is the "vector" $-\mathbf{f} = -f\left(x\right)$.

We should make a couple of quick comments about this vector space before we move on. First, recall that the $\left[a,b\right]$ represents the interval $a \le x \le b$ (*i.e.* we include the endpoints). We could also look at the set $F\left(a,b\right)$ which is the set of all real valued functions that are defined on $\left(a,b\right)$ ($a < x < b$, no endpoints) or $F\left(-\infty,\infty\right)$ the set of all real valued functions defined on $\left(-\infty,\infty\right)$ and we'll still have a vector space.

Also, depending upon the interval we choose to work with we may have a different set of functions in the set. For instance, the function $\frac{1}{x}$ would be in $F\left[2,10\right]$ but not in $F\left[-3,6\right]$ because of division by zero.

In this case the "vectors" are now functions so again we need to be careful with the term vector. In can mean a lot of different things depending upon what type of vector space we're working with.

Both of the vector spaces from Examples 9 and 10 are fairly important vector spaces and as we'll look at them again in the next section where we'll see some examples of some related vector spaces.

There is one final example that we need to look at in this section.

Example 11 Let V consist of a single object, denoted by $\mathbf{0}$, and define

$$\mathbf{0}+\mathbf{0}=0 \qquad\qquad c\,\mathbf{0}=0$$

Then V is a vector space and is called the **zero vector space**.

The last thing that we need to do in this section before moving on is to get a nice set of facts that fall pretty much directly from the axioms and will be true for all vector spaces.

Theorem 1 Suppose that V is a vector space, $\mathbf{u}$ is a vector in V and c is any scalar. Then,

(a) $0\mathbf{u}=\mathbf{0}$

(b) $c\mathbf{0}=\mathbf{0}$

(c) $\left(-1\right)\mathbf{u}=-\mathbf{u}$

(d) If $c\,\mathbf{u}=\mathbf{0}$ then either $c=0$ or $\mathbf{u}=\mathbf{0}$

The proofs of these are really quite simple, but they only appear that way after you've seen them. Coming up with them on your own can be difficult sometimes. We'll give the proof for two of them and you should try and prove the other two on your own.

Proof :
(a) Now, this can seem tricky, but each of these steps will come straight from a property of real numbers or one of the axioms above. We'll start with $0\mathbf{u}$ and use the fact that we can always write $0=0+0$ and then we'll use axiom **(h)**.

$$0\mathbf{u}=\left(0+0\right)\mathbf{u}=0\mathbf{u}+0\mathbf{u}$$

This may have seemed like a silly and/or strange step, but it was required. We couldn't just add a $0\mathbf{u}$ onto one side because this would, in essence, be using the fact that $0\mathbf{u}=\mathbf{0}$ and that's what we're trying to prove!

So, while we don't know just what $0\mathbf{u}$ is as a vector, it is in the vector space and so we know from axiom **(f)** that it has a negative which we'll denote by $-0\mathbf{u}$. Add the negative to both sides and then use axiom **(f)** again to say that $0\mathbf{u}+\left(-0\mathbf{u}\right)=\mathbf{0}$

$$0\mathbf{u}+\left(-0\mathbf{u}\right)=0\mathbf{u}+0\mathbf{u}+\left(-0\mathbf{u}\right)$$
$$\mathbf{0}=0\mathbf{u}+\mathbf{0}$$

Finally, use axiom **(e)** on the right side to get,

$$\mathbf{0}=0\mathbf{u}$$

and we've proven **(a)**.

(c) In this case if we can show that $\mathbf{u}+(-1)\mathbf{u}=\mathbf{0}$ then from axiom **(f)** we'll know that $(-1)\mathbf{u}$ is the negative of **u**, or in other words that $(-1)\mathbf{u}=-\mathbf{u}$. This isn't too hard to show. We'll start with $\mathbf{u}+(-1)\mathbf{u}$ and use axiom **(j)** to rewrite the first **u** as follows,

$$\mathbf{u}+(-1)\mathbf{u}=1\mathbf{u}+(-1)\mathbf{u}$$

Next, use axiom **(h)** on the right side and then a nice property of real numbers.

$$\begin{aligned}\mathbf{u}+(-1)\mathbf{u}&=(1+(-1))\mathbf{u}\\&=0\mathbf{u}\end{aligned}$$

Finally, use part **(a)** of this theorem on the right side and we get,

$$\mathbf{u}+(-1)\mathbf{u}=\mathbf{0}$$

■

Subspaces

Let's go back to the previous section for a second and examine Example 1 and Example 6. In Example 1 we saw that $\mathbb{R}^n$ was a vector space with the standard addition and scalar multiplication for any positive integer *n*. So, in particular $\mathbb{R}^2$ is a vector space with the standard addition and scalar multiplication. In Example 6 we saw that the set of points on a line through the origin in $\mathbb{R}^2$ with the standard addition and vector space multiplication was also a vector space.

So, just what is so important about these two examples? Well first notice that they both are using the same addition and scalar multiplication. In and of itself that isn't important, but it will be important for the end result of what we want to discus here. Next, the set of points in the vector space of Example 6 are also in the set of points in the vector space of Example 1. While it's not important to the discussion here note that the opposite isn't true, given a line we can find points in $\mathbb{R}^2$ that aren't on the line.

What we've seen here is that, at least for some vector spaces, it is possible to take certain subsets of the original vector space and as long as we retain the definition of addition and scalar multiplication we will get a new vector space. Of course, it is possible for some subsets to not be a new vector space. To see an example of this see Example 7 from the previous section. In that example we've got a subset of $\mathbb{R}^2$ with the standard addition and scalar multiplication and yet it's not a vector space.

We want to investigate this idea in more detail and we'll start off with the following definition.

Definition 1 Suppose that *V* is a vector space and *W* is a subset of *V*. If, under the addition and scalar multiplication that is defined on *V*, *W* is also a vector space then we call *W* a **subspace** of *V*.

Now, technically if we wanted to show that a subset W of a vector space V was a subspace we'd need to show that all 10 of the axioms from the definition of a vector space are valid, however, in reality that doesn't need to be done.

Many of the axioms (**c**, **d**, **g**, **h**, **i**, and **j**) deal with how addition and scalar multiplication work, but W is inheriting the definition of addition and scalar multiplication from V. Therefore, since elements of W are also elements of V the six axioms listed above are guaranteed to be valid on W.

The only ones that we really need to worry about are the remaining four, all of which require something to be in the subset W. The first two (**a**, and **b**) are the closure axioms that require that the sum of any two elements from W is back in W and that the scalar multiple of any element from W will be back in W. Note that the sum and scalar multiple will be in V we just don't know if it will be in W. We also need to verify that the zero vector (axiom **e**) is in W and that each element of W has a negative that is also in W (axiom **f**).

As the following theorem shows however, the only two axioms that we really need to worry about are the two closure axioms. Once we have those two axioms valid, we will get the zero vector and negative vector for free.

Theorem 1 Suppose that W is a non-empty (*i.e.* at least one element in it) subset of the vector space V then W will be a subspace if the following two conditions are true.

(a) If $\mathbf{u}$ and $\mathbf{v}$ are in W then $\mathbf{u}+\mathbf{v}$ is also in W (*i.e.* W is closed under addition).

(b) If $\mathbf{u}$ is in W and c is any scalar then $c\mathbf{u}$ is also in W (*i.e.* W is closed under scalar multiplication).

Where the definition of addition and scalar multiplication on W are the same as on V.

Proof : To prove this theorem all we need to do is show that if we assume the two closure axioms are valid the other 8 axioms will be given to us for free. As we discussed above the axioms **c**, **d**, **g**, **h**, **i**, and **j** are true simply based on the fact that W is a subset of V and it uses the same addition and scalar multiplication and so we get these for free.

We only need to verify that assuming the two closure condition we get axioms **e** and **f** as well. From the second condition above we see that we are assuming that W is closed under scalar multiplication and so both $0\mathbf{u}$ and $(-1)\mathbf{u}$ must be in W, but from Theorem 1 from the previous section we know that,

$$0\mathbf{u}=\mathbf{0} \qquad\qquad (-1)\mathbf{u}=-\mathbf{u}$$

But this means that the zero vector and the negative of $\mathbf{u}$ must be in W and so we're done.
■

Be careful with this proof. On the surface it may look like we never used the first condition of closure under addition and we didn't use that to show that axioms **e** and **f** were valid. However, in order for W to be a vector space it must be closed under addition and so without that first condition we can't know whether or not W is in fact a vector space. Therefore, even though we didn't explicitly use it in the proof it was required in order to guarantee that we have a vector space.

Next we should acknowledge the following fact.

Fact

Every vector space, V, has at least two subspaces. Namely, V itself and $W = \{\mathbf{0}\}$ (the zero space).

Because V can be thought of as a subset of itself we can also think of it as a subspace of itself. Also, the zero space which is the vector space consisting only of the zero vector, $W = \{\mathbf{0}\}$ is a subset of V and is a vector space in its own right and so will be a subspace of V.

At this point we should probably take a look at some examples. In all of these examples we assume that the standard addition and scalar multiplication are being used in each case unless otherwise stated.

Example 1 Determine if the given set is a subspace of the given vector space.

(a) Let W be the set of all points, (x, y), from $\mathbb{R}^2$ in which $x \geq 0$. Is this a subspace of $\mathbb{R}^2$?

(b) Let W be the set of all points from $\mathbb{R}^3$ of the form $(0, x_2, x_3)$. Is this a subspace of $\mathbb{R}^3$?

(c) Let W be the set of all points from $\mathbb{R}^3$ of the form $(1, x_2, x_3)$. Is this a subspace of $\mathbb{R}^3$?

Solution

In each of these cases we need to show either that the set is closed under addition and scalar multiplication or it is not closed for at least one of those.

(a) This set is closed under addition because,

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and since $x_1, x_2 \geq 0$ we also have $x_1 + x_2 \geq 0$ and so the resultant point is back in W.

However, this set is not closed under scalar multiplication. Let c be any *negative* scalar and further assume that $x > 0$ then,

$$c(x, y) = (cx, cy)$$

Then because $x > 0$ and $c < 0$ we must have $cx < 0$ and so the resultant point is not in W because the first component is neither zero nor positive.

Therefore, W is not a subspace of V.

(b) This one is fairly simple to check a point will be in *W* if the first component is zero. So, let $\mathbf{x} = \left(0, x_2, x_3\right)$ and $\mathbf{y} = \left(0, y_2, y_3\right)$ be any two points in *W* and let *c* be any scalar then,

$$\mathbf{x}+\mathbf{y} = \left(0, x_2, x_3\right) + \left(0, y_2, y_3\right) = \left(0, x_2+y_2, x_3+y_3\right)$$
$$c\mathbf{x} = \left(0, cx_2, cx_3\right)$$

So, both $\mathbf{x}+\mathbf{y}$ and $c\mathbf{x}$ are in *W* and so *W* is closed under addition and scalar multiplication and so *W* is a subspace.

(c) This one is here just to keep us from making any assumptions based on the previous part. This set is closed under neither addition nor scalar multiplication. In order for points to be in *W* in this case the first component must be a 1. However, if $\mathbf{x} = \left(1, x_2, x_3\right)$ and $\mathbf{y} = \left(1, y_2, y_3\right)$ be any two points in *W* and let *c* be any scalar other than 1 we get,

$$\mathbf{x}+\mathbf{y} = \left(1, x_2, x_3\right) + \left(1, y_2, y_3\right) = \left(2, x_2+y_2, x_3+y_3\right)$$
$$c\mathbf{x} = \left(c, cx_2, cx_3\right)$$

Neither of which is in *W* and so *W* is not a subspace.

Example 2 Determine if the given set is a subspace of the given vector space.

(a) Let *W* be the set of diagonal matrices of size $n \times n$. Is this a subspace of M_{nn}?

(b) Let *W* be the set of matrices of the form $\begin{bmatrix} 0 & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$.Is this a subspace of M_{32}?

(c) Let *W* be the set of matrices of the form $\begin{bmatrix} 2 & a_{12} \\ 0 & a_{22} \end{bmatrix}$.Is this a subspace of M_{22}?

Solution

(a) Let **u** and **v** be any two $n \times n$ diagonal matrices and *c* be any scalar then,

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_1 & 0 & \cdots & 0 \\ 0 & u_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_n \end{bmatrix} + \begin{bmatrix} v_1 & 0 & \cdots & 0 \\ 0 & v_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} u_1+v_1 & 0 & \cdots & 0 \\ 0 & u_2+v_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_n+v_n \end{bmatrix}$$

$$c\,\mathbf{u} = \begin{bmatrix} c\,u_1 & 0 & \cdots & 0 \\ 0 & c\,u_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c\,u_n \end{bmatrix}$$

Both $\mathbf{u}+\mathbf{v}$ and $c\mathbf{u}$ are also diagonal $n \times n$ matrices and so *W* is closed under addition and scalar multiplication and so is a subspace of M_{nn}.

(b) Let **u** and **v** be any two matrices from *W* and *c* be any scalar then,

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix} 0 & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \end{bmatrix}+\begin{bmatrix} 0 & v_{12} \\ v_{21} & v_{22} \\ v_{31} & v_{32} \end{bmatrix}=\begin{bmatrix} 0 & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \\ u_{31}+v_{31} & u_{32}+v_{32} \end{bmatrix}$$

$$c\,\mathbf{u}=\begin{bmatrix} 0 & cu_{12} \\ cu_{21} & cu_{22} \\ cu_{31} & cu_{32} \end{bmatrix}$$

Both $\mathbf{u}+\mathbf{v}$ and $c\mathbf{u}$ are also in *W* and so *W* is closed under addition and scalar multiplication and hence is a subspace of M_{32}.

(c) Let **u** and **v** be any two matrices from *W* then,

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix} 2 & u_{12} \\ 0 & u_{22} \end{bmatrix}+\begin{bmatrix} 2 & v_{12} \\ 0 & v_{22} \end{bmatrix}=\begin{bmatrix} 4 & u_{12}+v_{12} \\ 0 & u_{22}+v_{22} \end{bmatrix}$$

So, $\mathbf{u}+\mathbf{v}$ isn't in *W* since the entry in the first row and first column isn't a 2. Therefore, *W* is not closed under addition. You should also verify for yourself that *W* is not closed under scalar multiplication either.

In either case *W* is not a subspace of M_{22}.

Do not read too much into the result from part **(c)** of this example. In general the set of upper triangular $n\times n$ matrices (without restrictions, unlike part **(c)** from above) is a subspace of M_{nn} and the set of lower triangular $n\times n$ matrices is also a subspace of M_{nn}. You should verify this for the practice.

Example 3 Determine if the given set is a subspace of the given vector space.

(a) Let $C[a,b]$ be the set of all continuous functions on the interval $[a,b]$. Is this a subspace of $F[a,b]$, the set of all real valued functions on the interval $[a,b]$.

(b) Let P_n be the set of all polynomials of degree *n* or less. Is this a subspace of $F[a,b]$?

(c) Let *W* be the set of all polynomials of degree exactly *n*. Is this a subspace of $F[a,b]$?

(d) Let *W* be the set of all functions such that $f(6)=10$. Is this a subspace of $F[a,b]$ where we have $a\le 6\le b$?

Solution

(a) Okay, if you've not had Calculus you may not know what a continuous function is. A quick and dirty definition of continuity (not mathematically correct, but useful if you haven't had Calculus) is that a function is continuous on $[a,b]$ if there are no holes or breaks in the graph. Put in another way. You can sketch the graph of the function from *a* to *b* without ever picking up your pencil of pen.

A fact from Calculus (which if you haven't had please just believe this) is that the sum of two continuous functions is continuous and multiplying a continuous function by a constants will give a new continuous function. So, what this fact tells us is that the set of continuous functions is closed under standard function addition and scalar multiplication and that is what we're working with here.

So, $C[a,b]$ is a subspace of $F[a,b]$.

(b) First recall that a polynomial is said to have degree *n* if its largest exponent is *n*. Okay, let $\mathbf{u} = a_n x^n + \cdots + a_1 x + a_0$ and $\mathbf{v} = b_n x^n + \cdots + b_1 x + b_0$ and let *c* be any scalar. Then,

$$\mathbf{u} + \mathbf{v} = (a_n + b_n)x^n + \cdots + (a_1 + b_1)x + a_0 + b_0$$
$$c\mathbf{u} = ca_n x^n + \cdots + ca_1 x + ca_0$$

In both cases the degree of the new polynomial is not greater than *n*. Of course in the case of scalar multiplication it will remain degree *n*, but with the sum, it is possible that some of the coefficients cancel out to zero and hence reduce the degree of the polynomial.

The point is that P_n is closed under addition and scalar multiplication and so will be a subspace of $F[a,b]$.

(c) In this case *W* is not closed under addition. To see this let's take a look at the $n = 2$ case to keep things simple (the same argument will work for other values of *n*) and consider the following two polynomials,

$$\mathbf{u} = ax^2 + bx + c \qquad\qquad \mathbf{v} = -ax^2 + dx + e$$

where *a* is not zero, we know this is true because each polynomial must have degree 2. The other constants may or may not be zero. Both are polynomials of exactly degree 2 (since *a* is not zero) and if we add them we get,

$$\mathbf{u} + \mathbf{v} = (b + d)x + c + e$$

So, the sum had degree 1 and so is not in *W*. Therefore for $n = 2$ *W* is not closed under addition.

We looked at $n = 2$ only to make it somewhat easier to write down the two example polynomials. We could just have easily done the work for general *n* and we'd get the same result and so *W* is not a subspace.

(d) First notice that if we don't have $a \le 6 \le b$ then this problem makes no sense, so we will assume that $a \le 6 \le b$.

In this case suppose that we have two elements from *W*, $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$. This means that $f(6) = 10$ and $g(6) = 10$. In order for *W* to be a subspace we'll need to show that the sum and a scalar multiple will also be in *W*. In other words, if we evaluate the sum or the scalar multiple at 6 we'll get a result of 10. However, this won't happen. Let's take a look at the sum. The sum is,

$$(\mathbf{f} + \mathbf{g})(6) = f(6) + g(6) = 10 + 10 = 20 \neq 10$$

and so the sum will not be in *W*. Likewise, if *c* is any scalar that isn't 1 we'll have,

$$(c\mathbf{f})(6) = cf(6) = c(10) \neq 10$$

and so the scalar is not in *W* either.

Therefore *W* is not closed under addition or scalar multiplication and so is not a subspace.

Before we move on let's make a couple of observations about some of the sets we looked at in this example.

First, we should just point out that the set of all continuous functions on the interval $[a,b]$, $C[a,b]$, is a fairly important vector space in its own right to many areas of mathematical study.

Next, we saw that the set of all polynomials of degree less then or equal to *n*, P_n, was a subspace of $F[a,b]$. However, if you've had Calculus you'll know that polynomials are continuous and so P_n can also be thought of as a subspace of $C[a,b]$ as well. In other words, subspaces can have subspaces themselves.

Finally, here is something for you to think about. In the last part we saw that the set of all functions for which $f(6) = 10$ was not a subspace of $F[a,b]$ with $a \le 6 \le b$. Let's take a more general look at this. For some fixed number *k* let *W* be the set of all real valued functions for which $f(6) = k$. Are there any values of *k* for which *W* will be a subspace of $F[a,b]$ with $a \le 6 \le b$? Go back and think about how we did the work for that part and that should show you that there is one value of *k* (and only one) for which *W* will be a subspace. Can you figure out what that number has to be?

We now need to look at a fairly important subspace of $\mathbb{R}^m$ that we'll be seeing in future sections.

Definition 2 Suppose *A* is an $n \times m$ matrix. The **null space** of *A* is the set of all $\mathbf{x}$ in $\mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{0}$.

Let's see some examples of null spaces that are easy to find.

Example 4 Determine the null space of each of the following matrices.

(a) $A = \begin{bmatrix} 2 & 0 \\ -4 & 10 \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & -7 \\ -3 & 21 \end{bmatrix}$

(c) $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Solution

(a) To find the null space of A we'll need to solve the following system of equations.

$$\begin{bmatrix} 2 & 0 \\ -4 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad \begin{aligned} 2x_1 &= 0 \\ -4x_1 + 10x_2 &= 0 \end{aligned}$$

We've given this in both matrix form and equation form. In equation form it is easy to see that the only solution is $x_1 = x_2 = 0$. In terms of vectors from $\mathbb{R}^2$ the solution consists of the single vector $\{\mathbf{0}\}$ and hence the null space of A is $\{\mathbf{0}\}$.

(b) Here is the system that we need to solve for this part.

$$\begin{bmatrix} 1 & -7 \\ -3 & 21 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad \begin{aligned} x_1 - 7x_2 &= 0 \\ -3x_1 + 21x_2 &= 0 \end{aligned}$$

Now, we can see that these two equations are in fact the same equation and so we know there will be infinitely many solutions and that they will have the form,

$$x_1 = 7t \qquad x_2 = t \qquad t \text{ is any real number}$$

If you need a refresher on solutions to system take a look at the first section of the first chapter.

So, the since the null space of B consists of all the solutions to $B\mathbf{x} = \mathbf{0}$. Therefore, the null space of B will consist of all the vectors $\mathbf{x} = (x_1, x_2)$ from $\mathbb{R}^2$ that are in the form,

$$\mathbf{x} = (7t, t) = t(7,1) \qquad t \text{ is any real number}$$

We'll see a better way to write this answer in the next section.

In terms of equations, rather than vectors in $\mathbb{R}^2$, let's note that the null space of B will be all of the points that are on the equation through the origin give by $x_1 - 7x_2 = 0$.

(c) In this case we're going to be looking for solutions to

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

However, if you think about it, every vector $\mathbf{x}$ in $\mathbb{R}^2$ will be a solution to this system since we are multiplying $\mathbf{x}$ by the zero matrix.

Hence the null space of $\mathbf{0}$ is all of $\mathbb{R}^2$.

To see some examples of a more complicated null space check out Example 7 from the section on Basis and Example 2 in the Fundamental Subspace section. Both of these examples have more going on in them, but the first step is to write down the null space of a matrix so you can check out the first step of the examples and then ignore the remainder of the examples.

Now, let's go back and take a look at all the null spaces that we saw in the previous example. The null space for the first matrix was $\{\mathbf{0}\}$. For the second matrix the null space was the line through the origin given by $x_1 - 7x_2 = 0$. The null space for the zero matrix was all of $\mathbb{R}^2$. Thinking back to the early parts of this section we can see that all of these are in fact subspaces of $\mathbb{R}^2$.

In fact, this will always be the case as the following theorem shows.

Theorem 2 Suppose that A is an $n \times m$ matrix then the null space of A will be a subspace of $\mathbb{R}^m$.

Proof : We know that the subspace of A consists of all the solution to the system $A\mathbf{x} = \mathbf{0}$. First, we should point out that the zero vector, $\mathbf{0}$, in $\mathbb{R}^m$ will be a solution to this system and so we know that the null space is not empty. This is a good thing since a vector space (subspace or not) must contain at least one element.

Now that we know that the null space is not empty let $\mathbf{x}$ and $\mathbf{y}$ be two elements from the null space and let c be any scalar. We just need to show that the sum and scalar multiple of these are also in the null space and we'll be done.

Let's start with the sum.

$$A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

The sum, $\mathbf{x}+\mathbf{y}$ is a solution to $A\mathbf{x} = \mathbf{0}$ and so is in the null space. The null space is therefore closed under addition.

Next, let's take a look at the scalar multiple.

$$A(c\,\mathbf{x}) = c\,A\mathbf{x} = c\,\mathbf{0} = \mathbf{0}$$

The scalar multiple is also in the null space and so the null space is closed under scalar multiplication.

Therefore the null space is a subspace of $\mathbb{R}^m$.

■

Span

In this section we will cover a topic that we'll see off and on over the course of this chapter. Let's start off by going back to part **(b)** of Example 4 from the previous section. In that example we saw that the null space of the given matrix consisted of all the vectors of the form

$$\mathbf{x} = (7t, t) = t(7,1) \qquad\qquad t \text{ is any real number}$$

We would like a more compact way of stating this result and by the end of this section we'll have that.

Let's first revisit an idea that we saw quite some time ago. In the section on Matrix Arithmetic we looked at linear combinations of matrices and columns of matrices. We can also talk about linear combinations of vectors.

> ***Definition 1*** We say the vector $\mathbf{w}$ from the vector space V is a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$,all from V, if there are scalars $c_1, c_2, \ldots, c_n$ so that $\mathbf{w}$ can be written
> $$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

So, we can see that the null space we were looking at above is in fact all the linear combinations of the vector (7,1). It may seem strange to talk about linear combinations of a single vector since that is really scalar multiplication, but we can think of it as that if we need to.

The null space above was not the first time that we've seen linear combinations of vectors however. When we were looking at Euclidean *n*-space we introduced these things called the standard basis vectors. The standard basis vectors for $\mathbb{R}^n$ were defined as,

$$\mathbf{e}_1 = (1,0,0,\ldots,0) \qquad \mathbf{e}_2 = (0,1,0,\ldots,0) \qquad \cdots \qquad \mathbf{e}_n = (0,0,0,\ldots,1)$$

We saw that we could take any vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ from $\mathbb{R}^n$ and write it as,

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + \cdots + u_n\mathbf{e}_n$$

Or, in other words, we could write $\mathbf{u}$ and a linear combination of the standard basis vectors, $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. We will be revisiting this idea again in a couple of sections, but the point here is simply that we've seen linear combinations of vectors prior to us actually discussing them here.

Let's take a look at an example or two.

> ***Example 1*** Determine if the vector is a linear combination of the two given vectors.
>
> **(a)** Is $\mathbf{w} = (-12, 20)$ a linear combination of $\mathbf{v}_1 = (-1, 2)$ and $\mathbf{v}_2 = (4, -6)$?
>
> **(b)** Is $\mathbf{w} = (4, 20)$ a linear combination of $\mathbf{v}_1 = (2, 10)$ and $\mathbf{v}_2 = (-3, -15)$?

(c) Is $\mathbf{w} = (1,-4)$ a linear combination of $\mathbf{v}_1 = (2,10)$ and $\mathbf{v}_2 = (-3,-15)$?

Solution

(a) In each of these cases we'll need to set up and solve the following equation,

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$$
$$(-12,20) = c_1(-1,2) + c_2(4,-6)$$

Then set coefficients equal to arrive at the following system of equations,

$$-c_1 + 4c_2 = -12$$
$$2c_1 - 6c_2 = 20$$

If the system is consistent (*i.e.* has at least one solution then **w** is a linear combination of the two vectors. If there is no solution then **w** is not a linear combination of the two vectors.

We'll leave it to you to verify that the solution to this system is $c_1 = 4$ and $c_2 = -2$. Therefore, **w** is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ and we can write $\mathbf{w} = 4\mathbf{v}_1 - 2\mathbf{v}_2$.

(b) For this part we'll need to the same kind of thing so here is the system.

$$2c_1 - 3c_2 = 4$$
$$10c_1 - 15c_2 = 20$$

The solution to this system is,

$$c_1 = 2 + \frac{3}{2}t \qquad c_2 = t \qquad\qquad t \text{ is any real number}$$

This means **w** is linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. However, unlike the previous part there are literally an infinite number of ways in which we can write the linear combination. So, any of the following combinations would work for instance.

$$\mathbf{w} = 2\mathbf{v}_1 + (0)\mathbf{v}_2 \qquad\qquad \mathbf{w} = (0)\mathbf{v}_1 - \frac{4}{3}\mathbf{v}_2$$
$$\mathbf{w} = 8\mathbf{v}_1 + 4\mathbf{v}_2 \qquad\qquad \mathbf{w} = -\mathbf{v}_1 - 2\mathbf{v}_2$$

There are of course many more. There are just a few of the possibilities.

(c) Here is the system we'll need to solve for this part.

$$2c_1 - 3c_2 = 1$$
$$10c_1 - 15c_2 = -4$$

This system does not have a solution and so **w** is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

So, this example was kept fairly simple, but if we add in more components and/or more vectors to the set the problem will work in essentially the same manner.

Now that we've seen how linear combinations work and how to tell if a vector is a linear combination of a set of other vectors we need to move into the real topic of this section. In the opening of this section we recalled a null space that we'd looked at in the previous section. We can now see that the null space from that example is nothing more than all the linear combinations of the vector (7,1) (and again, it is kind of strange to be talking about linear combinations of a single vector).

As pointed out at the time we're after a more compact notation for denoting this. It is now time to give that notation.

Definition 2 Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of vectors in a vector space *V* and let *W* be the set of all linear combinations of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. The set *W* is the **span** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and is denoted by

$$W = \operatorname{span}(S) \qquad \text{OR} \qquad W = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

We also say that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ **span** *W*.

So, with this notation we can now see that the null space that we examined at the start of this section is now nothing more than,

$$\operatorname{span}\{(7,1)\}$$

Before we move on to some examples we should get a nice theorem out of the way.

Theorem 1 Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be vectors in a vector space *V* and let their span be $W = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ then,

(a) *W* is a subspace of *V*.
(b) *W* is the smallest subspace of *V* that contains all of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

Proof :
(a) So, we need to show that *W* is closed under addition and scalar multiplication. Let **u** and **w** be any two vectors from *W*. Now, since *W* is the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ that means that both **u** and **w** must be a linear combination of these vectors. So, there are scalars $c_1, c_2, \ldots, c_n$ and $k_1, k_2, \ldots, k_n$ so that,

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \qquad \text{and} \qquad \mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$$

Now, let's take a look at the sum.

$$\mathbf{u} + \mathbf{w} = (c_1 + k_1)\mathbf{v}_1 + (c_2 + k_2)\mathbf{v}_2 + \cdots + (c_n + k_n)\mathbf{v}_n$$

So the sum, $\mathbf{u} + \mathbf{w}$, is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and hence must be in *W* and so *W* is closed under addition.

Now, let *k* be any scalar and let's take a look at,

$$k\,\mathbf{u} = (k\,c_1)\mathbf{v}_1 + (k\,c_2)\mathbf{v}_2 + \cdots + (k\,c_n)\mathbf{v}_n$$

As we can see the scalar multiple, $k\mathbf{u}$, is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and hence must be in *W* and so *W* is closed under scalar multiplication.

Therefore, *W* must be a vector space.

(b) In these cases when we say that *W* is the smallest vector space that contains the set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ we're really saying that if W' is also a vector space that contains $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ then it will also contain a complete copy of *W* as well.

So, let's start this off by noticing that *W* does in fact contain each of the $\mathbf{v}_i$'s since,

$$\mathbf{v}_i = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 1\mathbf{v}_i + \cdots + 0\mathbf{v}_n$$

Now, let W' be a vector space that contains $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and consider any vector $\mathbf{u}$ from *W*. If we can show that $\mathbf{u}$ must also be in W' then we'll have shown that W' contains a copy of *W* since it will contain all the vectors in *W*. Now, $\mathbf{u}$ is in *W* and so must be a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$,

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

Each of the terms in this sum, $c_i\mathbf{v}_i$, is a scalar multiple of a vector that is in W' and since W' is a vector space it must be closed under scalar multiplication and so each $c_i\mathbf{v}_i$ is in W'. But this means that $\mathbf{u}$ is the sum of a bunch of vectors that are in W' which is closed under addition and so that means that $\mathbf{u}$ must in fact be in W'.

We've now shown that W' contains every vector from *W* and so must contain *W* itself.

■

Now, let's take a look at some examples of spans.

Example 2 Describe the span of each of the following sets of vectors.

(a) $\mathbf{v}_1 = (1,0,0)$ and $\mathbf{v}_2 = (0,1,0)$.

(b) $\mathbf{v}_1 = (1,0,1,0)$ and $\mathbf{v}_2 = (0,1,0,-1)$

Solution

(a) The span of this set of vectors, $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, is the set of all linear combinations and we can write down a general linear combination for these two vectors.

$$a\,\mathbf{v}_1 + b\,\mathbf{v}_2 = (a,0,0) + (0,b,0) = (a,b,0)$$

So, it looks like $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ will be all of the vectors from $\mathbb{R}^3$ that are in the form $(a,b,0)$ for any choices of *a* and *b*.

(b) This one is fairly similar to the first one. A general linear combination will look like,

$$a\,\mathbf{v}_1 + b\,\mathbf{v}_2 = (a,0,a,0) + (0,b,0,-b) = (a,b,a,-b)$$

So, $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ will be all the vectors from $\mathbb{R}^4$ of the form $(a,b,a,-b)$ for any choices of *a* and *b*.

Example 3 Describe the span of each of the following sets of "vectors".

(a) $\mathbf{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\mathbf{v}_1 = 1$, $\mathbf{v}_2 = x$, and $\mathbf{v}_3 = x^3$

Solution

These work exactly the same as the previous set of examples worked. The only difference is that this time we aren't working in $\mathbb{R}^n$ for this example.

(a) Here is a general linear combination of these "vectors".

$$a\,\mathbf{v}_1 + b\,\mathbf{v}_2 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

Here it looks like $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ will be all the diagonal matrices in M_{22}.

(b) A general linear combination in this case is,

$$a\,\mathbf{v}_1 + b\,\mathbf{v}_2 + c\,\mathbf{v}_3 = a + bx + cx^3$$

In this case $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ will be all the polynomials from P_3 that do not have a quadratic term.

Now, let's see if we can determine a set of vectors that will span some of the common vector spaces that we've seen. What we'll need in each of these examples is a set of vectors with which we can write a general vector from the space as a linear combination of the vectors in the set.

Example 4 Determine a set of vectors that will exactly span each of the following vector spaces.

(a) $\mathbb{R}^n$

(b) M_{22}

(c) P_n

Solution

Okay, before we start this let's think about just what we need to show here. We'll need to find a set of vectors so that the span of that set will be exactly the space given. In other words, we need to show that the span of our proposed set of vectors is in fact the same set as the vector space.

So just what do we need to do to mathematically show that two sets are equal? Let's suppose that we want to show that *A* and *B* are equal sets. To so this we'll need to show that each **a** in *A* will be in *B* and in doing so we'll have shown that *B* will at the least contain all of *A*. Likewise, we'll need to show that each **b** in *B* will be in *A* and in doing that we'll have shown that *A* will contain all of *B*. However, the only way that *A* can contain all of *B* and *B* can contain all of *A* is for *A* and *B* to be the same set.

So, for our example we'll need to determine a possible set of spanning vectors show that every vector from our vector space is in the span of our set of vectors. Next we'll need to show that each vector in our span will also be in the vector space.

(a) We've pretty much done this one already. Earlier in the section we showed that the any vector from $\mathbb{R}^n$ can be written as a linear combination of the standard basis vectors, $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ and so at the least the span of the standard basis vectors will contain all of $\mathbb{R}^n$. However, since any linear combination of the standard basis vectors is going to be a vector in $\mathbb{R}^n$ we can see that $\mathbb{R}^n$ must also contain the span of the standard basis vectors.

Therefore, the span of the standard basis vectors must be $\mathbb{R}^n$.

(b) We can use result of Example 3(a) above as a guide here. In that example we saw a set of matrices that would span all the diagonal matrices in M_{22} and so we can do a natural extension to get a set that will span all of M_{22}. It looks like the following set should do it.

$$\mathbf{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ \mathbf{v}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Clearly any linear combination of these four matrices will be a 2×2 matrix and hence in M_{22} and so the span of these matrices must be contained in M_{22}.

Likewise, given any matrix from M_{22},

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

we can write it as the following linear combination of these "vectors".

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = a\,\mathbf{v}_1 + b\,\mathbf{v}_2 + c\,\mathbf{v}_3 + d\,\mathbf{v}_4$$

and so M_{22} must be contained in the span of these vectors and so these vectors will span M_{22}.

(c) We can use Example 3(b) to help with this one. First recall that P_n is the set of all polynomials of degree *n* or less. Using Example 3(b) as a guide it looks like the following set of "vectors" will work for us.

$$\mathbf{v}_0 = 1, \mathbf{v}_1 = x, \mathbf{v}_2 = x^2, \ldots, \mathbf{v}_n = x^n$$

Note that used subscripts that matched the degree of the term and so started at $\mathbf{v}_0$ instead of the usual $\mathbf{v}_1$.

It should be clear (hopefully) that a linear combination of these is a polynomial of degree *n* or less and so will be in P_n. Therefore the span of these vectors will be contained in P_n.

Likewise, we can write a general polynomial of degree *n* or less,

$$\mathbf{p} = a_0 + a_1 x + \cdots + a_n x^n$$

as the following linear combination

$$\mathbf{p} = a_0 \mathbf{v}_0 + a_1 \mathbf{v}_1 + \cdots + a_n \mathbf{v}_n$$

Therefore P_n is contained in the span of these vectors and this means that the span of these vectors is exactly P_n.

There is one last idea about spans that we need to discuss and its best illustrated with an example.

Example 5 Determine if the following sets of vectors will span $\mathbb{R}^3$.

(a) $\mathbf{v}_1 = (2,0,1)$, $\mathbf{v}_2 = (-1,3,4)$, and $\mathbf{v}_3 = (1,1,-2)$.

(b) $\mathbf{v}_1 = (1,2,-1)$, $\mathbf{v}_2 = (3,-1,1)$, and $\mathbf{v}_3 = (-3,8,-5)$.

Solution

(a) Okay let's think about how we've got to approach this. Clearly the span of these vectors will be in $\mathbb{R}^3$ since they are vectors from $\mathbb{R}^3$. The real question is whether or not $\mathbb{R}^3$ will be contained in the span of these vectors, $\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. In the previous example our set of vectors contained vectors that we could easily show this. However, in this case its not so clear. So to answer that question here we'll do the following.

Choose a general vector from $\mathbb{R}^3$, $\mathbf{u} = (u_1, u_2, u_3)$, and determine if we can find scalars c_1, c_2, and c_3 so that $\mathbf{u}$ is a linear combination of the given vectors. Or,

$$(u_1, u_2, u_3) = c_1(2,0,1) + c_2(-1,3,4) + c_3(1,1,-2)$$

If we set components equal we arrive at the following system of equations,

$$\begin{aligned} 2c_1 - c_2 + c_3 &= u_1 \\ 3c_2 + c_3 &= u_2 \\ c_1 + 4c_2 - 2c_3 &= u_3 \end{aligned}$$

In matrix form this is,

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 1 \\ 1 & 4 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

What we need to do is to determine if this system will be consistent (*i.e.* have at least one solution) for every possible choice of $\mathbf{u} = (u_1, u_2, u_3)$. Nicely enough this is very easy to do if you recall Theorem 9 from the section on Determinant Properties. This theorem tells us that this system will be consistent for every choice of $\mathbf{u} = (u_1, u_2, u_3)$ provided the coefficient matrix is invertible and we can check that be doing a quick determinant computation. So, if we denote the coefficient matrix as *A* we'll leave it to you to verify that $\det(A) = -24$.

Therefore the coefficient matrix is invertible and so this system will have a solution for every choice of $\mathbf{u} = (u_1, u_2, u_3)$. This in turn tells us that $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is contained in $\mathbb{R}^3$ and so we've now shown that

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3$$

(b) We'll do this one a little quicker. As with the first part, let's choose a general vector $\mathbf{u} = (u_1, u_2, u_3)$ form $\mathbb{R}^3$ and form up the system that we need to solve. We'll leave it to you to verify that the matrix form of this system is,

$$\begin{bmatrix} 1 & 3 & -3 \\ 2 & -1 & 8 \\ -1 & 1 & -5 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

This system will have a solution for every choice of $\mathbf{u} = (u_1, u_2, u_3)$ if the coefficient matrix, *A*, is invertible. However, in this case we have $\det(A) = 0$ (you should verify this) and so the coefficient matrix is not invertible.

This in turn tells us that there is at least one choice of $\mathbf{u} = (u_1, u_2, u_3)$ for which this system will not have a solution and so $\mathbf{u} = (u_1, u_2, u_3)$ cannot be written as a linear combination of these three vectors. Note that there are in fact infinitely many choices of $\mathbf{u} = (u_1, u_2, u_3)$ that will not yield solutions!

Now, we know that $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is contained in $\mathbb{R}^3$, but we've just shown that there is at least one vector from $\mathbb{R}^3$ that is not contained in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and so the span of these three vectors will not be all of $\mathbb{R}^3$.

This example has shown us two things. First, it has shown us that we can't just write down any set of three vectors and expect to get those three vectors to span $\mathbb{R}^3$. This is an idea we're going to be looking at in much greater detail in the next couple of sections.

Secondly, we've now seen at least two different sets of vectors that will span $\mathbb{R}^3$. There are the three vectors from Example 5(a) as well as the standard basis vectors for $\mathbb{R}^3$. This tells us that the set of vectors that will span a vector space are not unique. In other words, we can have more that one set of vectors span the same vector space.

Linear Independence

In the previous section we saw several examples of writing a particular vector as a linear combination of other vectors. However, as we saw in Example 1(b) of that section there is sometimes more than one linear combination of the same set of vectors can be used for a given vector. We also saw in the previous section that some sets of vectors, $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, can span a vector space. Recall that by span we mean that every vector in the space can be written as a linear combination of the vectors in *S*. In this section we'd like to start looking at when it will be possible to express a given vector from a vector space as exactly one linear combinations of the set *S*.

We'll start this section off with the following definition.

Definition 1 Suppose $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a non-empty set of vectors and form the vector equation,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

This equation has at least one solution, namely, $c_1 = 0$, $c_2 = 0$, … , $c_n = 0$. This solution is called the **trivial solution**.

If the trivial solution is the only solution to this equation then the vectors in the set *S* are called **linearly independent** and the set is called a **linearly independent set**. If there is another solution then the vectors in the set *S* are called **linearly dependent** and the set is called a **linearly dependent set**.

Let's take a look at some examples.

Example 1 Determine if each of the following sets of vectors are linearly independent or linearly dependent.

(a) $\mathbf{v}_1 = (3,-1)$ and $\mathbf{v}_2 = (-2,2)$.

(b) $\mathbf{v}_1 = (12,-8)$ and $\mathbf{v}_2 = (-9,6)$.

(c) $\mathbf{v}_1 = (1,0,0)$, $\mathbf{v}_2 = (0,1,0)$, and $\mathbf{v}_3 = (0,0,1)$

(d) $\mathbf{v}_1 = (2,-2,4)$, $\mathbf{v}_2 = (3,-5,4)$, and $\mathbf{v}_3 = (0,1,1)$

Solution

To answer the question here we'll need to set up the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

for each part, combine the left side into a single vector and the set all the components of the vector equal to zero (since it must be the zero vector, **0**). At this point we've got a system of equations that we can solve. If we only get the trivial solution the vectors will be linearly independent and if we get more than one solution the vectors will be linearly dependent.

(a) We'll do this one in detail and then do the remaining parts quicker. We'll first set up the equation and get the left side combined into a single vector.

$$c_1\left(3,-1\right)+c_2\left(-2,2\right)=\mathbf{0}$$
$$\left(3c_1-2c_2,-c_1+2c_2\right)=\left(0,0\right)$$

Now, set each of the components equal to zero to arrive at the following system of equations.

$$\begin{aligned}3c_1-2c_2&=0\\-c_1+2c_2&=0\end{aligned}$$

Solving this system gives to following solution (we'll leave it to you to verify this),

$$c_1=0 \qquad\qquad c_2=0$$

The trivial solution is the only solution and so these two vectors are linearly independent.

(b) Here is the vector equation we need to solve.

$$c_1\left(12,-8\right)+c_2\left(-9,6\right)=\mathbf{0}$$

The system of equations that we'll need to solve is,

$$\begin{aligned}12c_1-9c_2&=0\\-8c_1+6c_2&=0\end{aligned}$$

and the solution to this system is,

$$c_1=\frac{3}{4}t \qquad\qquad c_2=t \qquad\qquad t \text{ is any real number}$$

We've got more than the trivial solution (note however that the trivial solution IS still a solution, there's just more than that this time) and so these vectors are linearly dependent.

(c) The only difference between this one and the previous two are the fact that we now have three vectors out of $\mathbb{R}^3$. Here is the vector equation for this part.

$$c_1\left(1,0,0\right)+c_2\left(0,1,0\right)+c_3\left(0,0,1\right)=\mathbf{0}$$

The system of equations to solve for this part is,

$$\begin{aligned}c_1&=0\\c_2&=0\\c_3&=0\end{aligned}$$

So, not much solving to do this time. It is clear that the only solution will be the trivial solution and so these vectors are linearly independent.

(d) Here is the vector equation for this final part.

$$c_1\left(2,-2,4\right)+c_2\left(3,-5,4\right)+c_3\left(0,1,1\right)=\mathbf{0}$$

The system of equations that we'll need to solve here is,

$$\begin{aligned} 2c_1+3c_2&=0\\ -2c_1-5c_2+c_3&=0\\ 4c_1+4c_2+c_3&=0 \end{aligned}$$

The solution to this system is,

$$c_1=-\frac{3}{4}t \qquad c_2=\frac{1}{2}t \qquad c_3=t \qquad t \text{ is any real number}$$

We've got more than just the trivial solution and so these vectors are linearly dependent.

Note that we didn't really need to solve any of the systems above if we didn't want to. All we were interested in it was whether or not the system had only the trivial solution or if there were more solutions in addition to the trivial solution. Theorem 9 from the Properties of the Determinant section can help us answer this question without solving the system. This theorem tells us that if the determinant of the coefficient matrix is non-zero then the system will have exactly one solution, namely the trivial solution. Likewise, it can be shown that if the determinant is zero then the system will have infinitely many solutions.

Therefore, once the system is set up if the coefficient matrix is square all we really need to do is take the determinant of the coefficient matrix and if it is non-zero the set of vectors will be linearly independent and if the determinant is zero then the set of vectors will be linearly dependent. If the coefficient matrix is not square then we can't take the determinant and so we'll not have a choice but to solve the system.

This does not mean however, that the actual solution to the system isn't ever important as we'll see towards the end of the section.

Before proceeding on we should point out that the vectors from part **(c)** of this were actually the standard basis vectors for $\mathbb{R}^3$. In fact the standard basis vectors for $\mathbb{R}^n$,

$$\mathbf{e}_1=\left(1,0,0,\ldots,0\right),\ \mathbf{e}_2=\left(0,1,0,\ldots,0\right),\ldots,\mathbf{e}_n=\left(0,0,0,\ldots,1\right)$$

will be linearly independent.

The vectors in the previous example all had the same number of components as vectors, *i.e.* two vectors from $\mathbb{R}^2$ or three vectors from $\mathbb{R}^3$. We should work a couple of examples that does not fit this mold to make sure that you understand that we don't need to have the same number of vectors as components.

Example 2 Determine if the following sets of vectors are linearly independent or linearly dependent.

(a) $\mathbf{v}_1 = (1,-3)$, $\mathbf{v}_2 = (-2,2)$ and $\mathbf{v}_3 = (4,-1)$

(b) $\mathbf{v}_1 = (-2,1)$, $\mathbf{v}_2 = (-1,-3)$ and $\mathbf{v}_3 = (4,-2)$

(c) $\mathbf{v}_1 = (1,1,-1,2)$, $\mathbf{v}_2 = (2,-2,0,2)$ and $\mathbf{v}_3 = (2,-8,3,-1)$

(d) $\mathbf{v}_1 = (1,-2,3,-4)$, $\mathbf{v}_2 = (-1,3,4,2)$ and $\mathbf{v}_3 = (1,1,-2,-2)$

Solution

These will work in pretty much the same manner as the previous set of examples worked. Again, we'll do the first part in some detail and then leave it to you to verify the details in the remaining parts. Also, we'll not be showing the details of solving the systems of equations so you should verify all the solutions for yourself.

(a) Here is the vector equation we need to solve.

$$c_1(1,-3) + c_2(-2,2) + c_3(4,-1) = \mathbf{0}$$
$$(c_1 - 2c_2 + 4c_3, -3c_1 + 2c_2 - c_3) = (0,0)$$

The system of equations that we need to solve is,

$$\begin{aligned} c_1 - 2c_2 + 4c_3 &= 0 \\ -3c_1 + 2c_2 - c_3 &= 0 \end{aligned}$$

and this has the solution,

$$c_1 = \frac{3}{2}t \qquad c_2 = \frac{11}{4}t \qquad c_3 = t \qquad t \text{ is any real number}$$

We've got more than the trivial solution and so these vectors are linearly dependent.

Note that we didn't really need to solve this system to know that they were linearly dependent. From Theorem 2 in the solving systems of equations section we know that if there are more unknowns than equations in a homogeneous system then we will have infinitely many solutions.

(b) Here is the vector equation for this part.

$$c_1(-2,1) + c_2(-1,-3) + c_3(4,-2) = \mathbf{0}$$

The system of equations we'll need to solve is,

$$\begin{aligned} -2c_1 - c_2 + 4c_3 &= 0 \\ c_1 - 3c_2 - 2c_3 &= 0 \end{aligned}$$

Now, technically we don't need to solve this system for the same reason we really didn't need to solve the system in the previous part. There are more unknowns than equations so the system will have infinitely many solutions (so more than the trivial solution) and therefore the vectors will be linearly dependent.

However, let's solve anyway since there is an important idea we need to see in this part. Here is the solution.

$$c_1 = 2t \qquad c_2 = 0 \qquad c_3 = t \qquad t \text{ is any real number}$$

In this case one of the scalars was zero. There is nothing wrong with this. We still have solutions other than the trivial solution and so these vectors are linearly dependent. Note that was it does say however, is that $\mathbf{v}_1$ and $\mathbf{v}_3$ are linearly dependent themselves regardless of $\mathbf{v}_2$.

(c) Here is the vector equation for this part.

$$c_1\left(1,1,-1,2\right) + c_2\left(2,-2,0,2\right) + c_3\left(2,-8,3,-1\right) = \mathbf{0}$$

The system of equations that we'll need to solve this time is,

$$\begin{aligned} c_1 + 2c_2 + 2c_3 &= 0 \\ c_1 - 2c_2 - 8c_3 &= 0 \\ -c_1 + 3c_3 &= 0 \\ 2c_1 + 2c_2 - c_3 &= 0 \end{aligned}$$

The solution to this system is,

$$c_1 = 3t \qquad c_2 = -\frac{5}{2}t \qquad c_3 = t \qquad t \text{ is any real number}$$

We've got more solutions than the trivial solution and so these three vectors are linearly dependent.

(d) The vector equation for this part is,

$$c_1\left(1,-2,3,-4\right) + c_2\left(-1,3,4,2\right) + c_3\left(1,1,-2,-2\right) = \mathbf{0}$$

The system of equations is,

$$\begin{aligned} c_1 - c_2 + c_3 &= 0 \\ -2c_1 + 3c_2 + c_3 &= 0 \\ 3c_1 + 4c_2 - 2c_3 &= 0 \\ -4c_1 + 2c_2 - 2c_3 &= 0 \end{aligned}$$

This system has only the trivial solution and so these three vectors are linearly independent.

We should make one quick remark about part **(b)** of this problem. In this case we had a set of three vectors and one of the scalars was zero. This will happen on occasion and as noted this only means that the vectors with the zero scalars are not really required in order to make the set linearly dependent. This part has shown that if you have a set of vectors and a subset is linearly dependent then the whole set will be linearly dependent.

Often the only way to determine if a set of vectors is linearly independent or linearly dependent is to set up a system as above and solve it. However, there are a couple of cases were we can get the answer just be looking at the set of vectors.

Theorem 1 A finite set of vectors that contains the zero vector will be linearly dependent.

Proof : This is a fairly simply proof. Let $S = \{\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be any set of vectors that contains the zero vector as shown. We can then set up the following equation.

$$1(\mathbf{0}) + 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n = \mathbf{0}$$

We can see from this that we have a non-trivial solution to this equation and so the set of vectors is linearly dependent.

■

Theorem 2 Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in $\mathbb{R}^n$. If $k > n$ then the set of vectors is linearly dependent.

We're not going to prove this one but we will outline the basic proof. In fact, we saw how to prove this theorem in parts **(a)** and **(b)** from Example 2. If we set up the system of equations corresponding to the equation,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

we will get a system of equation that has more unknowns than equations (you should verify this) and this means that the system will infinitely many solutions. The vectors will therefore be linearly dependent.

To this point we've only seen examples of linear independence/dependence with sets of vectors in $\mathbb{R}^n$. We should now take a look at some examples of vectors from some other vector spaces.

Example 3 Determine if the following sets of vectors are linearly independent or linearly dependent.

(a) $\mathbf{v}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

(b) $\mathbf{v}_1 = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 4 & 1 \\ 0 & -3 \end{bmatrix}$.

(c) $\mathbf{v}_1 = \begin{bmatrix} 8 & -2 \\ 10 & 0 \end{bmatrix}$ and $\mathbf{v}_1 = \begin{bmatrix} -12 & 3 \\ -15 & 0 \end{bmatrix}$.

Solution

Okay, the basic process here is pretty much the same as the previous set of examples. It just may not appear that way at first however. We'll need to remember that this time the zero vector, $\mathbf{0}$, is in fact the zero matrix of the same size as the vectors in the given set.

(a) We'll first need to set of the "vector" equation,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

$$c_1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Next, combine the "vectors" (okay, they're matrices so let's call them that….) on the left into a single matrix using basic matrix scalar multiplication and addition.

$$\begin{bmatrix} c_1 & 0 & c_2 \\ 0 & c_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, we need both sides to be equal. This means that the three entries in the matrix on the left that are not already zero need to be set equal to zero. This gives the following system of equations.

$$\begin{aligned} c_1 &= 0 \\ c_2 &= 0 \\ c_3 &= 0 \end{aligned}$$

Of course this isn't really much of a system as it tells us that we must have the trivial solution and so these matrices (or vectors if you want to be exact) are linearly independent.

(b) So we can see that for the most part these problems work the same way as the previous problems did. We just need to set up a system of equations and solve. For the remainder of these problems we'll not put in the detail that did in the first part.

Here is the vector equation we need to solve for this part.

$$c_1 \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} + c_2 \begin{bmatrix} 4 & 1 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The system of equations we need to solve here is,

$$\begin{aligned} c_1 + 4c_2 &= 0 \\ 2c_1 + c_2 &= 0 \\ -c_1 - 3c_3 &= 0 \end{aligned}$$

We'll leave it to you to verify that the only solution to this system is the trivial solution and so these matrices are linearly independent.

(c) Here is the vector equation for this part

$$c_1 \begin{bmatrix} 8 & -2 \\ 10 & 0 \end{bmatrix} + c_2 \begin{bmatrix} -12 & 3 \\ -15 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and the system of equations is,

$$\begin{aligned} 8c_1 - 12c_2 &= 0 \\ -2c_1 + 3c_2 &= 0 \\ 10c_1 - 15c_3 &= 0 \end{aligned}$$

The solution to this system is,

$$c_1 = \frac{3}{2}t \qquad c_2 = t \qquad t \text{ is any real number}$$

So, we've got solutions other than the trivial solution and so these vectors are linearly dependent.

Example 4 Determine if the following sets of vectors are linearly independent or linearly dependent.

(a) $\mathbf{p}_1 = 1$, $\mathbf{p}_2 = x$, and $\mathbf{p}_3 = x^2$ in P_2.

(b) $\mathbf{p}_1 = x-3$, $\mathbf{p}_2 = x^2 + 2x$, and $\mathbf{p}_3 = x^2 + 1$ in P_2.

(c) $\mathbf{p}_1 = 2x^2 - x + 7$, $\mathbf{p}_2 = x^2 + 4x + 2$, and $\mathbf{p}_3 = x^2 - 2x + 4$ in P_2.

Solution

Again, these will work in essentially the same manner as the previous problems. In this problem set the zero vector, **0**, is the zero function. Since we're actually working in P_2 for all these parts we can think of this as the following polynomial.

$$\mathbf{0} = 0 + 0x + 0x^2$$

In other words a second degree polynomial with zero coefficients.

(a) Let's first set up the equation that we need to solve.

$$c_1\,\mathbf{p}_1 + c_2\,\mathbf{p}_2 + c_3\,\mathbf{p}_3 = \mathbf{0}$$
$$c_1(1) + c_2 x + c_3 x^2 = 0 + 0x + 0x^2$$

Now, we could set up a system of equations here, however we don't need to. In order for these two second degree polynomials to be equal the coefficient of each term must be equal. At this point is it should be pretty clear that the polynomial on the left will only equal if all the coefficients of the polynomial on the left are zero. So, the only solution to the vector equation will be the trivial solution and so these polynomials (or vectors if you want to be precise) are linearly independent.

(b) The vector equation for this part is,

$$c_1(x-3) + c_2(x^2+2x) + c_3(x^2+1) = 0 + 0x + 0x^2$$
$$(c_2+c_3)x^2 + (c_1+2c_2)x + (-3c_1+c_3) = 0 + 0x + 0x^2$$

Now, as with the previous part the coefficients of each term on the left must be zero in order for this polynomial to be the zero vector. This leads to the following system of equations.

$$\begin{aligned} c_2 + c_3 &= 0 \\ c_1 + 2c_2 &= 0 \\ -3c_1 + c_3 &= 0 \end{aligned}$$

The only solution to this system is the trivial solution and so these polynomials are linearly independent.

(c) In this part the vector equation is,

$$c_1(2x^2 - x + 7) + c_2(x^2 + 4x + 2) + c_3(x^2 - 2x + 4) = 0 + 0x + 0x^2$$
$$(2c_1 + c_2 + c_3)x^2 + (-c_1 + 4c_2 - 2c_3)x + (7c_1 + 2c_2 + 4c_3) = 0 + 0x + 0x^2$$

The system of equation we need to solve is,

$$\begin{aligned} 2c_1 + c_2 + c_3 &= 0 \\ -c_1 + 4c_2 - 2c_3 &= 0 \\ 7c_1 + 2c_2 + 4c_3 &= 0 \end{aligned}$$

The solution to this system is,

$$c_1 = -\frac{2}{3}t \qquad c_2 = \frac{1}{3}t \qquad c_3 = t \qquad t \text{ is any real number}$$

So, we have more solutions than the trivial solution and so these polynomials are linearly dependent.

Now that we've seen quite a few examples of linearly independent and linearly dependent vectors we've got one final topic that we want to discuss in this section. Let's go back and examine the results of the very first example that we worked in this section and in particular let's start with the final part.

In this part we looked at the vectors $\mathbf{v}_1 = (2,-2,4)$, $\mathbf{v}_2 = (3,-5,4)$, and $\mathbf{v}_3 = (0,1,1)$ and determined that they were linearly dependent. We did this by solving the vector equation,

$$c_1(2,-2,4) + c_2(3,-5,4) + c_3(0,1,1) = \mathbf{0}$$

and found that it had the solution,

$$c_1 = -\frac{3}{4}t \qquad c_2 = \frac{1}{2}t \qquad c_3 = t \qquad t \text{ is any real number}$$

We knew that the vectors were linearly dependent because there were solutions to the equation other than the trivial solution. Let's take a look at one of them. Say,

$$c_1 = -\frac{3}{2} \qquad c_2 = 1 \qquad c_3 = 2$$

In fact, let's plug these values into the vector equation above.

$$-\frac{3}{2}(2,-2,4) + (3,-5,4) + 2(0,1,1) = \mathbf{0}$$

Now, if we rearrange this a little we arrive at,

$$(3,-5,4) = \frac{3}{2}(2,-2,4) - 2(0,1,1)$$

or, in a little more compact form : $\mathbf{v}_2 = \frac{3}{2}\mathbf{v}_1 - 2\mathbf{v}_3$.

So, we were able to write one of the vectors as a linear combination of the other two. Notice as well that we could have just as easily written $\mathbf{v}_1$ and a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$ or $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ if we'd wanted to.

Let's see if we can do this with the three vectors from the third part of this example. In this part we were looking at the three vectors $\mathbf{v}_1 = (1,0,0)$, $\mathbf{v}_2 = (0,1,0)$, and $\mathbf{v}_3 = (0,0,1)$ and in that part we determined that these vectors were linearly independent. Let's see if we can write $\mathbf{v}_1$ and a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$. If we can we'll be able to find constants c_1 and c_2 that will make the following equation true.

$$(1,0,0) = c_1(0,1,0) + c_2(0,0,1) = (0, c_1, c_2)$$

Now, while we can find values of c_1 and c_2 that will make the second and third entries zero as we need them to we're in some pretty serious trouble with the first entry. In the vector on the left we've got a 1 in the first entry and in the vector on the right we've got a 0 in the first entry. So, there is no way we can write the first vector as a linear combination of the other two. You should also verify we also do this in any of the other combinations either.

So, what have we seen here with these two examples. With a set of linearly dependent vectors we were able to write at least one of them as a linear combination of the other vectors in the set and with a set of linearly independent vectors we were not able to do this for any of the vectors. This will always be the case.

With a set of linearly independent vectors we will never be able to write one of the vectors as a linear combination of the other vectors in the set. On the other hand, if we have a set of linearly dependent vectors then at least one of them can be written as a linear combination of the remaining vectors.

In the example of linearly dependent vectors we were looking at above we could write any of the vectors as a linear combination of the others. This will not always be the case, to see this take a look at Example 2(b). In this example we determined that the vectors $\mathbf{v}_1 = (-2,1)$, $\mathbf{v}_2 = (-1,-3)$ and $\mathbf{v}_3 = (4,-2)$ were linearly dependent. We also saw that the solution to the equation,

$$c_1(-2,1) + c_2(-1,-3) + c_3(4,-2) = \mathbf{0}$$

was given by

$$c_1 = 2t \qquad c_2 = 0 \qquad c_3 = t \qquad t \text{ is any real number}$$

and as we saw above we can always use this to determine how to write at least one of the vectors as a linear combination of the remaining vectors. Simply pick a value of *t* and the rearrange as you need to. Doing this in our case we see that we can do one of the following.

$$(4,-2) = -2(-2,1) - (0)(-1,-3)$$

$$(-2,1) = -(0)(-1,-3) - \frac{1}{2}(4,-2)$$

It's easy in this case to write the first or the third vector as a combination of the other vectors. However, because the coefficient of the second vector is zero, there is no way that we can write the second vector as a linear combination of the first and third vectors.

What that means here is that the first and third vectors are linearly dependent by themselves (as we pointed out in that example) but the first and second are linearly independent vectors as are the second and third if we just look at them as a pair of vectors (you should verify this).

This can be a useful idea about linearly independent/dependent vectors on occasion.

Basis and Dimension

In this section we're going to take a look at an important idea in the study of vector spaces. We will also be drawing heavily on the ideas from the previous two sections and so make sure that you are comfortable with the ideas of span and linear independence.

We'll start this section off with the following definition.

Definition 1 Suppose $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set of vectors from the vector space *V*. Then *S* is called a **basis** (plural is **bases**) for *V* if both of the following conditions hold.

(a) $\text{span}(S) = V$, *i.e. S* spans the vector space *V*.

(b) *S* is a linearly independent set of vectors.

Let's take a look at some examples.

Example 1 Determine if each of the sets of vectors will be a basis for $\mathbb{R}^3$.

(a) $\mathbf{v}_1 = (1,-1,1)$, $\mathbf{v}_2 = (0,1,2)$ and $\mathbf{v}_3 = (3,0,-1)$.

(b) $\mathbf{v}_1 = (1,0,0)$, $\mathbf{v}_2 = (0,1,0)$ and $\mathbf{v}_3 = (0,0,1)$.

(c) $\mathbf{v}_1 = (1,1,0)$ and $\mathbf{v}_2 = (-1,0,0)$.

(d) $\mathbf{v}_1 = (1,-1,1)$, $\mathbf{v}_2 = (-1,2,-2)$ and $\mathbf{v}_3 = (-1,4,-4)$

Solution

(a) Now, let's see what we've got to do here to determine whether or not this set of vectors will be a basis for $\mathbb{R}^3$. First, we'll need to show that these vectors span $\mathbb{R}^3$ and from the section on Span we know that to do this we need to determine if we can find scalars c_1, c_2, and c_3 so that a general vector $\mathbf{u} = (u_1, u_2, u_3)$ from $\mathbb{R}^3$ can be expressed as a linear combination of these three vectors or,

$$c_1(1,-1,1) + c_2(0,1,2) + c_3(3,0,-1) = (u_1, u_2, u_3)$$

As we saw in the section on Span all we need to do is convert this to a system of equations, in matrix form, and then determine if the coefficient matrix has a non-zero determinant or not. If the determinant of the coefficient matrix is non-zero then the set will span the given vector space and if the determinant of the coefficient matrix is zero then it will not span the given vector space. Recall as well that if the determinant of the coefficient matrix is non-zero then there will be exactly one solution to this system for each $\mathbf{u}$.

The matrix form of the system is,

$$\begin{bmatrix} 1 & 0 & 3 \\ -1 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Before we get the determinant of the coefficient matrix let's also take a look at the other condition that must be met in order for this set to be a basis for $\mathbb{R}^3$. In order for these vectors to be a basis for $\mathbb{R}^3$ then they must be linearly independent. From the section on Linear Independence we know that to determine this we need to solve the following equation,

$$c_1(1,-1,1) + c_2(0,1,2) + c_3(3,0,-1) = \mathbf{0} = (0,0,0)$$

If this system has only the trivial solution the vectors will be linearly independent and if it has solutions other than the trivial solution then the vectors will be linearly dependent.

Note however, that this is really just a specific case of the system that we need to solve for the span question. Namely here we need to solve,

$$\begin{bmatrix} 1 & 0 & 3 \\ -1 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Also, as noted above, if these vectors will span $\mathbb{R}^3$ then there will be exactly one solution to the system for each $\mathbf{u}$. In this case we know that the trivial solution will be a solution, our only question is whether or not it is the only solution.

So, all that we need to do here is compute the determinant of the coefficient matrix and if it is non-zero then the vectors will both span $\mathbb{R}^3$ and be linearly independent and hence the vectors will be a basis for $\mathbb{R}^3$. On the other hand, if the determinant is zero then the vectors will not span $\mathbb{R}^3$ and will not be linearly dependent and so they won't be a basis for $\mathbb{R}^3$.

So, here is the determinant of the coefficient matrix for this problem.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix} \qquad \Rightarrow \qquad \det(A) = -10 \neq 0$$

So, these vectors will form a basis for $\mathbb{R}^3$.

(b) Now, we could use a similar path for this one as we did earlier. However, in this case, we've done all the work for this one in previous sections. In Example 4(a) of the section on Span we determined that the standard basis vectors (Interesting name isn't it? We'll come back to this in a bit) $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ will span $\mathbb{R}^3$. Notice that while we've changed the notation a little just for this problem we are working with the standard basis vectors here and so we know that they will span $\mathbb{R}^3$.

Likewise, in Example 1(c) from the section on Linear Independence we saw that these vectors are linearly independent.

Hence based on all this previous work we know that these three vectors will form a basis for $\mathbb{R}^3$.

(c) We can't use the method from part (a) here because the coefficient matrix wouldn't be square and so we can't take the determinant of it. So, let's just start this out be checking to see if these two vectors will span $\mathbb{R}^3$. If these two vectors will span $\mathbb{R}^3$ then for each $\mathbf{u} = (u_1, u_2, u_3)$ in $\mathbb{R}^3$ there must be scalars c_1 and c_2 so that,

$$c_1(1,1,0) + c_2(-1,0,0) = (u_1, u_2, u_3)$$

However, we can see right away that there will be problems here. The third component of the each of these vectors is zero and hence the linear combination will never have any non-zero third component. Therefore, if we choose $\mathbf{u} = (u_1, u_2, u_3)$ to be any vector in $\mathbb{R}^3$ with $u_3 \neq 0$ we will not be able to find scalars c_1 and c_2 to satisfy the equation above.

Therefore, these two vectors do not span $\mathbb{R}^3$ and hence can not be a basis for $\mathbb{R}^3$.

Note however, that these two vectors are linearly independent (you should verify that). Despite this however, the vectors are still not a basis for $\mathbb{R}^3$ since they do not span $\mathbb{R}^3$.

(d) In this case we've got three vectors with three components and so we can use the same method that we did in the first part. The general equation that needs solved here is,

$$c_1(1,-1,1) + c_2(-1,2,-2) + c_3(-1,4,-4) = (u_1, u_2, u_3)$$

and the matrix form of this is,

$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 4 \\ 1 & -2 & -4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We'll leave it to you to verify that $\det(A) = 0$ and so these three vectors do not span $\mathbb{R}^3$ and are not linearly independent. Either of which will mean that these three vectors are not a basis for $\mathbb{R}^3$.

Before we move on let's go back and address something we pointed out in Example 1(b). As we pointed out at the time the three vectors we were looking at were the standard basis vectors for $\mathbb{R}^3$. We should discuss the name a little more at this point and we'll do it a little more generally than in $\mathbb{R}^3$.

The vectors

$$\mathbf{e}_1 = (1,0,0,\ldots,0) \qquad \mathbf{e}_2 = (0,1,0,\ldots,0) \qquad \cdots \qquad \mathbf{e}_n = (0,0,0,\ldots,1)$$

will span $\mathbb{R}^n$ as we saw in the section on Span and it is fairly simple to show that these vectors are linearly independent (you should verify this) and so they form a basis for $\mathbb{R}^n$. In some way this set of vectors is the simplest (we'll see this in a bit) and so we call them the **standard basis vectors** for $\mathbb{R}^n$.

We also have a set of standard basis vectors for a couple of the other vector spaces we've been looking at occasionally. Let's take a look at each of them.

Example 2 The set $\mathbf{p}_0 = 1$, $\mathbf{p}_1 = x$, $\mathbf{p}_2 = x^2$, … , $\mathbf{p}_n = x^n$ is a basis for P_n and is usually called the **standard basis** for P_n.

In Example 4 of the section on Span we showed that this set will span P_n. In Example 4 of the section on Linear Independence we shows that for $n = 3$ these form a linearly independent set in P_3. A similar argument can be used for the general case here and we'll leave it to you to go through that argument.

So, this set of vectors is in fact a basis for P_n.

Example 3 The set $\mathbf{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is a basis for M_{22} and is usually called the **standard basis** for M_{22}.

In Example 4 of the section on Span we showed that this set will span M_{22}. We have yet so show that they are linearly in dependent however. So, following the procedure from the last section we know that we need to set up the following equation,

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c_1 & c_3 \\ c_2 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So, the only way the matrix on the left can be the zero matrix is for all the scalars to be zero. In other words, this equation has only the trivial solution and so the matrices are linearly independent.

This combined with the fact that they span M_{22} shows that they are in fact a basis for M_{22}.

Note that we only looked at the standard basis vectors for M_{22}, but you should be able to modify this appropriately to arrive at a set of standard basis vector for M_{nm} in general.

Next let's take a look at the following theorem that gives us one of the reasons for being interested in a set of basis vectors.

Theorem 1 Suppose that the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for the vector space *V* then every vector **u** from *V* can be expressed as a linear combination of the vectors from *S* in exactly one way.

Proof : First, since we know that the vectors in *S* are a basis for *V* then for any vector **u** in *V* we can write it as a linear combination as follows,

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

Now, let's suppose that it is also possible to write it as the following linear combination,

$$\mathbf{u} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$$

If we take the difference of these two linear combinations we get,

$$\mathbf{0} = \mathbf{u} - \mathbf{u} = (c_1 - k_1)\mathbf{v}_1 + (c_2 - k_2)\mathbf{v}_2 + \cdots + (c_n - k_n)\mathbf{v}_n$$

However, because the vectors in *S* are a basis they are linearly independent. That means that this equation can only have the trivial solution. Or, in other words, we must have,

$$c_1 - k_1 = 0,\ c_2 - k_2 = 0,\ \ldots\ ,\ c_n - k_n = 0$$

But this means that,

$$c_1 = k_1,\ \ c_2 = k_2,\ \ \ldots\ ,\ \ c_n = k_n$$

and so the two linear combinations were in fact the same linear combination.

■

We also have the following fact. It probably doesn't really rise to the level of a theorem, but we'll call it that anyway.

Theorem 2 Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set of linearly independent vectors then *S* is a basis for the vector space $V = \text{span}(S)$

The proof here is so simple that we're not really going to give it. By assumption the set is linearly independent and by definition *V* is the span of *S* and so the set must be a basis for *V*.

We now need to take a look at the following definition.

Definition 2 Suppose that V is a non-zero vector space and that S is a set of vectors from V that for a basis for V. If S contains a finite number of vectors, say $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, then we call V a **finite dimensional** vector space and we say that the **dimension** of V, denoted by $\dim(V)$, is n (*i.e.* the number of basis elements in S. If V is not a finite dimensional vector space (so S does not have a finite number of vectors) then we call it an **infinite dimensional** vector space.

By definition the dimension of the zero vector space (*i.e.* the vector space consisting solely of the zero vector) is zero.

Here are the dimensions of some of the vector spaces we've been dealing with to this point.

Example 4 Dimensions of some vector spaces.

(a) $\dim(\mathbb{R}^n) = n$ since the standard basis vectors for $\mathbb{R}^n$ are,

$$\mathbf{e}_1 = (1,0,0,\ldots,0) \qquad \mathbf{e}_2 = (0,1,0,\ldots,0) \qquad \cdots \qquad \mathbf{e}_n = (0,0,0,\ldots,1)$$

(b) $\dim(P_n) = n+1$ since the standard basis vectors for P_n are,

$$\mathbf{p}_0 = 1 \qquad \mathbf{p}_1 = x \qquad \mathbf{p}_2 = x^2 \qquad \cdots \qquad \mathbf{p}_n = x^n$$

(c) $\dim(M_{22}) = (2)(2) = 4$ since the standard basis vectors for M_{22} are,

$$\mathbf{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(d) $\dim(M_{nm}) = nm$. This follows from the natural extension of the previous part. The set of standard basis vectors will be a set of vectors that are zero in all entries except one entry which is a 1. There are nm possible positions of the 1 and so there must be nm basis vectors

(e) The set of real valued functions on a interval, $F[a,b]$, and the set of continuous functions on an interval, $C[a,b]$, are infinite dimensional vector spaces. This is not easy to show at this point, but here is something to think about. If we take all the polynomials (of all degrees) then we can form a set (see part **(b)** above for elements of that set) that does not have a finite number of elements in it and yet is linearly independent. This set will be in either of the two vector spaces above and in the following theorem we can show that there will be no finite basis set for these vector spaces.

We now need to take a look at several important theorems about vector spaces. The first couple of theorems will give us some nice ideas about linearly independent/dependent sets and spans. One of the more important uses of these two theorems is constructing a set of basis vectors as we'll see eventually.

Theorem 3 Suppose that *V* is a vector space and that $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is any basis for *V*.

(a) If a set has more than *n* vectors then it is linearly dependent.
(b) If a set has fewer than *n* vectors then it does not span *V*.

Proof :

(a) Let $R = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ and suppose that $m > n$. Since *S* is a basis of *V* every vector in *R* can be written as a linear combination of vectors from *S* as follows,

$$\begin{aligned} \mathbf{w}_1 &= a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2 + \cdots + a_{n1}\mathbf{v}_n \\ \mathbf{w}_2 &= a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2 + \cdots + a_{n2}\mathbf{v}_n \\ &\vdots \\ \mathbf{w}_m &= a_{1m}\mathbf{v}_1 + a_{2m}\mathbf{v}_2 + \cdots + a_{nm}\mathbf{v}_n \end{aligned}$$

Now, we want to show that the vectors in *R* are linearly dependent. So, we'll need to show that there are more solutions than just the trivial solution to the following equation.

$$k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \cdots + k_m\mathbf{w}_m = 0$$

If we plug in the set of linear combinations above for the $\mathbf{w}_i$'s in this equation and collect all the coefficients of the $\mathbf{v}_j$'s we arrive at.

$$\begin{aligned} \left(a_{11}k_1 + a_{12}k_2 + \cdots + a_{1m}k_m\right)\mathbf{v}_1 + \left(a_{21}k_1 + a_{22}k_2 + \cdots + a_{2m}k_m\right)\mathbf{v}_2 + \cdots \\ + \left(a_{n1}k_1 + a_{n2}k_2 + \cdots + a_{nm}k\right)\mathbf{v}_n = \mathbf{0} \end{aligned}$$

Now, the $\mathbf{v}_j$'s are linearly independent and so we know that the coefficients of each of the $\mathbf{v}_j$ in this equation must be zero. This gives the following system of equations.

$$\begin{aligned} a_{11}k_1 + a_{12}k_2 + \cdots + a_{1m}k_m &= 0 \\ a_{21}k_1 + a_{22}k_2 + \cdots + a_{2m}k_m &= 0 \\ &\vdots \\ a_{n1}k_1 + a_{n2}k_2 + \cdots + a_{nm}k_m &= 0 \end{aligned}$$

Now, in this system the a_{ij}'s are known scalars from the linear combinations above and the k_i's are unknowns. So we can see that there are *n* equations and *m* unknowns. However, because $m > n$ there are more unknowns than equations and so by Theorem 2 in the solving systems of equations section we know that if there are more unknowns than equations in a homogeneous system, as we have here, there will be infinitely many solutions.

Therefore the equation,

$$k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \cdots + k_m\mathbf{w}_m = 0$$

will have more solutions than the trivial solution and so the vectors in *R* must be linearly dependent.

(b) The proof of this part is very similar to the previous part. Let's start with the set $R = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ and this time we're going to assume that $m < n$. It's not so easy to show directly that *R* will not span *V*, but if we assume for a second that *R* does span *V* we'll see that we'll run into some problems with our basis set *S*. This is called a proof by contradiction. We'll assume the opposite of what we want to prove and show that this will lead to a contradiction of something that we know is true (in this case that *S* is a basis for *V*).

So, we'll assume that *R* will span *V*. This means that all the vectors in *S* can be written as a linear combination of the vectors in *R* or,

$$\begin{gathered}
\mathbf{v}_1 = a_{11}\mathbf{w}_1 + a_{21}\mathbf{w}_2 + \cdots + a_{m1}\mathbf{w}_m \\
\mathbf{v}_2 = a_{12}\mathbf{w}_1 + a_{22}\mathbf{w}_2 + \cdots + a_{m2}\mathbf{w}_m \\
\vdots \\
\mathbf{v}_n = a_{1n}\mathbf{w}_1 + a_{2n}\mathbf{w}_2 + \cdots + a_{mn}\mathbf{w}_m
\end{gathered}$$

Let's now look at the equation,

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n = 0$$

Now, because *S* is a basis we know that the $\mathbf{v}_i$'s must be linearly independent and so the only solution to this must be the trivial solution. However, if we substitute the linear combinations of the $\mathbf{v}_i$'s into this, rearrange as we did in part **(a)** and then setting all the coefficients equal to zero gives the following system of equations.

$$\begin{gathered}
a_{11}k_1 + a_{12}k_2 + \cdots + a_{1n}k_n = 0 \\
a_{21}k_1 + a_{22}k_2 + \cdots + a_{2n}k_n = 0 \\
\vdots \\
a_{m1}k_1 + a_{m2}k_2 + \cdots + a_{mn}k_n = 0
\end{gathered}$$

Again, there are more unknowns than equations here and so there are infinitely many solutions. This contradicts the fact that we know the only solution to the equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n = 0$$

is the trivial solution.

So, our original assumption that *R* spans *V* must be wrong. Therefore *R* will not span *V*.
■

Theorem 4 Suppose *S* is a non-empty set of vectors in a vector space *V*.

(a) If *S* is linearly independent and **u** is any vector in *V* that is not in $\operatorname{span}(S)$

then the set $R = S \cup \{\mathbf{u}\}$ (*i.e.* the set of *S* and **u**) is also a linearly independent set.

(b) If **u** is any vector in *S* that can be written as a linear combination of the other vectors in *S* let $R = S - \{\mathbf{u}\}$ be the set we get by removing **u** from *S*. Then,

$$\operatorname{span}(S) = \operatorname{span}(R)$$

In other words, *S* and $S - \{\mathbf{u}\}$ will span the same space.

Proof :

(a) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ we need to show that the set $R = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \mathbf{u}\}$ is linearly independent. So, let's form the equation,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n + c_{n+1}\mathbf{u} = \mathbf{0}$$

Now, if c_{n+1} is not zero we will be able to write **u** as a linear combination of the $\mathbf{v}_i$'s but this contradicts the fact that **u** is not in $\operatorname{span}(S)$. Therefore we must have $c_{n+1} = 0$ and our equation is now,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

But the vectors in *S* are linearly dependent and so the only solution to this is the trivial solution,

$$c_1 = 0 \quad c_2 = 0 \quad \cdots \quad c_n = 0$$

So, we've shown that the only solution to

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n + c_{n+1}\mathbf{u} = \mathbf{0}$$

is

$$c_1 = 0 \quad c_2 = 0 \quad \cdots \quad c_n = 0 \quad c_{n+1} = 0$$

Therefore, the vectors in *R* are linearly independent.

(b) Let's suppose that our set is $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \mathbf{u}\}$ and so we have $R = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. First, by assumption **u** is a linear combination of the remaining vectors in *S* or,

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

Next let **w** be any vector in $\operatorname{span}(S)$. So, **w** can be written as a linear combination of all the vectors in *S* or,

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n + k_{n+1}\mathbf{u}$$

Now plug in the expression for **u** above to get,

$$\begin{aligned}\mathbf{w} &= k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n + k_{n+1}\left(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n\right)\\ &= \left(k_1 + k_{n+1}c_1\right)\mathbf{v}_1 + \left(k_2 + k_{n+1}c_2\right)\mathbf{v}_2 + \cdots + \left(k_n + k_{n+1}c_n\right)\mathbf{v}_n\end{aligned}$$

So, **w** is a linear combination of vectors only in *R* and so at the least every vector that is in $\text{span}(S)$ must also be in $\text{span}(R)$.

Finally, if **w** is any vector in $\text{span}(R)$ then it can be written as a linear combination of vectors from *R*, but since these are also vectors in *S* we see that **w** can also, by default, be written as a linear combination of vectors from *S* and so is also in $\text{span}(S)$. We've just shown that every vector in $\text{span}(R)$ must also be in $\text{span}(S)$.

Since we've shown that $\text{span}(S)$ must be contained in $\text{span}(R)$ and that every vector in $\text{span}(R)$ must also be contained in $\text{span}(S)$ this can only be true if $\text{span}(S) = \text{span}(R)$.

■

We can use the previous two theorems to get some nice ideas about the basis of a vector space.

Theorem 5 Suppose that *V* is a vector space then all the bases for *V* contain the same number of vectors.

Proof : Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for *V*. Now, let *R* be any other basis for *V*. Then by Theorem 3 above if *R* contains more than *n* elements it can't be a linearly independent set and so can't be a basis. So, we know that, at the least *R* can't contain more than *n* elements. However, Theorem 3 also tells us that if *R* contains less than *n* elements then it won't span *V* and hence can't be a basis for *V*. Therefore the only possibility is that *R* must contain exactly *n* elements.

■

Theorem 6 Suppose that *V* is a vector space and that $\dim(V) = n$. Also suppose that *S* is a set that contains exactly *n* vectors. *S* will be a basis for *V* if either $V = \text{span}\{S\}$ or *S* is linearly independent.

Proof : First suppose that $V = \text{span}\{S\}$. If *S* is linearly dependent then there must be some vector **u** in *S* that can be written as a linear combination of other vectors in *S* and so by Theorem 4(b) we can remove **u** from *S* and our new set of $n-1$ vectors will still span *V*. However, Theorem 3(b) tells us that any set with fewer vectors than a basis (*i.e.* less than *n* in this case) can't span *V*. Therefore, *S* must be linearly independent and hence *S* is a basis for *V*.

Now, let's suppose that *S* is linearly independent. If *S* does not span *V* then there must be a vector **u** that is not in $\text{span}(S)$. If we add **u** to *S* the resulting set with $n+1$ vectors must be linearly independent by Theorem 4(a). On the other hand, Theorem 3(a) tells us that any set with more vectors than the basis (*i.e.* greater than n) can't be linearly independent. Therefore, S must span V and hence S is a basis for V.

■

Theorem 7 Suppose that V is a finite dimensional vector space with $\dim(V) = n$ and that S is any finite set of vectors from V.

- **(a)** If S spans V but is not a basis for V then it can be reduced to a basis for V by removing certain vectors from S.
- **(b)** If S is linearly independent but is not a basis V then it can be enlarged to a basis for V by adding in certain vectors from V.

Proof :

(a) If S spans V but is not a basis for V then it must be a linearly dependent set. So, there is some vector $\mathbf{u}$ in S that can be written as a linear combination of the other vectors in S. Let R be the set that results from removing $\mathbf{u}$ from S. Then by Theorem 4(b) R will still span V. If R is linearly independent then we have a basis for V and if it is still linearly dependent we can remove another element to form a new set R' that will still span V. We continue in this way until we've reduced S down to a set of linearly independent vectors and at that point we will have a basis of V.

(b) If S is linearly independent but not a basis then it must not span V. Therefore, there is a vector $\mathbf{u}$ that is not in $\text{span}(S)$. So, add $\mathbf{u}$ to S to form the new set R. Then by Theorem 4(a) the set R is still linearly independent. If R now spans V we've got a basis for V and if not add another element to form the new linearly independent set R'. Continue in this fashion until we reach a set with n vectors and then by Theorem 6 this set must be a basis for V.

■

Okay we should probably see some examples of some of these theorems in action.

Example 5 Reduce each of the following sets of vectors to obtain a basis for the given vector space.

- **(a)** $\mathbf{v}_1 = (1,0,0)$, $\mathbf{v}_2 = (0,1,-1)$, $\mathbf{v}_3 = (0,4,-3)$ and $\mathbf{v}_4 = (0,2,0)$ for $\mathbb{R}^3$.
- **(b)** $\mathbf{p}_0 = 2$, $\mathbf{p}_1 = -4x$, $\mathbf{p}_2 = x^2 + x + 1$, $\mathbf{p}_3 = 2x + 7$ and $\mathbf{p}_4 = 5x^2 - 1$ for P_2

Solution

First, notice that provided each of these sets of vectors spans the given vector space Theorem 7(a) tells us that this can in fact be done.

(a) We will leave it to you to verify that this set of vectors does indeed span $\mathbb{R}^3$ and since we know that $\dim(\mathbb{R}^3) = 3$ we can see that we'll need to remove one vector from the list in order to get down to a basis. However, we can't just remove any of the vectors. For instance if we removed $\mathbf{v}_1$ the set would no longer span $\mathbb{R}^3$. You should verify this, but you can also quickly see that only $\mathbf{v}_1$ has a non-zero first component and so will be required for the vectors to span $\mathbb{R}^3$.

Theorem 4(b) tells us that if we remove a vector that is a linear combination of some of the other vectors we won't change the span of the set. So, that is what we need to look for. Now, it looks like the last three vectors are probably linearly dependent so if we set up the following equation

$$c_1\mathbf{v}_2 + c_2\mathbf{v}_3 + c_3\mathbf{v}_4 = \mathbf{0}$$

and solve it,

$$c_1 = 6t \qquad c_2 = -2t \qquad c_3 = t \qquad t \text{ is any real number}$$

we can see that these in fact are linearly dependent vectors. This means that we can remove any of these since we could write any one of them as a linear combination of the other two. So, let's remove $\mathbf{v}_3$ for no other reason that the entries in this vector are larger than the others.

The following set the still spans $\mathbb{R}^3$ and has exactly 3 vectors and so by Theorem 6 it must be a basis for $\mathbb{R}^3$.

$$\mathbf{v}_1 = (1,0,0) \qquad \mathbf{v}_2 = (0,1,-1) \qquad \mathbf{v}_4 = (0,2,0)$$

For the practice you should verify that this set does span $\mathbb{R}^3$ and is linearly independent.

(b) We'll go through this one a little faster. First, you should verify that the set of vectors does indeed span P_2. Also, because $\dim(P_2) = 3$ we know that we'll need to remove two of the vectors. Again, remember that each vector we remove must be a linear combination of some of the other vectors.

First, it looks like $\mathbf{p}_3$ is a linear combination of $\mathbf{p}_0$ and $\mathbf{p}_1$ (you should verify this) and so we can remove $\mathbf{p}_3$ and the set will still span P_2. This leaves us with the following set of vectors.

$$\mathbf{p}_0 = 2 \qquad \mathbf{p}_1 = -4x \qquad \mathbf{p}_2 = x^2 + x + 1 \qquad \mathbf{p}_4 = 5x^2 - 1$$

Now, it looks like $\mathbf{p}_2$ can easily be written as a linear combination of the remaining vectors (again, please verify this) and so we can remove that one as well.

We now have the following set,

$$\mathbf{p}_0 = 2 \qquad \mathbf{p}_1 = -4x \qquad \mathbf{p}_4 = 5x^2 - 1$$

which has 3 vectors and will span P_2 and so it must be a basis for P_2 by Theorem 6.

Example 6 Expand each of the following sets of vectors into a basis for the given vector space.

(a) $\mathbf{v}_1 = (1,0,0,0)$, $\mathbf{v}_2 = (1,1,0,0)$, $\mathbf{v}_3 = (1,1,1,0)$ in $\mathbb{R}^4$.

(b) $\mathbf{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix}$ in M_{22}.

Solution

Theorem 7(b) tells us that this is possible to do provided the sets are linearly independent.

(a) We'll leave it to you to verify that these vectors are linearly independent. Also, $\dim\left(\mathbb{R}^4\right)=4$ and so it looks like we'll just need to add in a single vector to get a basis. Theorem 4(a) tells us that provided the vector we add in is not in the span of the original vectors we can retain the linear independence of the vectors. This will in turn give us a set of 4 linearly independent vectors and so by Theorem 6 will have to be a basis for $\mathbb{R}^4$.

Now, we need to find a vector that is not in the span of the given vectors. This is easy to do provided you notice that all of the vectors have a zero in the fourth component. This means that all the vectors that are in $\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ will have a zero in the fourth component. Therefore, all that we need to do is take any vector that has a non-zero fourth component and we'll have a vector that is outside $\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$. Here are some possible vectors we could use,

$$(0,0,0,1) \qquad (0,-4,0,2) \qquad (6,-3,2,-1) \qquad (1,1,1,1)$$

The last one seems to be in keeping with the pattern of the original three vectors so we'll use that one to get the following set of four vectors.

$$\mathbf{v}_1=(1,0,0,0) \quad \mathbf{v}_2=(1,1,0,0) \quad \mathbf{v}_3=(1,1,1,0) \quad \mathbf{v}_4=(1,1,1,1)$$

Since this set is still linearly independent and now has 4 vectors by Theorem 6 this set is a basis for $\mathbb{R}^4$ (you should verify this).

(b) The two vectors here are linearly independent (verify this) and $\dim\left(M_{22}\right)=4$ and so we'll need to add in two vectors to get a basis. We will have to do this in two steps however. The first vector we add cannot be in $\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2\}$ and the second vector we can cannot be in $\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ where $\mathbf{v}_3$ is the new vector we added in the first step.

So, first notice that all the vectors in $\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2\}$ will have zeroes in the second column so anything that doesn't have a zero in at least one entry in the second column will work for $\mathbf{v}_3$. We'll choose the following for $\mathbf{v}_3$.

$$\mathbf{v}_3=\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}$$

Note that this is probably not the best choice since its got non-zero entries in both entries of the second column. It would have been easier to choose something that had a zero in one of the entries of the second column. However, if we don't do that this will allow us make a point about choosing the second vector. Here is the list of vectors that we've got to this point.

$$\mathbf{v}_1=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} \qquad \mathbf{v}_2=\begin{bmatrix}2 & 0\\ -1 & 0\end{bmatrix} \qquad \mathbf{v}_3=\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}$$

Now, we need to find a fourth vector and it needs to be outside of $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Now, let's again note that because of our choice of $\mathbf{v}_3$ all the vectors in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ will have identical numbers in both entries of the second column and so we can chose any new vector that does not have identical entries in the second column and we'll have something that is outside of $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Again, we'll go with something that is probably not the best choice if we had to work with this basis, but let's not get too locked into always taking the easy choice. There are, on occasion, reason to choose vectors other than the "obvious" and easy choices. In this case we'll use,

$$\mathbf{v}_4 = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$

This gives us the following set of vectors,

$$\mathbf{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$

and they will be a basis for M_{22} since these are four linearly independent vectors in a vector space with dimension of 4.

We'll close out this section with a couple of theorems and an example that will relate the dimensions of subspaces of a vector space to the dimension of the vector space itself.

Theorem 8 Suppose that *W* is a subspace of a finite dimensional vector space *V* then *W* is also finite dimensional.

Proof : Suppose that $\dim(V) = n$. Let's also suppose that *W* is not finite dimensional and suppose that *S* is a basis for *W*. Since we've assumed that *W* is not finite dimensional we know that *S* will not have a finite number of vectors in it. However, since *S* is a basis for *W* we know that they must be linearly independent and we also know that they must be vectors in *V*. This however, means that we've got a set of more than *n* vectors that is linearly independent and this contradicts the results of Theorem 3(a).

Therefore *W* must be finite dimensional as well.

■

We can actually go a step further here than this theorem.

Theorem 9 Suppose that *W* is a subspace of a finite dimensional vector space *V* then $\dim(W) \le \dim(V)$ and if $\dim(W) = \dim(V)$ then in fact we have $W = V$.

Proof : By Theorem 8 we know that *W* must be a finite dimensional vector space and so let's suppose that $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ is a basis for *W*. Now, *S* is either a basis for *V* or it isn't a basis for *V*.

If S is a basis for V then by Theorem 5 we have that $\dim(V) = \dim(W) = n$.

On the other hand, if S is not a basis for V by Theorem 7(b) (the vectors of S must be linearly independent since they form a basis for W) it can be expanded into a basis for V and so we then know that $\dim(W) < \dim(V)$.

So, we've shown that in every case we must have $\dim(W) \le \dim(V)$.

Now, let's just assume that all we know is that $\dim(W) = \dim(V)$. In this case S will be a set of n linearly independent vectors in a vector space of dimension n (since $\dim(W) = \dim(V)$) and so by Theorem 6, S must be a basis for V as well. This means that any vector **u** from V can be written as a linear combination of vectors from S. However, since S is also a basis for W this means that **u** must also be in W.

So, we've just shown that every vector in V must also be in W, and because W is a subspace of V we know that every vector in W is also in V. The only way for this to be true is if we have $W = V$.

■

We should probably work one quick example illustrating this theorem.

Example 7 Determine a basis and dimension for the null space of

$$A = \begin{bmatrix} 7 & 2 & -2 & -4 & 3 \\ -3 & -3 & 0 & 2 & 1 \\ 4 & -1 & -8 & 0 & 20 \end{bmatrix}$$

Solution

First recall that to find the null space of a matrix we need to solve the following system of equations,

$$\begin{bmatrix} 7 & 2 & -2 & -4 & 3 \\ -3 & -3 & 0 & 2 & 1 \\ 4 & -1 & -8 & 0 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We solved a similar system back in Example 7 of the Solving Systems of Equation section so we'll leave it to you to verify that the solution is,

$$x_1 = \frac{2}{3}t + \frac{1}{3}s \qquad x_2 = 0 \qquad x_3 = \frac{1}{3}t + \frac{8}{3}s$$

$$x_4 = t \qquad x_5 = s \qquad s \text{ and } t \text{ are any numbers}$$

Now, recall that the null space of an $n \times m$ matrix will be a subspace of $\mathbb{R}^m$ so the null space of this matrix must be a subspace of $\mathbb{R}^5$ and so its dimension should be 5 or less.

To verify this we'll need the basis for the null space. This is actually easier to find than you might think. The null space will consist of all vectors in $\mathbb{R}^5$ that have the form,

$$\begin{aligned}\mathbf{x} &= \left(x_1, x_2, x_3, x_4, x_5\right)\\ &= \left(\frac{2}{3}t+\frac{1}{3}s,\ 0,\ \frac{1}{3}t+\frac{8}{3}s,\ t,\ s\right)\end{aligned}$$

Now, split this up into two vectors. One that contains only terms with a *t* in them and one that contains only term with an *s* in them. Then factor the *t* and *s* out of the vectors.

$$\begin{aligned}\mathbf{x} &= \left(\frac{2}{3}t,\ 0,\ \frac{1}{3}t,\ t,\ 0\right)+\left(\frac{1}{3}s,\ 0,\ \frac{8}{3}s,\ 0,\ s\right)\\ &= t\left(\frac{2}{3},\ 0,\ \frac{1}{3},\ 1,\ 0\right)+s\left(\frac{1}{3},\ 0,\ \frac{8}{3},\ 0,\ 1\right)\end{aligned}$$

So, we can see that the null space is the space that is the set of all vectors that are a linear combination of

$$\mathbf{v}_1 = \left(\frac{2}{3},\ 0,\ \frac{1}{3},\ 1,\ 0\right) \qquad \mathbf{v}_2 = \left(\frac{1}{3},\ 0,\ \frac{8}{3},\ 0,\ 1\right)$$

and so the null space of *A* is spanned by these two vectors. You should also verify that these two vectors are linearly independent and so they in fact form a basis for the null space of *A*. This also means that the null space of *A* has a dimension of 2 which is less than 5 as Theorem 9 suggests it should be.

Change of Basis

In Example 1 of the previous section we saw that the vectors $\mathbf{v}_1 = (1,-1,1)$, $\mathbf{v}_2 = (0,1,2)$ and $\mathbf{v}_3 = (3,0,-1)$ formed a basis for $\mathbb{R}^3$. This means that every vector in $\mathbb{R}^3$, for example the vector $\mathbf{x} = (10,5,0)$, can be written as a linear combination of these three vectors. Of course this is not the only basis for $\mathbb{R}^3$. There are many other bases for $\mathbb{R}^3$ out there in the world, not the least of which is the standard basis for $\mathbb{R}^3$,

$$\mathbf{e}_1 = (1,0,0) \qquad \mathbf{e}_2 = (0,1,0) \qquad \mathbf{e}_3 = (0,0,1)$$

The standard basis for any vector space is generally the easiest to work with, but unfortunately there are times when we need to work with other bases. In this section we're going to take a look at a way to move between two different bases for a vector space and see how to write a general vector as a linear combination of the vectors from each basis.

To start this section off we're going to first need a way to quickly distinguish between the various linear combinations we get from each basis. The following definition will help with this.

Definition 1 Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space *V* and that **u** is any vector from *V*. Since **u** is a vector in *V* it can be expressed as a linear combination of the vectors from *S* as follows,

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

The scalars $c_1, c_2, \ldots, c_n$ are called the **coordinates** of **u** relative to the basis *S*. The **coordinate vectors of *u* relative to *S*** is denoted by $(\mathbf{u})_S$ and defined to be the following vector in $\mathbb{R}^n$,

$$(\mathbf{u})_S = (c_1, c_2, \ldots, c_n)$$

Note that by Theorem 1 of the previous section we know that the linear combination of vectors from the basis will be unique for **u** and so the coordinate vector $(\mathbf{u})_S$ will also be unique.

Also, on occasion it will be convenient to think of the coordinate vector as a matrix. In these cases we will call it the **coordinate matrix of *u* relative to *S***. The coordinate matrix will be denoted and defined as follows,

$$[\mathbf{u}]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

At this point we should probably also give a quick warning about the coordinate vectors. In most cases, although not all as we'll see shortly, the coordinate vector/matrix is NOT the vector itself that we're after. It is nothing more than the coefficients of the basis vectors that we need in order to write the given vector as a linear combination of the basis vectors. It is very easy to confuse the coordinate vector/matrix with the vector itself if we aren't paying attention, so be careful.

Let's see some examples of coordinate vectors.

Example 1 Determine the coordinate vector of $\mathbf{x} = (10, 5, 0)$ relative to the following bases.

(a) The standard basis vectors for $\mathbb{R}^3$, $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

(b) The basis $A = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where, $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (0, 1, 2)$ and $\mathbf{v}_3 = (3, 0, -1)$.

Solution

In each case we'll need to determine who to write $\mathbf{x} = (10, 5, 0)$ as a linear combination of the given basis vectors.

(a) In this case the linear combination is simple to write down.

$$\mathbf{x} = (10, 5, 0) = 10\mathbf{e}_1 + 5\mathbf{e}_2 + 0\mathbf{e}_3$$

and so the coordinate vectors for **x** relative to the standard basis vectors for $\mathbb{R}^3$ is,

$$(\mathbf{x})_S = (10, 5, 0)$$

So, in the case of the standard basis vectors we've got that,

$$(\mathbf{x})_S = (10, 5, 0) = \mathbf{x}$$

this is, of course, what makes the standard basis vectors so nice to work with. The coordinate vectors relative to the standard basis vectors is just the vector itself.

(b) Now, in this case we'll have a little work to do. We'll first need to set up the following vector equation,

$$(10, 5, 0) = c_1(1, -1, 1) + c_2(0, 1, 2) + c_3(3, 0, -1)$$

and we'll need to determine the scalars c_1, c_2 and c_3. We saw how to solve this kind of vector equation in both the section on Span and the section on Linear Independence. We need to set up the following system of equations,

$$\begin{aligned} c_1 + 3c_3 &= 10 \\ -c_1 + c_2 &= 5 \\ c_1 + 2c_2 - c_3 &= 0 \end{aligned}$$

We'll leave it to you to verify that the solution to this system is,

$$c_1 = -2 \qquad c_2 = 3 \qquad c_3 = 4$$

The coordinate vector for **x** relative to A is then,

$$(\mathbf{x})_A = (-2, 3, 4)$$

As always we should do an example or two in a vector space other than $\mathbb{R}^n$.

Example 2 Determine the coordinate vector of $\mathbf{p} = 4 - 2x + 3x^2$ relative to the following bases.

(a) The standard basis for P_2 $S = \{1, x, x^2\}$.

(b) The basis for P_2, $A = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$, where $\mathbf{p}_1 = 2$, $\mathbf{p}_2 = -4x$, and $\mathbf{p}_3 = 5x^2 - 1$.

Solution

(a) So, we need to write **p** as a linear combination of the standard basis vectors in this case. However, it's already written in that way. So, the coordinate vector for **p** relative to the standard basis vectors is,

$$(\mathbf{p})_S = (4, -2, 3)$$

The ease with which we can write down this vector is why this set of vectors is standard basis vectors for P_2.

(b) Okay, this set is similar to the standard basis vectors, but they are a little different so we can expect the coordinate vector to change. Note as well that we proved in Example 5 of the previous section that this set is a basis.

We'll need to find scalars c_1, c_2 and c_3 for the following linear combination.

$$4-2x+3x^2 = c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = c_1(2) + c_2(-4x) + c_3(5x^2-1)$$

The will mean solving the following system of equations.

$$\begin{aligned} 2c_1 - c_3 &= 4 \\ -4c_2 &= -2 \\ 5c_3 &= 3 \end{aligned}$$

This is not a terribly difficult system to solve. Here is the solution,

$$c_1 = \frac{23}{10} \qquad c_2 = \frac{1}{2} \qquad c_3 = \frac{3}{5}$$

The coordinate vector for **p** relative to this basis is then,

$$(\mathbf{p})_A = \left(\frac{23}{10}, \frac{1}{2}, \frac{3}{5}\right)$$

Example 3 Determine the coordinate vector of $\mathbf{v} = \begin{bmatrix} -1 & 0 \\ 1 & -4 \end{bmatrix}$ relative to the following bases.

(a) The standard basis of M_{22}, $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

(b) The basis for M_{22}, $A = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ where $\mathbf{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$.

Solution

(a) As with the previous two examples the standard basis is called that for a reason. It is very easy to write any 2×2 matrix as a linear combination of these vectors. Here it is for this case.

$$\begin{bmatrix} -1 & 0 \\ 1 & -4 \end{bmatrix} = (-1)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (1)\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (0)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + (-4)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The coordinate vector for **v** relative to the standard basis is then,

$$(\mathbf{v})_S = (-1, 1, 0, -4)$$

(b) This one will be a little work, as usual, but won't be too bad. We'll need to find scalars c_1, c_2, c_3 and c_4 for the following linear combination.

$$\begin{bmatrix} -1 & 0 \\ 1 & -4 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + c_4 \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$

Adding the matrices on the right into a single matrix and setting components equal gives the following system of equations that will need to be solved.

$$\begin{aligned} c_1 + 2c_2 - 3c_4 &= -1 \\ -c_2 &= 1 \\ c_3 &= 0 \\ c_3 + 2c_4 &= -4 \end{aligned}$$

Not a bad system to solve. Here is the solution.

$$c_1 = -5 \qquad c_2 = -1 \qquad c_3 = 0 \qquad c_4 = -2$$

The coordinate vector for **v** relative to this basis is then,

$$(\mathbf{v})_A = (-5, -1, 0, -2)$$

Before we move on we should point out that the order in which we list our basis elements is important, to see this let's take a look at the following example.

Example 4 The vectors $\mathbf{v}_1 = (1,-1,1)$, $\mathbf{v}_2 = (0,1,2)$ and $\mathbf{v}_3 = (3,0,-1)$ form a basis for $\mathbb{R}^3$. Let $A = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $B = \{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_1\}$ be different orderings of these vectors and determine the vector in $\mathbb{R}^3$ that has the following coordinate vectors.

(a) $(\mathbf{x})_A = (3,-1,8)$

(b) $(\mathbf{x})_B = (3,-1,8)$

Solution

So, these are both the same coordinate vector, but they are relative to different orderings of the basis vectors. Determining the vector in $\mathbb{R}^3$ for each is a simple thing to do. Recall that the coordinate vector is nothing more than the scalars in the linear combination and so all we need to do is reform the linear combination and then multiply and add everything out to determine the vector.

The one thing that we need to be careful of order however. The first scalar is the coefficient of the first vector listed in the set, the second scalar in the coordinate vector is the coefficient for the second vector listed, *etc*.

(a) Here is the work for this part.

$$\mathbf{x} = 3(1,-1,1) + (-1)(0,1,2) + (8)(3,0,-1) = (27,-4,-7)$$

(b) And here is the work for this part.

$$\mathbf{x} = 3(0,1,2) + (-1)(3,0,-1) + (8)(1,-1,1) = (5,-5,15)$$

So, we clearly get different vectors simply be rearranging the order of the vectors in our basis.

Now that we've got the coordinate vectors out of the way we want to find a quick and easy way to convert between the coordinate vectors from one basis to a different basis. This is called a change of basis. Actually, it will be easier to convert the coordinate matrix for a vector, but these are essentially the same thing as the coordinate vectors so if we can convert one we can convert the other.

We will develop the method for vectors in a 2-dimensional space (not necessarily $\mathbb{R}^2$) and in the process we will see how to do this for any vector space. So let's start off and assume that *V* is a vector space and that $\dim(V)=2$. Let's also suppose that we have two bases for *V*. The "old" basis,

$$B=\{\mathbf{v}_1,\mathbf{v}_2\}$$

and the "new" basis,

$$C=\{\mathbf{w}_1,\mathbf{w}_2\}$$

Now, because *B* is a basis for *V* we can write each of the basis vectors from *C* as a linear combination of the vectors from *B*.

$$\begin{aligned}\mathbf{w}_1&=a\,\mathbf{v}_1+b\,\mathbf{v}_2\\ \mathbf{w}_2&=c\,\mathbf{v}_1+d\,\mathbf{v}_2\end{aligned}$$

This means that the coordinate matrices of the vectors from *C* relative to the basis *B* are,

$$[\mathbf{w}_1]_B=\begin{bmatrix}a\\b\end{bmatrix}\qquad\qquad [\mathbf{w}_2]_B=\begin{bmatrix}c\\d\end{bmatrix}$$

Next, let **u** be any vector in *V*. In terms of the new basis, *C*, we can write **u** as,

$$\mathbf{u}=c_1\mathbf{w}_1+c_2\mathbf{w}_2$$

and so its coordinate matrix relative to *C* is,

$$[\mathbf{u}]_C=\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

Now, we know how to write the basis vectors from *C* as linear combinations of the basis vectors from *B* so substitute these into the linear combination for **u** above. This gives,

$$\mathbf{u}=c_1\left(a\,\mathbf{v}_1+b\,\mathbf{v}_2\right)+c_2\left(c\,\mathbf{v}_1+d\,\mathbf{v}_2\right)$$

Rearranging gives the following equation.

$$\mathbf{u}=\left(a\,c_1+c\,c_2\right)\mathbf{v}_1+\left(b\,c_1+d\,c_2\right)\mathbf{v}_2$$

We now know the coordinate matrix of **u** is relative to the "old" basis *B*. Namely,

$$[\mathbf{u}]_B=\begin{bmatrix}a\,c_1+c\,c_2\\ b\,c_1+d\,c_2\end{bmatrix}$$

We can now do a little rewrite as follows,

$$[\mathbf{u}]_B = \begin{bmatrix} a\,c_1 + c\,c_2 \\ b\,c_1 + d\,c_2 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}[\mathbf{u}]_C$$

So, if we define P to be the matrix,

$$P = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

where the columns of P are the coordinate matrices for the basis vectors of C relative to B, we can convert the coordinate matrix for **u** relative to the new basis C into a coordinate matrix for **u** relative to the old basis B as follows,

$$[\mathbf{u}]_B = P[\mathbf{u}]_C$$

Note that this may seem a little backwards at this point. We're converting to a new basis C and yet we've found a way to instead find the coordinate matrix for **u** relative to the old basis B and not the other way around. However, as we'll see we can use this process to go the other way around. Also, it could be that we have a coordinate matrix for a vector relative to the new basis and we need to determine what the coordinate matrix relative to the old basis will be and this will allow us to do that.

Here is the formal definition of how to perform a change of basis between two basis sets.

Definition 2 Suppose that V is a n-dimensional vector space and further suppose that $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $C = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ are two bases for V. The **transition matrix** from C to B is defined to be,

$$P = \left[\, [\mathbf{w}_1]_B \;\middle|\; [\mathbf{w}_2]_B \;\middle|\; \cdots \;\middle|\; [\mathbf{w}_n]_B \,\right]$$

where the i^{th} column of P is the coordinate matrix of $\mathbf{w}_i$ relative to B.

The coordinate matrix of a vector **u** in V, relative to B, is then related to the coordinate matrix of **u** relative to C by the following equation.

$$[\mathbf{u}]_B = P[\mathbf{u}]_C$$

We should probably take a look at an example or two at this point.

Example 5 Consider the standard basis for $\mathbb{R}^3$, $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, and the basis $C = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where, $\mathbf{v}_1 = (1,-1,1)$, $\mathbf{v}_2 = (0,1,2)$ and $\mathbf{v}_3 = (3,0,-1)$.

(a) Find the transition matrix from C to B.
(b) Find the transition matrix from B to C.
(c) Use the result of part **(a)** to compute $[\mathbf{u}]_B$ given $(\mathbf{u})_C = (-2,3,4)$.
(d) Use the result of part **(a)** to compute $[\mathbf{u}]_B$ given $(\mathbf{u})_C = (9,-1,-8)$.

(e) Use the result of part **(b)** to compute $\left[\mathbf{u}\right]_C$ given $\left(\mathbf{u}\right)_B = \left(10, 5, 0\right)$.

(f) Use the result of part **(b)** to compute $\left[\mathbf{u}\right]_C$ given $\left(\mathbf{u}\right)_B = \left(-6, 7, 2\right)$.

Solution

Note as well that we gave the coordinate vector in the last four parts of the problem statement to conserve on space. When we go to work with them we'll need to convert to them to a coordinate matrix.

(a) When the basis we're going to (*B* in this case) is the standard basis vectors for the vector space computing the transition matrix is generally pretty simple. Recall that the columns of *P* are just the coordinate matrices of the vectors in *C* relative to *B*. However, when *B* is the standard basis vectors we saw in Example 1 above that the coordinate vector (and hence the coordinate matrix) is simply the vector itself. Therefore, the coordinate matrix in this case is,

$$P = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

(b) First, do not make the mistake of thinking that the transition matrix here will be the same as the transition matrix from part **(a)**. It won't be. To find this transition matrix we need the coordinate matrices of the standard basis vectors relative to *C*. This means that we need to write each of the standard basis vectors as linear combinations of the basis vectors from *C*. We will leave it to you to verify the following linear combinations.

$$\mathbf{e}_1 = \frac{1}{10}\mathbf{v}_1 + \frac{1}{10}\mathbf{v}_2 + \frac{3}{10}\mathbf{v}_3$$

$$\mathbf{e}_2 = -\frac{3}{5}\mathbf{v}_1 + \frac{2}{5}\mathbf{v}_2 + \frac{1}{5}\mathbf{v}_3$$

$$\mathbf{e}_3 = \frac{3}{10}\mathbf{v}_1 + \frac{3}{10}\mathbf{v}_2 - \frac{1}{10}\mathbf{v}_3$$

The coordinate matrices for each of this is then,

$$\left[\mathbf{e}_1\right]_C = \begin{bmatrix} \frac{1}{10} \\ \frac{1}{10} \\ \frac{3}{10} \end{bmatrix} \qquad \left[\mathbf{e}_2\right]_C = \begin{bmatrix} -\frac{3}{5} \\ \frac{2}{5} \\ \frac{1}{5} \end{bmatrix} \qquad \left[\mathbf{e}_3\right]_C = \begin{bmatrix} \frac{3}{10} \\ \frac{3}{10} \\ -\frac{1}{10} \end{bmatrix}$$

The transition matrix from *C* to *B* is then,

$$P' = \begin{bmatrix} \frac{1}{10} & -\frac{3}{5} & \frac{3}{10} \\ \frac{1}{10} & \frac{2}{5} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{5} & -\frac{1}{10} \end{bmatrix}$$

So, a significantly different matrix as suggested at the start of this problem. Also, notice we used a slightly different notation for the transition matrix to make sure that we can keep the two transition matrices separate for this problem.

(c) Okay, we've done most of the work for this problem. The remaining steps are just doing some matrix multiplication. Note as well that we already know what the answer to this is from Example 1 above. Here is the matrix multiplication for this part.

$$[\mathbf{u}]_B = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 0 \end{bmatrix}$$

Sure enough we got the coordinate matrix for the point that we converted to get $(\mathbf{u})_C = (-2,3,4)$ from Example 1.

(d) The matrix multiplication for this part is,

$$[\mathbf{u}]_B = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 9 \\ -1 \\ -8 \end{bmatrix} = \begin{bmatrix} -15 \\ -10 \\ 15 \end{bmatrix}$$

So, what have we learned here? Well, we were given the coordinate vector of a point relative to *C*. Since the vectors in *C* are not the standard basis vectors we don't really have a frame of reference for what this vector might actually look like. However, with this computation we know now the coordinates of the vectors relative to the standard basis vectors and this means that we actually know what the vector is. In this case the vector is,

$$\mathbf{u} = (-15,-10,15)$$

So, as you can see, even though we're considering *C* to be the "new" basis here, we really did need to determine the coordinate matrix of the vector relative to the "old' basis here since that allowed us to quickly determine just what the vector was. Remember that the coordinate matrix/vector is not the vector itself, only the coefficients for the linear combination of the basis vectors.

(e) Again, here we are really just verifying the result of Example 1 in this part. Here is the matrix multiplication for this part.

$$[\mathbf{u}]_C = \begin{bmatrix} \frac{1}{10} & -\frac{3}{5} & \frac{3}{10} \\ \frac{1}{10} & \frac{2}{5} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{5} & -\frac{1}{10} \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}$$

And again, we got the result that we would expect to get.

(f) Here is the matrix multiplication for this part.

$$[\mathbf{u}]_C = \begin{bmatrix} \frac{1}{10} & -\frac{3}{5} & \frac{3}{10} \\ \frac{1}{10} & \frac{2}{5} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{5} & -\frac{1}{10} \end{bmatrix} \begin{bmatrix} -6 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{27}{5} \\ \frac{8}{5} \\ -\frac{1}{5} \end{bmatrix}$$

So what does this give us? We'll first we know that $\left(\mathbf{u}\right)_C = \left(-\frac{27}{5}, \frac{8}{5}, -\frac{1}{5}\right)$. Also, since B is the standard basis vectors we know that the vector from $\mathbb{R}^3$ that we're starting with is $\left(-6,7,2\right)$. Recall that when dealing with the standard basis vectors for $\mathbb{R}^3$ the coordinate matrix/vector just also happens to be the vector itself. Again, do not always expect this to happen.

The coordinate matrix/vector that we just found tells us how to write the vector as a linear combination of vectors from the basis C. Doing this gives,

$$\left(-6,7,2\right) = -\frac{27}{5}\mathbf{v}_1 + \frac{8}{5}\mathbf{v}_2 - \frac{1}{5}\mathbf{v}_3$$

Example 6 Consider the standard basis for P_2, $B = \left\{1, x, x^2\right\}$ and the basis $C = \left\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\right\}$, where $\mathbf{p}_1 = 2$, $\mathbf{p}_2 = -4x$, and $\mathbf{p}_3 = 5x^2 - 1$.

(a) Find the transition matrix from C to B.

(b) Determine the polynomial that has the coordinate vector $\left(\mathbf{p}\right)_C = \left(-4,3,11\right)$.

Solution

(a) Now, since B (the matrix we're going to) is the standard basis vectors writing down the transition matrix will be easy this time.

$$P = \begin{bmatrix} 2 & 0 & -1 \\ 0 & -4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Each column of P will be the coefficients of the vectors from C since those will also be the coordinates of each of those vectors relative to the standard basis vectors. The first row will be the constant terms from each basis vector, the second row will be the coefficient of x from each basis vector and the third column will be the coefficient of x^2 from each basis vector.

(b) We know what the coordinates of the polynomial are relative to C, but this is not the standard basis and so it is not really clear just what the polynomial is. One way to get the solution is to just form up the linear combination with the coordinates as the scalars in the linear combination and compute it.

However, it would be somewhat illustrative to use the transition matrix to answer this question. So, we need to find $\left[\mathbf{p}\right]_B$ and luckily we've got the correct transition matrix to do that for us. All we need to do is to do the following matrix multiplication.

$$[\mathbf{p}]_B = P[\mathbf{p}]_C$$
$$= \begin{bmatrix} 2 & 0 & -1 \\ 0 & -4 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} -4 \\ 3 \\ 11 \end{bmatrix} = \begin{bmatrix} -19 \\ -12 \\ 55 \end{bmatrix}$$

So, the coordinate vector for **u** relative to the standard basis vectors is
$$(\mathbf{p})_B = (-19, -12, 55)$$

Therefore, the polynomial is,
$$p(x) = -19 - 12x + 55x^2$$

Note that, as mentioned above we can also do this problem as follows,
$$p(x) = -4\mathbf{p}_1 + 3\mathbf{p}_2 + 11\mathbf{p}_3 = -4(2) + 3(-4x) + 11(5x^2 - 1) = -19 - 12x + 55x^2$$

The same answer with less work, but it won't always be less work to do it this way. We just wanted to point out the alternate method of working this problem.

Example 7 Consider the standard basis for M_{22}, $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, and the basis $C = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ where $\mathbf{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$.

(a) Find the transition matrix from *C* to *B*.

(b) Determine the matrix that has the coordinate vector $(\mathbf{v})_C = (-8, 3, 5, -2)$.

Solution

(a) Now, as with the previous couple of problems, *B* is the standard basis vectors but this time let's be a little careful. Let's find one of the columns of the transition matrix in detail to make sure we can quickly write down the remaining columns. Let's look at the fourth column. To find this we need to write $\mathbf{v}_4$ as a linear combination of the standard basis vectors. This is fairly simple to do.
$$\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} = (-3)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (0)\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (0)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + (2)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
So, the coordinate matrix for $\mathbf{v}_4$ relative to *B* and hence the fourth column of *P* is,
$$[\mathbf{v}_4]_B = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

So, each column will be the entries from the $\mathbf{v}_i$'s and with the standard basis vectors in the order that we've using them here, the first two entries is the first column of the $\mathbf{v}_i$ and the last two entries will be the second column of $\mathbf{v}_i$. Here is the transition matrix for this problem.

$$P = \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(b) So, just as with the previous problem we have the coordinate vector, but that is for the non-standard basis vectors and so it's not readily apparent what the matrix will be. As with the previous problem we could just write down the linear combination of the vectors from *C* and compute it directly, but let's go ahead and used the transition matrix.

$$[\mathbf{v}]_B = \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} -8 \\ 3 \\ 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 5 \\ 1 \end{bmatrix}$$

Now that we've got the coordinates for **v** relative to the standard basis we can write down **v**.

$$\mathbf{v} = \begin{bmatrix} 4 & 5 \\ -3 & 1 \end{bmatrix}$$

To this point we've only worked examples where one of the bases was the standard basis vectors. Let's work one more example and this time we'll avoid the standard basis vectors. In this example we'll just find the transition matrices.

Example 8 Consider the two bases for $\mathbb{R}^2$, $B = \{(1,-1), (0,6)\}$ and $C = \{(2,1), (-1,4)\}$.

(a) Find the transition matrix from *C* to *B*.
(b) Find the transition matrix from *B* to *C*.

Solution

Note that you should verify for yourself that these two sets of vectors really are bases for $\mathbb{R}^2$ as we claimed them to be.

(a) To do this we'll need to write the vectors from *C* as linear combinations of the vectors from *B*. Here are those linear combinations.

$$(2,1) = 2(1,-1) + \frac{1}{2}(0,6)$$

$$(-1,4) = -(1,-1) + \frac{1}{2}(0,6)$$

The two coordinate matrices are then,

$$[(2,1)]_B = \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix} \qquad\qquad [(-1,4)]_B = \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}$$

and the transition matrix is then,

$$P = \begin{bmatrix} 2 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(b) Okay, we'll need to do pretty much the same thing here only this time we need to write the vectors from B as linear combinations of the vectors from C. Here are the linear combinations.

$$(1,-1) = \frac{1}{3}(2,1) - \frac{1}{3}(-1,4)$$

$$(0,6) = \frac{2}{3}(2,1) + \frac{4}{3}(-1,4)$$

The coordinate matrices are,

$$[(1,-1)]_C = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} \qquad\qquad [(0,6)]_C = \begin{bmatrix} \frac{2}{3} \\ \frac{4}{3} \end{bmatrix}$$

The transition matrix is,

$$P' = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{4}{3} \end{bmatrix}$$

In Examples 5 and 8 above we computed both transition matrices for each direction. There is another way of computing the second transition matrix from the first and we will close out this section with the theorem that tells us how to do that.

Theorem 1 Suppose that V is a finite dimensional vector space and that P is the transition matrix from C to B then,

- **(a)** P is invertible and,
- **(b)** P^{-1} is the transition matrix from B to C.

You should go back to Examples 5 and 8 above and verify that the two transition matrices are in fact inverses of each other. Also, note that due to the difficulties sometimes present in finding the inverse of a matrix it might actually be easier to compute the second transition matrix as we did above.

Fundamental Subspaces

In this section we want to take a look at some important subspaces that are associated with matrices. In fact they are so important that they are often called the fundamental subspaces of a matrix. We've actually already seen one of the fundamental subspaces, the null space, previously although we will give its definition here again for the sake of completeness.

Before we give the formal definitions of the fundamental subspaces we need to quickly review a concept that we first saw back when we were looking at matrix arithmetic.

Given an $n \times m$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

The **row vectors** (we called them row matrices at the time) are the vectors in R^m formed out of the rows of A. The **column vectors** (again we called them column matrices at the time) are the vectors in $\mathbb{R}^n$ that are formed out of the columns of A.

Example 1 Write down the row vectors and column vectors for

$$A = \begin{bmatrix} -1 & 5 \\ 0 & -4 \\ 9 & 2 \\ 3 & -7 \end{bmatrix}$$

Solution

The row vectors are,

$$\mathbf{r}_1 = \begin{bmatrix} -1 & 5 \end{bmatrix} \qquad \mathbf{r}_2 = \begin{bmatrix} 0 & -4 \end{bmatrix} \qquad \mathbf{r}_3 = \begin{bmatrix} 9 & 2 \end{bmatrix} \qquad \mathbf{r}_4 = \begin{bmatrix} 3 & -7 \end{bmatrix}$$

The column vectors are

$$\mathbf{c}_1 = \begin{bmatrix} -1 \\ 0 \\ 9 \\ 3 \end{bmatrix} \qquad \mathbf{c}_2 = \begin{bmatrix} 5 \\ -4 \\ 2 \\ -7 \end{bmatrix}$$

Note that despite the fact that we're calling them vectors we are using matrix notation for them. The reason is two fold. First, they really are row/column matrices and so we may as well denote them as such and second in this way we can keep the "orientation" of each to remind us whether or not they are row vectors or column vectors. In other words, row vectors are listed horizontally and column vectors are listed vertically.

Because we'll be using the matrix notation for the row and column vectors we'll be using matrix notation for vectors in general in this section so we won't be mixing and matching the notations too much.

Here then are the definitions of the three fundamental subspaces that we'll be investigating in this section.

Definition 1 Suppose that *A* is an $n \times m$ matrix.

- **(a)** The subspace of R^m that is spanned by the row vectors of *A* is called the **row space** of *A*.
- **(b)** The subspace of R^n that is spanned by the column vectors of *A* is called the **column space** of *A*.
- **(c)** The set of all **x** in $\mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{0}$ (which is a subspace of $\mathbb{R}^m$ by Theorem 2 from the Subspaces section) is called the **null space** of *A*.

We are going to be particularly interested in the basis for each of these subspaces and that in turn means that we're going to be able to discuss the dimension of each of them. At this point we can give the notation for the dimension of the null space, but we'll need to wait a bit before we do so for the row and column spaces. The reason for the delay will be apparent once we reach that point. So, let's go ahead and give the notation for the null space.

Definition 2 The dimension of the null space of *A* is called the **nullity** of *A* and is denoted by $\text{nullity}(A)$.

We should work an example at this point. Because we've already seen how to find the basis for the null space (Example 4(b) in the Subspaces section and Example 7 of the Basis section) we'll do one example at this point and then devote the remainder of the discussion on basis/dimension of these subspaces to finding the basis/dimension for the row and column space. Note that we will see an example or two later in this section of null spaces as well.

Example 2 Determine a basis for the null space of the following matrix.

$$A = \begin{bmatrix} 2 & -4 & 1 & 2 & -2 & -3 \\ -1 & 2 & 0 & 0 & 1 & -1 \\ 10 & -4 & -2 & 4 & -2 & 4 \end{bmatrix}$$

Solution

So, to find the null space we need to solve the following system of equations.

$$\begin{aligned} 2x_1 - 4x_2 + x_3 + 2x_4 - 2x_5 - 3x_6 &= 0 \\ -x_1 + 2x_2 + x_5 - x_6 &= 0 \\ 10x_1 - 4x_2 - 2x_3 + 4x_4 - 2x_5 + 4x_6 &= 0 \end{aligned}$$

We'll leave it to you to verify that the solution is given by,

$$x_1 = -t + r \qquad x_2 = -\frac{1}{2}t - \frac{1}{2}s + r \qquad x_3 = -2t + 5r$$
$$x_4 = t \qquad x_5 = s \qquad x_6 = r \qquad t, s, r \text{ are any real numbers}$$

In matrix form the solution can be written as,

$$\mathbf{x} = \begin{bmatrix} -t + r \\ -\frac{1}{2}t - \frac{1}{2}s + r \\ -2t + 5r \\ t \\ s \\ r \end{bmatrix} = t\begin{bmatrix} -1 \\ -\frac{1}{2} \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} 1 \\ 1 \\ 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So, the solution can be written as a linear combination of the three linearly independent vectors (verify the linearly independent claim!)

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ -\frac{1}{2} \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and so these three vectors then form the basis for the null space since they span the null space and are linearly independent. Note that this also means that the null space has a dimension of 3 since there are three basis vectors for the null space and so we can see that

$$\text{nullity}(A) = 3$$

Again, remember that we'll be using matrix notation for vectors in this section.

Okay, now that we've gotten an example of the basis for the null space taken care of we need to move onto finding bases (and hence the dimensions) for the row and column spaces of a matrix. However, before we do that we first need a couple of theorems out of the way. The first theorem tells us how to find the basis for a matrix that is in row-echelon form.

Theorem 1 Suppose that the matrix U is in row-echelon form. The row vectors containing leading 1's (so the non-zero row vectors) will form a basis for the row space of U. The column vectors that contain the leading 1's from the row vectors will form a basis for the column space of U.

Example 3 Find the basis for the row and column space of the following matrix.

$$U = \begin{bmatrix} 1 & 5 & -2 & 3 & 5 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution

Okay, the basis for the row space is simply all the row vectors that contain a leading 1. So, for this matrix the basis for the row space is,

$$\mathbf{r}_1 = \begin{bmatrix} 1 & 5 & -2 & 3 & 5 \end{bmatrix} \qquad \mathbf{r}_2 = \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

$$\mathbf{r}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We can also see that the dimension of the row space will be 3.

The basis for the column space will be the columns that contain leading 1's and so for this matrix the basis for the column space will be,

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{c}_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{c}_5 = \begin{bmatrix} 5 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Note that we subscripted the vectors here with the column that each came out of. We will generally do that for these problems. Also note that the dimension of the column space is 3 as well.

Now, all of this is fine provided we have a matrix in row-echelon form. However, as we know, most matrices will not be in row-echelon form. The following two theorems will tell us how to find the basis for the row and column space of a general matrix.

Theorem 2 Suppose that A is a matrix and U is a matrix in row-echelon form that has been obtained by performing row operations on A. Then the row space of A and the row space of U are the same space.

So, how does this theorem help us? We'll if the matrix A and U have the same row space then if we know a basis for one of them we will have a basis for the other. Notice as well that we assumed the matrix U is in row-echelon form and we do know how to find a basis for its row space. Therefore, to find a basis for the row space of a matrix A we'll need to reduce it to row-echelon form. Once in row-echelon form we can write down a basis for the row space of U, but that is the same as the row space of A and so that set of vectors will also be a basis for the row space of A.

So, what about a basis for the column space? That's not quite as straight forward, but is almost as simple.

Theorem 3 Suppose that A and B are two row equivalent matrices (so we got from one to the other by row operations) then a set of column vectors from A will be a basis for the column space of A if and only if the corresponding columns from B will form a basis for the column space of *B*.

How does this theorem help us to find a basis for the column space of a general matrix? We'll let's start with a matrix *A* and reduce it to row-echelon form, *U*, (which we'll need for a basis for the row space anyway). Now, because we arrived at *U* by applying row operations to *A* we know that *A* and *U* are row equivalent. Next, from Theorem 1 we know how to identify the columns from *U* that will form a basis for the column space of *U*. These columns will probably not be a basis for the column space of *A* however, what Theorem 3 tells us is that corresponding columns from *A* will form a basis for the columns space of *A*. For example, suppose the columns 1, 2, 5 and 8 from *U* form a basis for the column space of *U* then columns 1, 2, 5 and 8 from *A* will form a basis for the column space of *A*.

Before we work an example we can now talk about the dimension of the row and column space of a matrix *A*. From our theorems above we know that to find a basis for both the row and column space of a matrix *A* we first need to reduce it to row-echelon form and we can get a basis for the row and column space from that.

Let's go back and take a look at Theorem 1 in a little more detail. According to this theorem the rows with leading 1's will form a basis for the row space and the columns that containing the same leading 1's will form a basis for the column space. Now, there are a fixed number of leading 1's and each leading 1 will be in a separate column. For example, there won't be two leading 1's in the second column because that would mean that the upper 1 (one) would not be a leading 1.

Think about this for a second. If there are *k* leading 1's in a row-echelon matrix then there will be *k* row vectors in a basis for the row space and so the row space will have a dimension of *k*. However, since each of the leading 1's will be in separate columns there will also be *k* column vectors that will form a basis for the column space and so the column space will also have a dimension of *k*. This will always happen and this is the reason that we delayed talking about the dimension of the row and column space above. We needed to get a couple of theorems out of the way so we could give the following theorem/definition.

Theorem 4 Suppose that *A* is a matrix then the row space of *A* and the column space of *A* will have the same dimension. We call this common dimension the **rank** of *A* and denote it by $\operatorname{rank}(A)$.

Note that if *A* is an $n \times m$ matrix we know that the row space will be a subspace of R^m and hence have a dimension of *m* or less and that the column space will be a subspace of R^n and hence have a dimension of *n* or less. Then, because we know that the dimension of the row and column space must be the same we have the following upper bound for the rank of a matrix.

$$\operatorname{rank}(A) \le \min(n, m)$$

We should now work an example.

Example 4 Find a basis for the row and column space of the matrix from Example 2 above. Determine the rank of the matrix.

Solution

Before starting this example let's note that by the upper bound for the rank above we know that the largest that the rank can be is 3 since that is the smaller of the number of rows and columns in A.

So, the first thing that we need to do is get the matrix into row-echelon form. We will leave it to you to verify that the following is one possible row echelon form for the matrix from Example 2 above. If you need a refresher on how to reduce a matrix to row-echelon form you can go back to the section on Solving Systems of Equations for a refresher. Also, recall that there is more than one possible row-echelon form for a given matrix.

$$U = \begin{bmatrix} 1 & -2 & 0 & 0 & -1 & 1 \\ 0 & 1 & -\frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -\frac{3}{8} \\ 0 & 0 & 1 & 2 & 0 & -5 \end{bmatrix}$$

So, a basis for the row space of the matrix will be every row that contains a leading 1 (all of them in this case). A basis for the row space is then,

$$\mathbf{r}_1 = \begin{bmatrix} 1 & -2 & 0 & 0 & -1 & 1 \end{bmatrix} \qquad \mathbf{r}_2 = \begin{bmatrix} 0 & 1 & -\frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -\frac{3}{8} \end{bmatrix}$$

$$\mathbf{r}_3 = \begin{bmatrix} 0 & 0 & 1 & 2 & 0 & -5 \end{bmatrix}$$

Next, the first three columns of U will form a basis for the column space of U since they all contain the leading 1's. Therefore the first three columns of A will form a basis for the column space of A. This gives the following basis for the column space of A.

$$\mathbf{c}_1 = \begin{bmatrix} 2 \\ -1 \\ 10 \end{bmatrix} \qquad \mathbf{c}_2 = \begin{bmatrix} -4 \\ 2 \\ -4 \end{bmatrix} \qquad \mathbf{c}_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

Now, as Theorem 4 suggested both the row space and the column space of A have dimension 3 and so we have that

$$\operatorname{rank}(A) = 3$$

Before going on to another example let's stop for a bit and take a look at the results of Examples 2 and 4. From these two examples we saw that the rank and nullity of the matrix used in those examples were both 3. The fact that they were the same won't always happen as we'll see shortly and so isn't all that important. What is important to note is that $3+3=6$ and there were 6 columns in this matrix. This in fact will always be the case.

Theorem 5 Suppose that A is an $n \times m$ matrix. Then,

$$\text{rank}(A) + \text{nullity}(A) = m$$

Let's take a look at a couple more examples now.

Example 5 Find a basis for the null space, row space and column space of the following matrix. Determine the rank and nullity of the matrix.

$$A = \begin{bmatrix} -1 & 2 & -1 & 5 & 6 \\ 4 & -4 & -4 & -12 & -8 \\ 2 & 0 & -6 & -2 & 4 \\ -3 & 1 & 7 & -2 & 12 \end{bmatrix}$$

Solution

Before we get started we can notice that the rank can be at most 4 since that is smaller of the number of rows and number of columns.

We'll find the null space first since that was the first thing asked for. To do this we'll need to solve the following system of equations.

$$\begin{aligned} -x_1 + 2x_2 - x_3 + 5x_4 + 6x_5 &= 0 \\ 4x_1 - 4x_2 - 4x_3 - 12x_4 - 8x_5 &= 0 \\ 2x_1 - 6x_3 - 2x_4 + 4x_5 &= 0 \\ -3x_1 + x_2 + 7x_3 - 2x_4 + 12x_5 &= 0 \end{aligned}$$

You should verify that the solution is,

$$x_1 = 3t \qquad x_2 = 2t - 8s \qquad x_3 = t \qquad x_4 = 2s \qquad x_5 = s$$

$$s \text{ and } t \text{ are any real numbers}$$

The null space is then given by,

$$\mathbf{x} = \begin{bmatrix} 3t \\ 2t-8s \\ t \\ 2s \\ s \end{bmatrix} = t\begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ -8 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

and so we can see that a basis for the null space is,

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} 0 \\ -8 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Therefore we now know that $\text{nullity}(A) = 2$. At this point we know the rank of A by Theorem 5 above. According to this theorem the rank must be,

$$\operatorname{rank}(A) = \#\ \text{columns} - \operatorname{nullity}(A) = 5 - 2 = 3$$

This will give us a nice check when we find a basis for the row space and the column space. We now know that each should contain three vectors.

Speaking of which, let's get a basis for the row space and the column space. We'll need to reduce A to row-echelon form first. We'll leave it to you to verify that a possible row-echelon form for A is,

$$U = \begin{bmatrix} 1 & -2 & 1 & -5 & -6 \\ 0 & 1 & -2 & 2 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rows containing leading 1's will form a basis for the row space of A and so this basis is,

$$\mathbf{r}_1 = \begin{bmatrix} 1 & -2 & 1 & -5 & -6 \end{bmatrix} \qquad \mathbf{r}_2 = \begin{bmatrix} 0 & 1 & -2 & 2 & 4 \end{bmatrix}$$
$$\mathbf{r}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Next, the first, second and fourth columns of U contain leading 1's and so will form a basis for the column space of U and this tells us that the first, second and fourth columns of A will form a basis for the column space of A. Here is that basis.

$$\mathbf{c}_1 = \begin{bmatrix} -1 \\ 4 \\ 2 \\ -3 \end{bmatrix} \qquad \mathbf{c}_2 = \begin{bmatrix} 2 \\ -4 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{c}_3 = \begin{bmatrix} 5 \\ -12 \\ -2 \\ -2 \end{bmatrix}$$

Note that the dimension of each of these is 3 as we noted it should be above.

Example 6 Find a basis for the null space, row space and column space of the following matrix. Determine the nullity and rank of this matrix.

$$A = \begin{bmatrix} 6 & -3 \\ -2 & 3 \\ -8 & 4 \end{bmatrix}$$

Solution

In this case we can notice that the rank of this matrix can be at most 2 since that is the minimum of the number of rows and number of columns.

To find the null space we'll need to solve the following system of equations,

$$\begin{aligned} 6x_1 - 3x_2 &= 0 \\ -2x_1 + 3x_2 &= 0 \\ -8x_1 + 4x_2 &= 0 \end{aligned}$$

We'll leave it to you to verify that the solution to this system is,

$$x_1 = 0 \qquad\qquad x_2 = 0$$

This is actually the point to this problem. There is only a single solution to the system above, namely the zero vector, **0**. Therefore the null space consists solely of the zero vector and vector spaces that consist solely of the zero vector do not have a basis and so we can't give one. Also, vector spaces consisting solely of the zero vectors are defined to have a dimension of zero. Therefore, the nullity of this matrix is zero. This also tells us that the rank of this matrix must be 2 by Theorem 5.

Let's now find a basis for the row space and the column space. You should verify that one possible row-reduced form for A is,

$$U = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

A basis for the row space of A is then,

$$\mathbf{r}_1 = \begin{bmatrix} 1 & -\frac{1}{2} \end{bmatrix} \qquad\qquad \mathbf{r}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

and since both columns of U form a basis for the column space of U both columns from A will form a basis for the column space of A. The basis for the column space of A is then,

$$\mathbf{c}_1 = \begin{bmatrix} 6 \\ -2 \\ -8 \end{bmatrix} \qquad\qquad \mathbf{c}_2 = \begin{bmatrix} -3 \\ 3 \\ 4 \end{bmatrix}$$

Once again, both have dimension of 2 as we knew they should from our use of Theorem 5 above.

In all of the examples that we've worked to this point in finding a basis for the row space and the column space we should notice that the basis we found for the column space consisted of columns from the original matrix while the basis we found for the row space did not consist of rows from the original matrix.

Also note that we can't necessarily use the same idea we used to get a basis for the column space to get a basis for the row space. For example let's go back and take a look at Example 5. The first three rows of U formed a basis for the row space, but that does not mean that the first three rows of A will also form a basis for the row space. In fact, in this case they won't. In this case the third row is twice the first row added onto the second row and so the first three rows are not linearly independent (which you'll recall is required for a set of vectors to be a basis).

So, what do we do if we do want rows from the original matrix to form our basis? The answer to this is surprisingly simple.

Example 7 Find a basis for the row space of the matrix in Example 5 that consists of rows from the original matrix.

Solution
The first thing that we'll do is take the transpose of *A*. In doing so the rows of *A* will become the columns of A^T. This means that the row space of *A* will become the column space of A^T. Recall as well that we find a basis for the column space in terms of columns from the original matrix (A^T in this case). So, we'll be finding a basis for the column space of A^T in terms of the columns of A^T, but the columns of A^T are the rows of *A* and the column space of A^T is the row space of *A*. Therefore, when this is all said and done by finding a basis for the column space of A^T we will also be finding a basis for the row space of *A* and it will be in terms of rows from *A* and not rows from the row-echelon form of the matrix.

So, here is the transpose of *A*.

$$A^T = \begin{bmatrix} -1 & 4 & 2 & -3 \\ 2 & -4 & 0 & 1 \\ -1 & -4 & -6 & 7 \\ 5 & -12 & -2 & -2 \\ 6 & -8 & 4 & 12 \end{bmatrix}$$

Here is a possible row-echelon form of the transpose (you should verify this).

$$U = \begin{bmatrix} 1 & -4 & -2 & 3 \\ 0 & 1 & 1 & -\frac{5}{4} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first, second and fourth columns of *U* form a basis for the column space of *U* and so a basis for the column space of A^T is,

$$\mathbf{c}_1 = \begin{bmatrix} -1 \\ 2 \\ -1 \\ 5 \\ 6 \end{bmatrix} \qquad \mathbf{c}_2 = \begin{bmatrix} 4 \\ -4 \\ -4 \\ -12 \\ -8 \end{bmatrix} \qquad \mathbf{c}_1 = \begin{bmatrix} -3 \\ 1 \\ 7 \\ -2 \\ 12 \end{bmatrix}$$

Again, however, the column space of A^T is nothing more than the row space of *A* and so these three column are rows from *A* and will also form a basis for the row space. So, let's change notation a little to make it clear that we're dealing with a basis for the row space and we'll be done. Here is a basis for the row space of A in terms of rows from A itself.

$$\mathbf{r}_1 = \begin{bmatrix} -1 & 2 & -1 & 5 & 6 \end{bmatrix} \qquad \mathbf{r}_2 = \begin{bmatrix} 4 & -4 & -4 & -12 & -8 \end{bmatrix}$$

$$\mathbf{r}_3 = \begin{bmatrix} -3 & 1 & 7 & -2 & 12 \end{bmatrix}$$

Next we want to give a quick theorem that gives a relationship between the solution to a system of equations and the column space of the coefficient matrix. This theorem can be useful on occasion.

Theorem 6 The system of linear equations $A\mathbf{x} = \mathbf{b}$ will be consistent (*i.e.* have at least one solution) if and only if **b** is in the column space of A.

Note that since the basis for the column space of a matrix is given in terms of the certain columns of A this means that a system of equations will be consistent if and only if **b** can be written as a linear combination of at least some of the columns of A. This should be clear from application of the Theorem above. This theorem tells us that **b** must be in the column space of A, but that means that it can be written as a linear combination of the basis vectors for the column space of A.

We'll close out this section with a couple of theorems relating the invertibility of a square matrix A to some of the ideas in this section.

Theorem 7 Let A be an $n \times n$ matrix. The following statements are equivalent.

(a) A is invertible.

(b) The null space of A is $\{\mathbf{0}\}$, *i.e.* just the zero vector.

(c) $\text{nullity}(A) = 0$.

(d) $\text{rank}(A) = n$.

(e) The columns vectors of A form a basis for $\mathbb{R}^n$.

(f) The row vectors of A form a basis for $\mathbb{R}^n$.

The proof of this theorem follows directly from Theorem 9 in the Properties of Determinants section and from the definitions of null space, rank and nullity so we're not going to give it here. We will point our however that if the rank of an $n \times n$ matrix is n then a basis for the row (column) space must contain n vectors, but there are only n rows (columns) in A and so all the rows (columns) of A must be in the basis. Also, the row (column) space is a subspace of $\mathbb{R}^n$ which also has a dimension of n. These ideas are helpful in showing that **(d)** will imply either **(e)** or **(f)**.

Finally, speaking of Theorem 9 in the Properties of Determinant section, this was also a theorem listing many equivalent statements on the invertibility of a matrix. We can merge that theorem with Theorem 7 above into the following theorem.

Theorem 8 Let A be an $n \times n$ matrix. The following statements are equivalent.

(a) A is invertible.
(b) The only solution to the system $A\mathbf{x} = 0$ is the trivial solution.
(c) A is row equivalent to I_n.
(d) A is expressible as a product of elementary matrices.
(e) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix **b**.
(f) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix **b**.
(g) $\det(A) \neq 0$
(h) The null space of A is $\{\mathbf{0}\}$, *i.e.* just the zero vector.
(i) $\text{nullity}(A) = 0$.
(j) $\text{rank}(A) = n$.
(k) The columns vectors of A form a basis for $\mathbb{R}^n$.
(l) The row vectors of A form a basis for $\mathbb{R}^n$.

Inner Product Spaces

If you go back to the Euclidean *n*-space chapter where we first introduced the concept of vectors you'll notice that we also introduced something called a dot product. However, in this chapter, where we're dealing with the general vector space, we have yet to introduce anything even remotely like the dot product. It is now time to do that. However, just as this chapter is about vector spaces in general, we are going to introduce a more general idea and it will turn out that a dot product will fit into this more general idea. Here is the definition of this more general idea.

Definition 1 Suppose **u**, **v**, and **w** are all vectors in a vector space V and c is any scalar. An **inner product** on the vector space V is a function that associates with each pair of vectors in V, say **u** and **v**, a real number denoted by $\langle \mathbf{u}, \mathbf{v} \rangle$ that satisfies the following axioms.

(a) $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
(b) $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
(c) $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$
(d) $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = 0$

A vector space along with an inner product is called an **inner product space**.

Note that we are assuming here that the scalars are real numbers in this definition. In fact we probably should have been using the terms "real vector space" and "real inner product space" in this definition to make it clear. If we were to allow the scalars to be complex numbers (*i.e.* dealing with a complex vector space) the axioms would change slightly.

Also, in the rest of this section if we say that V is an inner product space we are implicitly assuming that it is a vector space and that some inner product has been defined on it. If we do not explicitly give the inner product then the exact inner product that we are using is not important. It will only be important in these cases that there has been an inner product defined on the vector space.

Example 1 The Euclidean inner product as defined in the Euclidean n-space section is an inner product.

For reference purposes here is the Euclidean inner product. Given two vectors in $\mathbb{R}^n$, $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, the Euclidean inner product is defined to be,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \bullet \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

By Theorem 2 from the Euclidean n-space section we can see that this does in fact satisfy all the axioms of the definition. Therefore, $\mathbb{R}^n$ is an inner product space.

Here are some more examples of inner products.

Example 2 Suppose that $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are two vectors in $\mathbb{R}^n$ and that w_1, w_2, … , w_n are **positive** real numbers (called **weights**) then the weighted Euclidean inner product is defined to be,

$$\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \cdots + w_n u_n v_n$$

It is fairly simple to show that this is in fact an inner product. All we need to do is show that it satisfies all the axioms from Definition 1.

So, suppose that **u**, **v**, and **a** are all vectors in $\mathbb{R}^n$ and that c is a scalar.

First note that because we know that real numbers commute with multiplication we have,

$$\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \cdots + w_n u_n v_n = w_1 v_1 u_1 + w_2 v_2 u_2 + \cdots + w_n v_n u_n = \langle \mathbf{v}, \mathbf{u} \rangle$$

So, the first axiom is satisfied.

To show the second axiom is satisfied we just need to run through the definition as follows,

$$\begin{aligned}\langle \mathbf{u} + \mathbf{v}, \mathbf{a} \rangle &= w_1 (u_1 + v_1) a_1 + w_2 (u_2 + v_2) a_2 + \cdots + w_n (u_n + v_n) a_n \\ &= (w_1 u_1 a_1 + w_2 u_2 a_2 + \cdots + w_n u_n a_n) + (w_1 v_1 a_1 + w_2 v_2 a_2 + \cdots + w_n v_n a_n) \\ &= \langle \mathbf{u}, \mathbf{a} \rangle + \langle \mathbf{v}, \mathbf{a} \rangle\end{aligned}$$

and the second axiom is satisfied.

Here's the work for the third axiom.

$$\begin{aligned}\langle c\,\mathbf{u},\mathbf{v}\rangle &= w_1 c\,u_1 v_1 + w_2 c\,u_2 v_2 + \cdots + w_n c\,u_n v_n \\ &= c\left(w_1 u_1 v_1 + w_2 u_2 v_2 + \cdots + w_n u_n v_n\right) \\ &= c\langle \mathbf{u},\mathbf{v}\rangle\end{aligned}$$

Finally, for the fourth axiom there are two things we need to check. Here's the first,

$$\langle \mathbf{u},\mathbf{u}\rangle = w_1 u_1^2 + w_2 u_2^2 + \cdots + w_n u_n^2 \ge 0$$

Note that this is greater than or equal to zero because the weights w_1, w_2, … , w_n are positive numbers. If we hadn't made that assumption there would be no way to guarantee that this would be positive.

Now suppose that $\langle \mathbf{u},\mathbf{u}\rangle = 0$. Because each of the terms above is greater than or equal to zero the only way this can be zero is if each of the terms is zero itself. Again, however, the weights are positive numbers and so this means that

$$u_i^2 = 0 \qquad \Rightarrow \qquad u_i = 0, \qquad i = 1, 2, \ldots, n$$

We therefore must have $\mathbf{u} = 0$ if $\langle \mathbf{u},\mathbf{u}\rangle = 0$.

Likewise if $\mathbf{u} = 0$ then by plugging in we can see that we must also have $\langle \mathbf{u},\mathbf{u}\rangle = 0$ and so the fourth axiom is also satisfied.

Example 3 Suppose that $A = \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_3 \\ b_2 & b_4 \end{bmatrix}$ are two matrices in M_{22}. An inner product on M_{22} can be defined as,

$$\langle A, B\rangle = \operatorname{tr}\left(A^T B\right)$$

where $\operatorname{tr}(C)$ is the trace of the matrix *C*.

We will leave it to you to verify that this is in fact an inner product. This is not difficult once you show (you can do a direct computation to show this) that

$$\operatorname{tr}\left(A^T B\right) = \operatorname{tr}\left(B^T A\right) = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4$$

This formula is very similar to the Euclidean inner product formula and so showing that this is an inner product will be almost identical to showing that the Euclidean inner product is an inner product. There are differences, but for the most part it is pretty much the same.

The next two examples require that you've had Calculus and so if you haven't had Calculus you can skip these examples. Both of these however are very important inner products in some areas of mathematics, although we're not going to be looking at them much here because of the Calculus requirement.

Example 4 Suppose that $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ are two continuous functions on the interval $[a,b]$. In other words, they are in the vector space $C[a,b]$. An inner product on $C[a,b]$ can be defined as,

$$\langle \mathbf{f},\mathbf{g}\rangle = \int_a^b f(x)\,g(x)\,dx$$

Provided you remember your Calculus, showing this is an inner product is fairly simple. Suppose that **f**, **g**, and **h** are continuous functions in $C[a,b]$ and that c is any scalar.

Here is the work showing the first axiom is satisfied.

$$\langle \mathbf{f},\mathbf{g}\rangle = \int_a^b f(x)\,g(x)\,dx = \int_a^b g(x)\,f(x)\,dx = \langle \mathbf{g},\mathbf{f}\rangle$$

The second axiom is just as simple,

$$\begin{aligned}\langle \mathbf{f}+\mathbf{g},\mathbf{h}\rangle &= \int_a^b \left(f(x)+g(x)\right)h(x)\,dx\\ &= \int_a^b f(x)\,h(x)\,dx + \int_a^b g(x)\,h(x)\,dx = \langle \mathbf{f},\mathbf{h}\rangle + \langle \mathbf{g},\mathbf{h}\rangle\end{aligned}$$

Here's the third axiom.

$$\langle c\,\mathbf{f},\mathbf{g}\rangle = \int_a^b c\,f(x)\,g(x)\,dx = c\int_a^b f(x)\,g(x)\,dx = c\langle \mathbf{f},\mathbf{g}\rangle$$

Finally, the fourth axiom. This is the only one that really requires something that you may not remember from a Calculus class. The previous examples all used properties of integrals that you should remember.

First, we'll start with the following,

$$\langle \mathbf{f},\mathbf{f}\rangle = \int_a^b f(x)\,f(x)\,dx = \int_a^b f^2(x)\,dx$$

Now, recall that if you integrate a continuous function that is greater than or equal to zero then the integral must also be greater than or equal to zero. Hence,

$$\langle \mathbf{f},\mathbf{f}\rangle \geq 0$$

Next, if $\mathbf{f} = \mathbf{0}$ the clearly we'll have $\langle \mathbf{f},\mathbf{f}\rangle = 0$. Likewise, if we have $\langle \mathbf{f},\mathbf{f}\rangle = \int_a^b f^2(x)\,dx = 0$ then we must also have $\mathbf{f} = \mathbf{0}$.

Example 5 Suppose that $\mathbf{f} = f(x)$ and $\mathbf{g} = g(x)$ are two vectors in $C[a,b]$ and further suppose that $w(x) > 0$ is a continuous function called a **weight**. A weighted inner product on $C[a,b]$ can be defined as,

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x)\, g(x)\, w(x)\, dx$$

We'll leave it to you to verify that this is an inner product. It should be fairly simple if you've had calculus and you followed the verification of the weighted Euclidean inner product. The key is again the fact that the weight is a strictly positive function on the interval $[a,b]$.

Okay, once we have an inner product defined on a vector space we can define both a norm and distance for the inner product space as follows.

Definition 2 Suppose that V is an inner product space. The **norm** or **length** of a vector **u** in V is defined to be,

$$\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{\frac{1}{2}}$$

Definition 3 Suppose that V is an inner product space and that **u** and **v** are two vectors in V. The distance between **u** and **v**, denoted by $d(\mathbf{u}, \mathbf{v})$ is defined to be,

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

We're not going to be working many examples with actual numbers in them in this section, but we should work one or two so at this point let's pause and work an example. Note that part **(c)** in the example below requires Calculus. If you haven't had Calculus you should skip that part.

Example 6 For each of the following compute $\langle \mathbf{u}, \mathbf{v} \rangle$, $\|\mathbf{u}\|$ and $d(\mathbf{u}, \mathbf{v})$ for the given pair of vectors and inner product.

(a) $\mathbf{u} = (2,-1,4)$ and $\mathbf{v} = (3,2,0)$ in $\mathbb{R}^3$ with the standard Euclidean inner product.

(b) $\mathbf{u} = (2,-1,4)$ and $\mathbf{v} = (3,2,0)$ in $\mathbb{R}^3$ with the weighed Euclidean inner product using the weights $w_1 = 2$, $w_2 = 6$ and $w_3 = \frac{1}{5}$.

(c) $\mathbf{u} = x$ and $\mathbf{v} = x^2$ in $C[0,1]$ using the inner product defined in Example 4.

Solution

(a) There really isn't much to do here other than go through the formulas.

$$\langle \mathbf{u}, \mathbf{v} \rangle = (2)(3) + (-1)(2) + (4)(0) = 4$$

$$\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{\frac{1}{2}} = \sqrt{(2)^2 + (-1)^2 + (4^2)} = \sqrt{21}$$

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|(-1,-3,4)\| = \sqrt{(-1)^2 + (-3)^2 + (4)^2} = \sqrt{26}$$

(b) Again, not a lot to do other than formula work. Note however, that even though we've got the same vectors as part **(a)** we should expect to get different results because we are now working with a weighted inner product.

$$\langle \mathbf{u},\mathbf{v}\rangle = (2)(3)(2)+(-1)(2)(6)+(4)(0)\left(\frac{1}{5}\right)=0$$

$$\|\mathbf{u}\| = \langle \mathbf{u},\mathbf{u}\rangle^{\frac{1}{2}} = \sqrt{(2)^2(2)+(-1)^2(6)+(4^2)\left(\frac{1}{5}\right)} = \sqrt{\frac{86}{5}} = \sqrt{17.2}$$

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \|(-1,-3,4)\| = \sqrt{(-1)^2(2)+(-3)^2(6)+(4)^2\left(\frac{1}{5}\right)} = \sqrt{\frac{296}{5}} = \sqrt{59.2}$$

So, we did get different answers here. Note that in under this weighted norm **u** is "smaller" in some way than under the standard Euclidean norm and the distance between **u** and **v** is "larger" in some way than under the standard Euclidean norm.

(c) Okay, again if you haven't had Calculus this part won't make much sense and you should skip it. If you have had Calculus this should be a fairly simple example.

$$\langle \mathbf{u},\mathbf{v}\rangle = \int_0^1 x\left(x^2\right)dx = \int_0^1 x^3\,dx = \left.\frac{1}{4}x^4\right|_0^1 = \frac{1}{4}$$

$$\|\mathbf{u}\| = \langle \mathbf{u},\mathbf{u}\rangle^{\frac{1}{2}} = \sqrt{\int_0^1 x(x)\,dx} = \sqrt{\int_0^1 x^2\,dx} = \sqrt{\left.\frac{1}{3}x^3\right|_0^1} = \frac{1}{\sqrt{3}}$$

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \left\|x-x^2\right\| = \sqrt{\int_0^1 \left(x-x^2\right)^2 dx} = \sqrt{\left.\left(\frac{1}{5}x^5-\frac{1}{2}x^4+\frac{1}{3}x^3\right)\right|_0^1} = \frac{1}{\sqrt{30}}$$

Now, we also have all the same properties for the inner product, norm and distance that we had for the dot product back in the Euclidean *n*-space section. We'll list them all here for reference purposes and so you can see them with the updated inner product notation. The proofs for these theorems are practically identical to their dot product counterparts and so aren't shown here.

Theorem 1 Suppose **u**, **v**, and **w** are vectors in an inner product space and c is any scalar. Then,

(a) $\langle \mathbf{u},\mathbf{v}+\mathbf{w}\rangle = \langle \mathbf{u},\mathbf{v}\rangle + \langle \mathbf{u},\mathbf{w}\rangle$

(b) $\langle \mathbf{u}-\mathbf{v},\mathbf{w}\rangle = \langle \mathbf{u},\mathbf{w}\rangle - \langle \mathbf{v},\mathbf{w}\rangle$

(c) $\langle \mathbf{u},\mathbf{v}-\mathbf{w}\rangle = \langle \mathbf{u},\mathbf{v}\rangle - \langle \mathbf{u},\mathbf{w}\rangle$

(d) $\langle c\,\mathbf{u},\mathbf{v}\rangle = c\langle \mathbf{u},\mathbf{v}\rangle$

(e) $\langle \mathbf{u},\mathbf{0}\rangle = \langle \mathbf{0},\mathbf{u}\rangle = 0$

Theorem 2 **Cauchy-Schwarz Inequality :** Suppose **u** and **v** are two vectors in an inner product space then

$$\left|\langle \mathbf{u},\mathbf{v}\rangle\right| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$$

Theorem 3 Suppose **u** and **v** are two vectors in an inner product space and that c is a scalar then,

(a) $\|\mathbf{u}\| \geq 0$

(b) $\|\mathbf{u}\| = 0$ if and only if **u**=**0**.

(c) $\|c\,\mathbf{u}\| = |c|\,\|\mathbf{u}\|$

(d) $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ - Usually called the **Triangle Inequality**

Theorem 4 Suppose **u**, **v**, and **w** are vectors in an inner product space then,

(a) $d(\mathbf{u},\mathbf{v}) \geq 0$

(b) $d(\mathbf{u},\mathbf{v}) = 0$ if and only if **u**=**v**.

(c) $d(\mathbf{u},\mathbf{v}) = d(\mathbf{v},\mathbf{u})$

(d) $d(\mathbf{u},\mathbf{v}) \leq d(\mathbf{u},\mathbf{w}) + d(\mathbf{w},\mathbf{v})$ - Usually called the **Triangle Inequality**

There was also an important concept that we saw back in the Euclidean *n*-space section that we'll need in the next section. Here is the definition for this concept in terms of inner product spaces.

Definition 4 Suppose that **u** and **v** are two vectors in an inner product space. They are said to be **orthogonal** if $\langle \mathbf{u},\mathbf{v} \rangle = 0$.

Note that whether or not two vectors are orthogonal will depend greatly on the inner product that we're using. Two vectors may be orthogonal with respect to one inner product defined on a vector space, but not orthogonal with respect to a second inner product defined on the same vector space.

Example 7 The two vectors $\mathbf{u} = (2,-1,4)$ and $\mathbf{v} = (3,2,0)$ in $\mathbb{R}^3$ are not orthogonal with respect to the standard Euclidean inner product, but are orthogonal with respect to the weighted Euclidean inner product with weights $w_1 = 2$, $w_2 = 6$ and $w_3 = \frac{1}{5}$.

We saw the computations for these back in Example 6.

Now that we have the definition of orthogonality out of the way we can give the general version of the **Pythagorean Theorem** of an inner product space.

Theorem 5 Suppose that **u** and **v** are two orthogonal vectors in an inner product space then,

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

There is one final topic that we want to briefly touch on in this section. In previous sections we spent quite a bit of time talking about subspaces of a vector space. There are also subspaces that will only arise if we are working with an inner product space. The following definition gives one such subspace.

Definition 5 Suppose that W is a subspace of an inner product space V. We say that a vector $\mathbf{u}$ from V is **orthogonal to *W*** if it is orthogonal to every vector in W. The set of all vectors that are orthogonal to W is called the **orthogonal complement of *W*** and is denoted by $W^{\perp}$.

We say that W and $W^{\perp}$ are **orthogonal complements**.

We're not going to be doing much with the orthogonal complement in these notes, although they will show up on occasion. We just wanted to acknowledge that there are subspaces that are only going to be found in inner product spaces. Here are a couple of nice theorems pertaining to orthogonal complements.

Theorem 6 Suppose W is a subspace of an inner product space V. Then,

(a) $W^{\perp}$ is a subspace of V.

(b) Only the zero vector, $\mathbf{0}$, is common to both W and $W^{\perp}$.

(c) $\left(W^{\perp}\right)^{\perp} = W$. Or in other words, the orthogonal complement of $W^{\perp}$ is W.

Here is a nice theorem that relates some of the fundamental subspaces that we were discussing in the previous section.

Theorem 7 If A is an $n \times m$ matrix then,

(a) The null space of A and the row space of A are orthogonal complements in $\mathbb{R}^m$ with respect to the standard Euclidean inner product.

(b) The null space of A^T and the column space of A are orthogonal complements in $\mathbb{R}^n$ with respect to the standard Euclidean inner product.

Orthonormal Basis

We now need to come back and revisit the topic of basis. We are going to be looking at a special kind of basis in this section that can arise in an inner product space, and yes it does require an inner product space to construct. However, before we do that we're going to need to get some preliminary topics out of the way first.

We'll first need to get a set of definitions out of way.

Definition 1 Suppose that S is a set of vectors in an inner product space.

(a) If each pair of distinct vectors from S is orthogonal then we call S an **orthogonal set**.

(b) If S is an orthogonal set and each of the vectors in S also has a norm of 1 then we call S an **orthonormal set**.

Let's take a quick look at an example.

Example 1 Given the three vectors $\mathbf{v}_1 = (2,0,-1)$, $\mathbf{v}_2 = (0,-1,0)$ and $\mathbf{v}_3 = (2,0,4)$ in $\mathbb{R}^3$ answer each of the following.

(a) Show that they form an orthogonal set under the standard Euclidean inner product for $\mathbb{R}^3$ but not an orthonormal set.

(b) Turn them into a set of vectors that will form an orthonormal set of vectors under the standard Euclidean inner product for $\mathbb{R}^3$.

Solution

(a) All we need to do here to show that they form an orthogonal set is to compute the inner product of all the possible pairs and show that they are all zero.

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = (2)(0) + (0)(-1) + (-1)(0) = 0$$
$$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = (2)(2) + (0)(0) + (-1)(4) = 0$$
$$\langle \mathbf{v}_2, \mathbf{v}_3 \rangle = (0)(2) + (-1)(0) + (0)(4) = 0$$

So, they do form an orthogonal set. To show that they don't form an orthonormal set we just need to show that at least one of them does not have a norm of 1. For the practice we'll compute all the norms.

$$\|\mathbf{v}_1\| = \sqrt{(2)^2 + (0)^2 + (-1)^2} = \sqrt{5}$$
$$\|\mathbf{v}_2\| = \sqrt{(0)^2 + (-1)^2 + (0)^2} = 1$$
$$\|\mathbf{v}_3\| = \sqrt{(2)^2 + (0)^2 + (4)^2} = \sqrt{20} = 2\sqrt{5}$$

So, one of them has a norm of 1, but the other two don't and so they are not an orthonormal set of vectors.

(b) We've actually done most of the work here for this part of the problem already. Back when we were working in $\mathbb{R}^n$ we saw that we could turn any vector $\mathbf{v}$ into a vector with norm 1 by dividing by it's norm as follows,

$$\frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

This new vector will have a norm of 1. So, we can turn each of the vectors above into a set of vectors with norm 1.

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \frac{1}{\sqrt{5}}(2,0,-1) = \left(\frac{2}{\sqrt{5}}, 0, -\frac{1}{\sqrt{5}}\right)$$
$$\mathbf{u}_2 = \frac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2 = \frac{1}{1}(0,-1,0) = (0,-1,0)$$
$$\mathbf{u}_3 = \frac{1}{\|\mathbf{v}_3\|}\mathbf{v}_3 = \frac{1}{2\sqrt{5}}(2,0,4) = \left(\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right)$$

All that remains is to show that this new set of vectors is still orthogonal. We'll leave it to you to verify that,

$$\left\langle \mathbf{u}_1, \mathbf{u}_2 \right\rangle = \left\langle \mathbf{u}_1, \mathbf{u}_3 \right\rangle = \left\langle \mathbf{u}_2, \mathbf{u}_3 \right\rangle = 0$$

and so we have turned the three vectors into a set of vectors that form an orthonormal set.

We have the following very nice fact about orthogonal sets.

Theorem 1 Suppose $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of non-zero vectors in an inner product space, then S is also a set of linearly independent vectors.

Proof : Note that we need the vectors to be non-zero vectors because the zero vector could be in a set of orthogonal vectors and yet we know that if a set includes the zero vector it will be linearly dependent.

So, now that we know there is a chance that these vectors are linearly independent (since we've excluded the zero vector) let's form the equation,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

and we'll need to show that the only scalars that work here are $c_1 = 0$, $c_2 = 0$, … , $c_n = 0$.

In fact, we can do this in a single step. All we need to do is take the inner product of both sides with respect to $\mathbf{v}_i$, $i = 1, 2, \ldots, n$, and the use the properties of inner products to rearrange things a little.

$$\begin{aligned} \left\langle c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n, \mathbf{v}_i \right\rangle &= \left\langle \mathbf{0}, \mathbf{v}_i \right\rangle \\ \left\langle c_1\mathbf{v}_1, \mathbf{v}_i \right\rangle + \left\langle c_2\mathbf{v}_2, \mathbf{v}_i \right\rangle + \cdots + \left\langle c_n\mathbf{v}_n, \mathbf{v}_i \right\rangle &= 0 \\ c_1\left\langle \mathbf{v}_1, \mathbf{v}_i \right\rangle + c_2\left\langle \mathbf{v}_2, \mathbf{v}_i \right\rangle + \cdots + c_n\left\langle \mathbf{v}_n, \mathbf{v}_i \right\rangle &= 0 \end{aligned}$$

Now, because we know the vectors in S are orthogonal we know that $\left\langle \mathbf{v}_i, \mathbf{v}_j \right\rangle = 0$ if $i \neq j$ and so this reduced down to,

$$c_i\left\langle \mathbf{v}_i, \mathbf{v}_i \right\rangle = 0$$

Next, since we know that the vectors are all non-zero we have $\left\langle \mathbf{v}_i, \mathbf{v}_i \right\rangle > 0$ and so the only way that this can be zero is if $c_i = 0$. So, we've shown that we must have $c_1 = 0$, $c_2 = 0$, … , $c_n = 0$ and so these vectors are linearly independent.

■

Okay, we are now ready to move into the main topic of this section. Since a set of orthogonal vectors are also linearly independent if they just happen to span the vector space we are working on they will also form a basis for the vector space.

Definition 2 Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for an inner product space.

(a) If S is also an orthogonal set then we call S an **orthogonal basis**.

(b) If S is also an orthonormal set then we call S an **orthonormal basis**.

Note that we've been using an orthonormal basis already to this point. The standard basis vectors for $\mathbb{R}^n$ are an orthonormal basis.

The following fact gives us one of the very nice properties about orthogonal/orthonormal basis.

Theorem 2 Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for an inner product space and that $\mathbf{u}$ is any vector from the inner product space then,

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n$$

If in addition S is in fact an orthonormal basis then,

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n$$

Proof : We'll just show that the first formula holds. Once we have that the second will follow directly from the fact that all the vectors in an orthonormal set have a norm of 1.

So, given **u** we need to find scalars c_1, c_2, … , c_n so that,

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

To find these scalars simply take the inner product of both sides with respect to $\mathbf{v}_i$, $i = 1, 2, \ldots, n$.

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v}_i \rangle &= \langle c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n, \mathbf{v}_i \rangle \\ &= c_1\langle \mathbf{v}_1, \mathbf{v}_i \rangle + c_2\langle \mathbf{v}_2, \mathbf{v}_i \rangle + \cdots + c_n\langle \mathbf{v}_n, \mathbf{v}_i \rangle\end{aligned}$$

Now, since we have an orthogonal basis we know that $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ if $i \neq j$ and so this reduces to,

$$\langle \mathbf{u}, \mathbf{v}_i \rangle = c_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle$$

Also, because $\mathbf{v}_i$ is a basis vector we know that it isn't the zero vector and so we also know that $\langle \mathbf{v}_i, \mathbf{v}_i \rangle > 0$. This then gives us,

$$c_i = \frac{\langle \mathbf{u}, \mathbf{v}_i \rangle}{\langle \mathbf{v}_i, \mathbf{v}_i \rangle}$$

However, from the definition of the norm we see that we can also write this as,

$$c_i = \frac{\langle \mathbf{u}, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}$$

and so we're done.

■

What this theorem is telling us is that for any vector in an inner product space, with an orthogonal/orthonormal basis, it is very easy to write down the linear combination of basis vectors for that vector. In other words, we don't need to go through all the work to find the linear combinations that we were doing in earlier sections.

We would like to be able to construct an orthogonal/orthonormal basis for a finite dimensional vector space given any basis of that vector space. The following two theorems will help us to do that.

Theorem 3 Suppose that W is a finite dimensional subspace of an inner product space V and further suppose that **u** is any vector in V. Then **u** can be written as,

$$\mathbf{u} = \mathrm{proj}_W\, \mathbf{u} + \mathrm{proj}_{W^\perp}\, \mathbf{u}$$

where $\mathrm{proj}_W\, \mathbf{u}$ is a vector that is in W and is called the **orthogonal projection of *u* on *W*** and $\mathrm{proj}_{W^\perp}\, \mathbf{u}$ is a vector in $W^\perp$ (the orthogonal complement of W) and is called the **component of *u* orthogonal to *W***.

Note that this theorem is really an extension of the idea of projections that we saw when we first introduced the concept of the dot product. Also note that $\mathrm{proj}_{W^\perp}\, \mathbf{u}$ can be easily computed from $\mathrm{proj}_W\, \mathbf{u}$ by,

$$\mathrm{proj}_{W^\perp}\, \mathbf{u} = \mathbf{u} - \mathrm{proj}_W\, \mathbf{u}$$

This theorem is not really the one that we need to construct an orthonormal basis. We will use portions of this theorem, but we needed it more to acknowledge that we could do projections and to get the notation out of the way. The following theorem is the one that will be the main workhorse of the process.

Theorem 4 Suppose that W is a finite dimensional subspace of an inner product space V. Further suppose that W has an orthogonal basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and that **u** is any vector in V then,

$$\mathrm{proj}_W\, \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n$$

If in addition S is in fact an orthonormal basis then,

$$\mathrm{proj}_W\, \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n$$

So, just how does this theorem help us to construct an orthogonal/orthonormal basis? The following process, called the **Gram-Schmidt process**, will construct an orthogonal/orthonormal basis for a finite dimensional inner product space given any basis. We'll also be able to develop some very nice facts about the basis that we're going to be constructing as we go through the construction process.

Gram-Schmidt Process

Suppose that V is a finite dimensional inner product space and that $\left\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\right\}$ is a basis for V. The following process will construct an orthogonal basis for V, $\left\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\right\}$. To find an orthonormal basis simply divide the $\mathbf{u}_i$'s by their norms.

Step 1 : Let $\mathbf{u}_1 = \mathbf{v}_1$.

Step 2 : Let $W_1 = \operatorname{span}\left\{\mathbf{u}_1\right\}$ and then define $\mathbf{u}_2 = \operatorname{proj}_{W_1^\perp} \mathbf{v}_2$ (*i.e.* $\mathbf{u}_2$ is the portion of $\mathbf{v}_2$ that is orthogonal to $\mathbf{u}_1$). Technically, this is all there is to step 2 (once we show that $\mathbf{u}_2 \neq \mathbf{0}$ anyway) since $\mathbf{u}_2$ will be orthogonal to $\mathbf{u}_1$ because it is in $W_1^\perp$. However, this isn't terribly useful from a computational standpoint. Using the result of Theorem 3 and the formula from Theorem 4 gives us the following formula for $\mathbf{u}_2$,

$$\mathbf{u}_2 = \mathbf{v}_2 - \operatorname{proj}_{W_1} \mathbf{v}_2 = \mathbf{v}_2 - \frac{\left\langle \mathbf{v}_2, \mathbf{u}_1 \right\rangle}{\left\| \mathbf{u}_1 \right\|^2} \mathbf{u}_1$$

Next, we need to verify that $\mathbf{u}_2 \neq \mathbf{0}$ because the zero vector cannot be a basis vector. To see that $\mathbf{u}_2 \neq \mathbf{0}$ assume for a second that we do have $\mathbf{u}_2 = \mathbf{0}$. This would give us,

$$\mathbf{v}_2 = \frac{\left\langle \mathbf{v}_2, \mathbf{u}_1 \right\rangle}{\left\| \mathbf{u}_1 \right\|^2} \mathbf{u}_1 = \frac{\left\langle \mathbf{v}_2, \mathbf{u}_1 \right\rangle}{\left\| \mathbf{u}_1 \right\|^2} \mathbf{v}_1 \qquad \text{since } \mathbf{u}_1 = \mathbf{v}_1$$

But this tells us that $\mathbf{v}_2$ is a multiple of $\mathbf{v}_1$ which we can't have since they are both basis vectors and are hence linearly independent. So, $\mathbf{u}_2 \neq \mathbf{0}$.

Finally, let's observe an interesting consequence of how we found $\mathbf{u}_2$. Both $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal and so are linearly independent by Theorem 1 above and this means that they are a basis for the subspace $W_2 = \operatorname{span}\left\{\mathbf{u}_1, \mathbf{u}_2\right\}$ and this subspace has dimension of 2. However, they are also linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$ and so W_2 is a subspace of $\operatorname{span}\left\{\mathbf{v}_1, \mathbf{v}_2\right\}$ which also has dimension 2. Therefore, by Theorem 9 from the section on Basis we can see that we must in fact have,

$$\operatorname{span}\left\{\mathbf{u}_1, \mathbf{u}_2\right\} = \operatorname{span}\left\{\mathbf{v}_1, \mathbf{v}_2\right\}$$

So, the two new vectors, $\mathbf{u}_1$ and $\mathbf{u}_2$, will in fact span the same subspace as the two original vectors, $\mathbf{v}_1$ and $\mathbf{v}_2$, span. This is a nice consequence of the Gram-Schmidt process.

Step 3 : This step is really an extension of Step 2 and so we won't go into quite the detail here as we did in Step 2. First, define $W_2 = \text{span}\{\mathbf{u}_1, \mathbf{u}_2\}$ and then define $\mathbf{u}_3 = \text{proj}_{W_2^\perp} \mathbf{v}_3$ and so $\mathbf{u}_3$ will be the portion of $\mathbf{v}_3$ that is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$. We can compute $\mathbf{u}_3$ as follows,

$$\mathbf{u}_3 = \mathbf{v}_3 - \text{proj}_{W_2} \mathbf{v}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2}\mathbf{u}_2$$

Next, both $\mathbf{u}_1$ and $\mathbf{u}_2$ are linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$ and so $\mathbf{u}_3$ can be thought of as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. Then because $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent we know that we must have $\mathbf{u}_3 \neq \mathbf{0}$. You should probably go through the steps of verifying the claims made here for the practice.

With this step we can also note that because $\mathbf{u}_3$ is in the orthogonal complement of W_2 (by construction) and because we know that,

$$W_2 = \text{span}\{\mathbf{u}_1, \mathbf{u}_2\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$$

from the previous step we know as well that $\mathbf{u}_3$ must be orthogonal to all vectors in W_2. In particular $\mathbf{u}_3$ must be orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$.

Finally, following an argument similar to that in Step 2 we get that,

$$\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$$

Step 4 : Continue in this fashion until we've found $\mathbf{u}_n$.

There is the Gram-Schmidt process. Going through the process above, with all the explanation as we provided, can be a little daunting and can make the process look more complicated than it in fact is. Let's summarize the process before we go onto a couple of examples.

Gram-Schmidt Process

Suppose that *V* is a finite dimensional inner product space and that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for *V* then an orthogonal basis for *V*, $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$, can be found using the following process.

$$\mathbf{u}_1 = \mathbf{v}_1$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2}\mathbf{u}_2$$

$$\vdots \qquad\qquad \vdots$$

$$\mathbf{u}_n = \mathbf{v}_n - \frac{\langle \mathbf{v}_n, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 - \frac{\langle \mathbf{v}_n, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2}\mathbf{u}_2 - \cdots - \frac{\langle \mathbf{v}_n, \mathbf{u}_{n-1} \rangle}{\|\mathbf{u}_{n-1}\|^2}\mathbf{u}_{n-1}$$

To convert the basis to an orthonormal basis simply divide all the new basis vectors by their norm. Also, due to the construction process we have

$$\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\} = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \qquad k = 1, 2, \ldots, n$$

and u_k will be orthogonal to $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}\}$ for $k = 2, 3, \ldots n$.

Okay, let's go through a couple of examples here.

Example 2 Given that $\mathbf{v}_1 = (2,-1,0)$, $\mathbf{v}_2 = (1,0,-1)$, and $\mathbf{v}_3 = (3,7,-1)$ is a basis of $\mathbb{R}^3$ and assuming that we're working with the standard Euclidean inner product construct an orthogonal basis for $\mathbb{R}^3$.

Solution

You should verify that the set of vectors above is in fact a basis for $\mathbb{R}^3$. Now, we'll need to go through the Gram-Schmidt process a couple of times. The first step is easy.

$$\mathbf{u}_1 = \mathbf{v}_1 = (2,-1,0)$$

The remaining two steps are going to involve a little more work, but won't be all that bad. Here is the formula for the second vector in our orthogonal basis.

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1$$

and here is all the quantities that we'll need.

$$\langle \mathbf{v}_2, \mathbf{u}_1 \rangle = 2 \qquad\qquad \|\mathbf{u}_1\|^2 = 5$$

The second vector is then,

$$\mathbf{u}_2 = (1,0,-1) - \frac{2}{5}(2,-1,0) = \left(\frac{1}{5}, \frac{2}{5}, -1\right)$$

The formula for the third (and final vector) is,

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2}\mathbf{u}_2$$

and here are the quantities that we need for this step.

$$\langle \mathbf{v}_3, \mathbf{u}_1 \rangle = -1 \qquad \langle \mathbf{v}_3, \mathbf{u}_2 \rangle = \frac{22}{5} \qquad \|\mathbf{u}_1\|^2 = 5 \qquad \|\mathbf{u}_2\|^2 = \frac{6}{5}$$

The third vector is then,

$$\mathbf{u}_3 = (3,7,-1) - \frac{-1}{5}(2,-1,0) - \frac{\frac{22}{5}}{\frac{6}{5}}\left(\frac{1}{5},\frac{2}{5},-1\right) = \left(\frac{8}{3},\frac{16}{3},\frac{8}{3}\right)$$

So, the orthogonal basis that we've constructed is,

$$\mathbf{u}_1 = (2,-1,0) \qquad \mathbf{u}_2 = \left(\frac{1}{5},\frac{2}{5},-1\right) \qquad \mathbf{u}_3 = \left(\frac{8}{3},\frac{16}{3},\frac{8}{3}\right)$$

You should verify that these do in fact form an orthogonal set.

Example 3 Given that $\mathbf{v}_1 = (2,-1,0)$, $\mathbf{v}_2 = (1,0,-1)$, and $\mathbf{v}_3 = (3,7,-1)$ is a basis of $\mathbb{R}^3$ and assuming that we're working with the standard Euclidean inner product construct an orthonormal basis for $\mathbb{R}^3$.

Solution

First, note that this is almost the same problem as the previous one except this time we're looking for an *orthonormal* basis instead of an *orthogonal* basis. There are two ways to approach this. The first is often the easiest way and that is to acknowledge that we've got a orthogonal basis and we can turn that into an orthonormal basis simply by dividing by the norms of each of the vectors. Let's do it this way and see what we get.

Here are the norms of the vectors from the previous example.

$$\|\mathbf{u}_1\| = \sqrt{5} \qquad \|\mathbf{u}_2\| = \frac{\sqrt{30}}{5} \qquad \|\mathbf{u}_3\| = \frac{8\sqrt{6}}{3}$$

Note that in order to eliminate as many square roots as possible we rationalized the denominators of the fractions here.

Dividing by the norms gives the following set of vectors.

$$\mathbf{w}_1 = \left(\frac{2}{\sqrt{5}},-\frac{1}{\sqrt{5}},0\right) \qquad \mathbf{w}_2 = \left(\frac{1}{\sqrt{30}},\frac{2}{\sqrt{30}},-\frac{5}{\sqrt{30}}\right) \qquad \mathbf{w}_3 = \left(\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)$$

Okay that's the first way to do it. The second way is to go through the Gram-Schmidt process and this time divide by the norm as we find each new vector. This will have two effects. First, it will put a fair amount of roots into the vectors that we'll need to work with. Second, because we are turning the new vectors into vectors with length one the norm in the Gram-Schmidt formula will also be 1 and so isn't needed.

Let's go through this once just to show you the differences.

The first new vector will be,

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \frac{1}{\sqrt{5}}(2,-1,0) = \left(\frac{2}{\sqrt{5}},-\frac{1}{\sqrt{5}},0\right)$$

Now, to get the second vector we first need to compute,

$$\mathbf{w} = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 = \mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{u}_1 \rangle \mathbf{u}_1$$

however we won't call it $\mathbf{u}_2$ yet since we'll need to divide by it's norm once we're done. Also note that we've acknowledged that the norm of $\mathbf{u}_1$ is 1 and so we don't need that in the formula. Here is the dot product that we need for this step.

$$\langle \mathbf{v}_2, \mathbf{u}_1 \rangle = \frac{2}{\sqrt{5}}$$

Here is the new *orthogonal* vector.

$$\mathbf{w} = (1,0,-1) - \frac{2}{\sqrt{5}}\left(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, 0\right) = \left(\frac{1}{5}, \frac{2}{5}, -1\right)$$

Notice that this is the same as the second vector we found in Example 2. In this case we'll need to divide by its norm to get the vector that we want in this case.

$$\mathbf{u}_2 = \frac{1}{\|\mathbf{w}\|}\mathbf{w} = \frac{5}{\sqrt{30}}\left(\frac{1}{5}, \frac{2}{5}, -1\right) = \left(\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, -\frac{5}{\sqrt{30}}\right)$$

Finally, for the third *orthogonal* vector the formula will be,

$$\mathbf{w} = \mathbf{v}_3 - \langle \mathbf{v}_3, \mathbf{u}_1 \rangle \mathbf{u}_1 - \langle \mathbf{v}_3, \mathbf{u}_2 \rangle \mathbf{u}_2$$

and again we've acknowledged that the norms of the first two vectors will be 1 and so aren't needed in this formula. Here are the dot products that we'll need.

$$\langle \mathbf{v}_3, \mathbf{u}_1 \rangle = -\frac{1}{\sqrt{5}} \qquad \langle \mathbf{v}_3, \mathbf{u}_2 \rangle = \frac{22}{\sqrt{30}}$$

The *orthogonal* vector is then,

$$\mathbf{w} = (3,7,-1) - \left(-\frac{1}{\sqrt{5}}\right)\left(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, 0\right) - \frac{22}{\sqrt{30}}\left(\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, -\frac{5}{\sqrt{30}}\right) = \left(\frac{8}{3}, \frac{16}{3}. \frac{8}{3}\right)$$

Again, this is the third *orthogonal* vector that we found in Example 2. Here is the final step to get our third orthonormal vector for this problem.

$$\mathbf{u}_3 = \frac{1}{\|\mathbf{w}\|}\mathbf{w} = \frac{3}{8\sqrt{6}}\left(\frac{8}{3}, \frac{16}{3}, \frac{8}{3}\right) = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

So, we got exactly the same vectors as if we did when we just used the results of Example 2. Of course that is something that we should expect to happen here.

So, as we saw in the previous example there are two ways to get an orthonormal basis from any given basis. Each has its pros and cons and you'll need to decide which method to use. If we first compute the orthogonal basis and the divide all of them at then end by their norms we don't have to work much with square roots, however we do need to compute norms that we won't need otherwise. Again, it will be up to you to determine what the best method for you to use is.

Example 4 Given that $\mathbf{v}_1 = (1,1,1,1)$, $\mathbf{v}_2 = (1,1,1,0)$, $\mathbf{v}_3 = (1,1,0,0)$ and $\mathbf{v}_4 = (1,0,0,0)$ is a basis of $\mathbb{R}^4$ and assuming that we're working with the standard Euclidean inner product construct an orthonormal basis for $\mathbb{R}^4$.

Solution
Now, we're looking for an orthonormal basis and so we've got our two options on how to proceed here. In this case we'll construct an orthogonal basis and then convert that into an orthonormal basis at the very end.

The first vector is,

$$\mathbf{u}_1 = \mathbf{v}_1 = (1,1,1,1)$$

Here's the dot product and norm we need for the second vector.

$$\langle \mathbf{v}_2, \mathbf{u}_1 \rangle = 3 \qquad \|\mathbf{u}_1\|^2 = 4$$

The second orthogonal vector is then,

$$\mathbf{u}_2 = (1,1,1,0) - \frac{3}{4}(1,1,1,1) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{3}{4}\right)$$

For the third vector we'll need the following dot products and norms

$$\langle \mathbf{v}_3, \mathbf{u}_1 \rangle = 2 \qquad \langle \mathbf{v}_3, \mathbf{u}_2 \rangle = \frac{1}{2} \qquad \|\mathbf{u}_1\|^2 = 4 \qquad \|\mathbf{u}_2\|^2 = \frac{3}{4}$$

and the third orthogonal vector is,

$$\mathbf{u}_3 = (1,1,0,0) - \frac{2}{4}(1,1,1,1) - \frac{\frac{1}{2}}{\frac{3}{4}}\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{3}{4}\right) = \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, 0\right)$$

Finally, for the fourth orthogonal vector we'll need,

$$\langle \mathbf{v}_4, \mathbf{u}_1 \rangle = 1 \qquad \langle \mathbf{v}_4, \mathbf{u}_2 \rangle = \frac{1}{4} \qquad \langle \mathbf{v}_4, \mathbf{u}_3 \rangle = \frac{1}{3}$$

$$\|\mathbf{u}_1\|^2 = 4 \qquad \|\mathbf{u}_2\|^2 = \frac{3}{4} \qquad \|\mathbf{u}_3\|^2 = \frac{2}{3}$$

and the fourth vector in out new orthogonal basis is,

$$\mathbf{u}_4 = (1,0,0,0) - \frac{1}{4}(1,1,1,1) - \frac{\frac{1}{4}}{\frac{3}{4}}\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{3}{4}\right) - \frac{\frac{1}{3}}{\frac{2}{3}}\left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, 0\right) = \left(\frac{1}{2}, -\frac{1}{2}, 0, 0\right)$$

Okay, the orthogonal basis is then,

$$\mathbf{u}_1 = (1,1,1,1) \quad \mathbf{u}_2 = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{3}{4}\right) \quad \mathbf{u}_3 = \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, 0\right) \quad \mathbf{u}_4 = \left(\frac{1}{2}, -\frac{1}{2}, 0, 0\right)$$

Next, we'll need their norms so we can turn this set into an orthonormal basis.

$$\|\mathbf{u}_1\| = 2 \qquad \|\mathbf{u}_2\| = \frac{\sqrt{3}}{2} \qquad \|\mathbf{u}_3\| = \frac{\sqrt{6}}{3} \qquad \|\mathbf{u}_4\| = \frac{\sqrt{2}}{2}$$

The orthonormal basis is then,

$$\mathbf{w}_1 = \frac{1}{\|\mathbf{u}_1\|}\mathbf{u}_1 = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$$

$$\mathbf{w}_2 = \frac{1}{\|\mathbf{u}_2\|}\mathbf{u}_2 = \left(\frac{1}{2\sqrt{3}},\frac{1}{2\sqrt{3}},\frac{1}{2\sqrt{3}},-\frac{3}{2\sqrt{3}}\right)$$

$$\mathbf{w}_3 = \frac{1}{\|\mathbf{u}_3\|}\mathbf{u}_3 = \left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}},0\right)$$

$$\mathbf{w}_4 = \frac{1}{\|\mathbf{u}_4\|}\mathbf{u}_4 = \left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0,0\right)$$

Now, we saw how to expand a linearly independent set of vectors into a basis for a vector space. We can do the same thing here with orthogonal sets of vectors and the Gram-Schmidt process.

Example 5 Expand the vectors $\mathbf{v}_1 = (2,0,-1)$ and $\mathbf{v}_2 = (2,0,4)$ into an orthogonal basis for $\mathbb{R}^3$ and assume that we're working with the standard Euclidean inner product.

Solution

First notice that the two vectors are already orthogonal and linearly independent. Since they are linearly independent and we know that a basis for $\mathbb{R}^3$ will contain 3 vectors we know that we'll only need to add in one more vector. Next, since they are already orthogonal that will simplify some of the work.

Now, recall that in order to expand a linearly independent set into a basis for a vector space we need to add in a vector that is not in the span of the original vectors. Doing so will retain the linear independence of the set. Since both of these vectors have a zero in the second term we can add in any of the following to the set.

$$(0,1,0) \qquad (1,1,1) \qquad (1,1,0) \qquad (0,1,1)$$

If we used the first one we'd actually have an orthonormal set without any work, but that would be boring and defeat the purpose of the example. To make our life at least somewhat easier with the work let's add in the fourth on to get the set of vectors.

$$\mathbf{v}_1 = (2,0,-1) \qquad \mathbf{v}_2 = (2,0,4) \qquad \mathbf{v}_3 = (0,1,1)$$

Now, we know these are linearly independent and since there are three vectors by Theorem 6 from the section on Basis we know that they form a basis for $\mathbb{R}^3$. However, they don't form an orthogonal basis.

To get an orthogonal basis we would need to perform Gram-Schmidt on the set. However, since the first two vectors are already orthogonal performing Gram-Schmidt would not have any affect (you should verify this). So, let's just rename the first two vectors as,

$$\mathbf{u}_1 = (2,0,-1) \qquad \mathbf{u}_2 = (2,0,4)$$

and then just perform Gram-Schmidt for the third vector. Here are the dot products and norms that we'll need.

$$\langle \mathbf{v}_3, \mathbf{u}_1 \rangle = -1 \qquad \langle \mathbf{v}_3, \mathbf{u}_2 \rangle = 4 \qquad \|\mathbf{u}_1\|^2 = 5 \qquad \|\mathbf{u}_2\|^2 = 20$$

The third vector will then be,

$$\mathbf{u}_3 = (0,1,1) - \frac{-1}{5}(2,0,-1) - \frac{5}{20}(2,0,4) = \left(-\frac{1}{10}, 1, -\frac{1}{5}\right)$$

Least Squares

In this section we're going to take a look at an important application of orthogonal projections to inconsistent systems of equations. Recall that a system is called inconsistent if there are no solutions to the system. The natural question should probably arise at this point of just why we would care about this. Let's take a look at the following examples that we can use to motivate the reason for looking into this.

Example 1 Find the equation of the line that runs through the four points $(1,-1)$, $(4,11)$, $(-1,-9)$ and $(-2,-13)$.

Solution

So, what we're looking for are the values of *m* and *b* for which the line, $y = mx + b$ will run through the four points given above. If we plug these points into the line we arrive at the following system of equations.

$$\begin{aligned} m + b &= -1 \\ 4m + b &= 11 \\ -m + b &= -9 \\ -2m + b &= -13 \end{aligned}$$

The corresponding matrix form of this system is,

$$\begin{bmatrix} 1 & 1 \\ 4 & 1 \\ -1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 11 \\ -9 \\ -13 \end{bmatrix}$$

Solving this system (either the matrix form or the equations) gives us the solution,

$$m = 4 \qquad b = -5$$

So, the line $y = 4x - 5$ will run through the three points given above. Note that this makes this a consistent system.

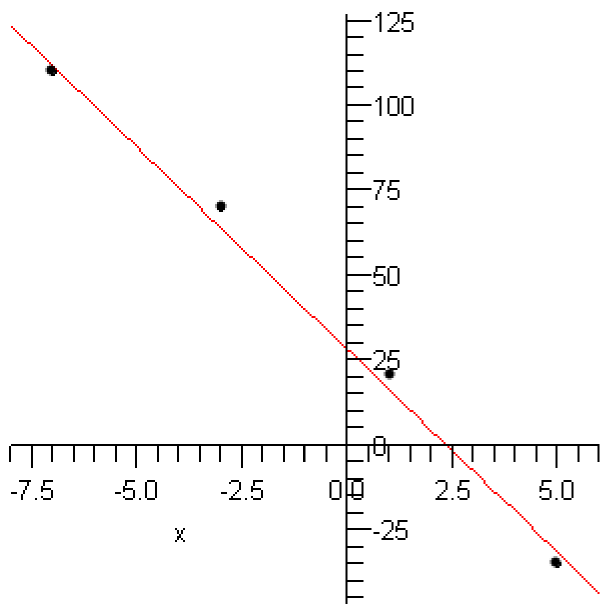
Example 2 Find the equation of the line that runs through the four points $(-3,70)$, $(1,21)$, $(-7,110)$ and $(5,-35)$.

Solution
So, this is essentially the same problem as in the previous example. Here are the system of equations and matrix form of the system of equations that we need to solve for this problem.

$$\begin{aligned} -3m+b&=70 \\ m+b&=21 \\ -7m+b&=110 \\ 5m+b&=-35 \end{aligned} \qquad \begin{bmatrix} -3 & 1 \\ 1 & 1 \\ -7 & 1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 70 \\ 21 \\ 110 \\ -35 \end{bmatrix}$$

Now, try as we might we won't find a solution to this system and so this system is inconsistent.

The previous two examples were asking for pretty much the same thing and in the first example we were able to answer the question while in the second we were not able to answer the question. It is the second example that we want to look at a little closer. Here is a graph of the points given in this example.



We can see that these points do almost fall on a line. Without the reference line that we put into the sketch it would not be clear that these points did not fall onto a line and so asking the question that we did was not totally unreasonable.

Let's further suppose that the four points that we have in this example came from some experiment and we know for some physical reason that the data should all lie on a straight line. However, due to inaccuracies in the measuring equipment caused some (or all) of the numbers to be a little off. In light of this the question in Example 2 is again not unreasonable and in fact we may still need to answer it in some way.

That is the point of this section. Given this set of data can we find the equation of a line that will as closely as possible (what ever this means…) approximate each of the data points. Or more generally, given an inconsistent system of equations, $A\mathbf{x} = \mathbf{b}$, can we find a vector, let's call it $\overline{\mathbf{x}}$, so that $A\overline{\mathbf{x}}$ will be as close to **b** as possible (again, what ever this means…).

To answer this question let's step back a bit and take a look at the general situation. So, we will suppose that we have an inconsistent system of *n* equations in *m* unknowns, $A\mathbf{x} = \mathbf{b}$, so the coefficient matrix, *A*, will have size $n \times m$. Let's rewrite the system a little and make the following definition.

$$\varepsilon = \mathbf{b} - A\mathbf{x}$$

We will call ε the **error vector** and we'll call $\|\varepsilon\| = \|\mathbf{b} - A\mathbf{x}\|$ the **error** since it will measure the distance between $A\mathbf{x}$ and **b** for any vector **x** in $\mathbb{R}^m$ (there are *m* unknowns and so **x** will be in $\mathbb{R}^m$). Note that we're going to be using the standard Euclidean inner product to compute the norm in these cases. The least squares problem is then the following problem.

Least Square Problem Given an inconsistent system of equations, $A\mathbf{x} = \mathbf{b}$, we want to find a vector, $\overline{\mathbf{x}}$, from $\mathbb{R}^m$ so that the error $\|\overline{\varepsilon}\| = \|\mathbf{b} - A\overline{\mathbf{x}}\|$ is the smallest possible error. The vector $\overline{\mathbf{x}}$ is called the **least squares solution**.

Solving this problem is actually easier than it might look at first. The first thing that we'll want to do is look at a more general situation. The following theorem will be useful in solving the least squares problem.

Theorem 1 Suppose that *W* is a finite dimensional subspace of an inner product space *V* and that **u** is any vector in *V*. The **best approximation** to **u** from *W* is then $\operatorname{proj}_W \mathbf{u}$. By best approximation we mean that for every **w** (that is not $\operatorname{proj}_W \mathbf{u}$) in *W* we will have,

$$\|\mathbf{u} - \operatorname{proj}_W \mathbf{u}\| < \|\mathbf{u} - \mathbf{w}\|$$

Proof : For any vector **w** in *W* we can write.

$$\mathbf{u} - \mathbf{w} = \left(\mathbf{u} - \operatorname{proj}_W \mathbf{u}\right) + \left(\operatorname{proj}_W \mathbf{u} - \mathbf{w}\right)$$

Notice that $\operatorname{proj}_W \mathbf{u} - \mathbf{w}$ is a difference of vectors in *W* and hence must also be in *W*. Likewise, $\mathbf{u} - \operatorname{proj}_W \mathbf{u}$ is in fact $\operatorname{proj}_{W^\perp} \mathbf{u}$, the component of **u** orthogonal to *W*, and so is orthogonal to any vector in *W*. Therefore $\operatorname{proj}_W \mathbf{u} - \mathbf{w}$ and $\mathbf{u} - \operatorname{proj}_W \mathbf{u}$ are orthogonal vectors. So, by the Pythagorean Theorem we have,

$$\|\mathbf{u}-\mathbf{w}\| = \left\|\left(\mathbf{u}-\text{proj}_W \mathbf{u}\right)+\left(\text{proj}_W \mathbf{u}-\mathbf{w}\right)\right\| = \|\mathbf{u}-\text{proj}_W \mathbf{u}\|+\|\text{proj}_W \mathbf{u}-\mathbf{w}\|$$

Or, upon dropping the middle term,

$$\|\mathbf{u}-\mathbf{w}\| = \|\mathbf{u}-\text{proj}_W \mathbf{u}\|+\|\text{proj}_W \mathbf{u}-\mathbf{w}\|$$

Finally, if we have $\mathbf{w} \neq \text{proj}_W \mathbf{u}$ then we know that $\|\text{proj}_W \mathbf{u}-\mathbf{w}\| > 0$ and so if we drop this term we get,

$$\|\mathbf{u}-\mathbf{w}\| > \|\mathbf{u}-\text{proj}_W \mathbf{u}\|$$

which is what we wanted to prove.

■

So, just what does this theorem do for us? Well for any vector $\mathbf{x}$ in $\mathbb{R}^m$ we know that $A\mathbf{x}$ will be a linear combination of the column vectors from A. Now, let W be the subspace of $\mathbb{R}^n$ (yes, $\mathbb{R}^n$ since each column of A has n entries) that is spanned by the column vectors of A. Then $A\mathbf{x}$ will not only be in W (since it's a linear combination of the column vectors) but as we let $\mathbf{x}$ range over all possible vectors in $\mathbb{R}^m$ $A\mathbf{x}$ will range over all of W.

Now, the least squares problem is asking us to find the vector $\mathbf{x}$ in $\mathbb{R}^m$, we're calling it $\overline{\mathbf{x}}$, so that $\overline{\varepsilon}$ is smaller than (*i.e.* smaller norm) than all other possible values of ε, *i.e.* $\|\overline{\varepsilon}\| < \|\varepsilon\|$. If we plug in for the definition of the errors we arrive at.

$$\|\mathbf{b}-A\overline{\mathbf{x}}\| < \|\mathbf{b}-A\mathbf{x}\|$$

With the least squares problem we are looking for the closest that we can get $A\mathbf{x}$ to $\mathbf{b}$. However, this is exactly the type of situation that Theorem 1 is telling us how to solve. The $A\mathbf{x}$ range over all possible vectors in W and we want the one that is closed to some vector $\mathbf{b}$ in $\mathbb{R}^n$. Theorem 1 tells us that the one that we're after is,

$$A\overline{\mathbf{x}} = \text{proj}_W \mathbf{b}$$

Of course we are actually after $\overline{\mathbf{x}}$ and not $A\overline{\mathbf{x}}$ but this does give us one way to find $\overline{\mathbf{x}}$. We could first compute $\text{proj}_W \mathbf{b}$ and then solve $A\overline{\mathbf{x}} = \text{proj}_W \mathbf{b}$ for $\overline{\mathbf{x}}$ and we'd have the solution that we're after. There is however a better way of doing this.

Before we give that theorem however, we'll need a quick fact.

Theorem 2 Suppose that A is an $n \times m$ matrix with linearly independent columns. Then, $A^T A$ is an invertible matrix.

Proof : From Theorem 8 in the Fundamental Subspaces section we know that if $A^T A\mathbf{x} = \mathbf{0}$ has only the trivial solution then $A^T A$ will be an invertible matrix. So, let's suppose that $A^T A\mathbf{x} = \mathbf{0}$. This tells us that $A\mathbf{x}$ is in the null space of A^T, but we also know that $A\mathbf{x}$ is in the column space of A. Theorem 7 from the section on Inner Product Spaces tells us that these two spaces are orthogonal complements and Theorem 6 from the same section tells us that the only vector in common to both must be the zero vector and so we know that $A\mathbf{x} = \mathbf{0}$. If $\mathbf{c}_1$, $\mathbf{c}_2$, … , $\mathbf{c}_m$ are the columns of A then we know that $A\mathbf{x}$ can be written as,

$$A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_m\mathbf{c}_m$$

Then using $A\mathbf{x} = \mathbf{0}$ we also know that,

$$A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_m\mathbf{c}_m = \mathbf{0}$$

However, since the columns of A are linearly independent this equations can only have the trivial solution, $\mathbf{x} = \mathbf{0}$.

Therefore $A^T A\mathbf{x} = \mathbf{0}$ has only the trivial solution and so $A^T A$ is an invertible matrix.

■

The following theorem will now give us a better method for finding the least squares solution to a system of equations.

Theorem 3 Given the system of equations $A\mathbf{x} = \mathbf{b}$, a least squares solution to the system denoted by $\overline{\mathbf{x}}$, will also be a solution to the **associated normal system**,

$$A^T A\mathbf{x} = A^T\mathbf{b}$$

Further if A has linearly independent columns then there is a unique least squares solution given by,

$$\overline{\mathbf{x}} = \left(A^T A\right)^{-1} A^T\mathbf{b}$$

Proof :
Let's suppose that $\overline{\mathbf{x}}$ is a least squares solution and so,

$$A\overline{\mathbf{x}} = \operatorname{proj}_W \mathbf{b}$$

Now, let's consider,

$$\mathbf{b} - A\overline{\mathbf{x}} = \mathbf{b} - \operatorname{proj}_W \mathbf{b}$$

However as pointed out in the proof of Theorem 1 we know that $\mathbf{b} - \operatorname{proj}_W \mathbf{b}$ is in the orthogonal complement of W. Next, W is the column space of A and by Theorem 7 from the section on Inner Product Spaces we know that the orthogonal complement of the column space of A is in fact the null space of A^T and so, $\mathbf{b} - \operatorname{proj}_W \mathbf{b}$ must be in the null space of A^T.

So, we must then have,

$$A^T\left(\mathbf{b} - \operatorname{proj}_W \mathbf{b}\right) = A^T\left(\mathbf{b} - A\overline{\mathbf{x}}\right) = \mathbf{0}$$

Or, with a little rewriting we arrive at,

$$A^T A\overline{\mathbf{x}} = A^T\mathbf{b}$$

and so we see that $\bar{\mathbf{x}}$ must also be a solution to the normal system of equations.

For the second part we don't have much to do. If the columns of A are linearly independent then $A^T A$ is invertible by Theorem 2 above. However, by Theorem 8 in the Fundamental Subspaces section this means that $A^T A\mathbf{x} = A^T\mathbf{b}$ has a unique solution.

To find the unique solution we just need to multiply both sides by the inverse of $A^T A$.

■

So, to find a least squares solution to $A\mathbf{x} = \mathbf{b}$ all we need to do is solve the normal system of equations,

$$A^T A\mathbf{x} = A^T\mathbf{b}$$

and we will have a least squares solution.

Now we should work a couple of examples. We'll start with Example 2 from above.

Example 3 Use least squares to find the equation of the line that will best approximate the points $(-3,70)$, $(1,21)$, $(-7,110)$ and $(5,-35)$.

Solution

The system of equations that we need to solve from Example 2 is,

$$\begin{bmatrix} -3 & 1 \\ 1 & 1 \\ -7 & 1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 70 \\ 21 \\ 110 \\ -35 \end{bmatrix}$$

So, we have,

$$A = \begin{bmatrix} -3 & 1 \\ 1 & 1 \\ -7 & 1 \\ 5 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} -3 & 1 & -7 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 70 \\ 21 \\ 110 \\ -35 \end{bmatrix}$$

The normal system that we need to solve is then,

$$\begin{bmatrix} -3 & 1 & -7 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 1 & 1 \\ -7 & 1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} -3 & 1 & -7 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 70 \\ 21 \\ 110 \\ -35 \end{bmatrix}$$

$$\begin{bmatrix} 84 & -4 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} -1134 \\ 166 \end{bmatrix}$$

This is a fairly simple system to solve and upon doing so we get,

$$m = -\frac{121}{10} = -12.1 \qquad\qquad b = \frac{147}{5} = 29.4$$

So, the line that best approximates all the points above is given by,

$$y = -12.1x + 29.4$$

The sketch of the line and points after Example 2 above shows this line in relation to the points.

Example 4 Find the least squares solution to the following system of equations.

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & -5 & 2 \\ -3 & 1 & -4 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 5 \\ -1 \end{bmatrix}$$

Solution

Okay there really isn't much to do here other than run through the formula. Here are the various matrices that we'll need here.

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & -5 & 2 \\ -3 & 1 & -4 \\ 1 & -1 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 1 & -3 & 1 \\ -1 & -5 & 1 & -1 \\ 1 & 2 & -4 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -4 \\ 2 \\ 5 \\ -1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 15 & -11 & 17 \\ -11 & 28 & -16 \\ 17 & -16 & 22 \end{bmatrix} \qquad A^T\mathbf{b} = \begin{bmatrix} -22 \\ 0 \\ -21 \end{bmatrix}$$

The normal system if then,

$$\begin{bmatrix} 15 & -11 & 17 \\ -11 & 28 & -16 \\ 17 & -16 & 22 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -22 \\ 0 \\ -21 \end{bmatrix}$$

This system is a little messier to solve than the previous example, but upon solving we get,

$$x_1 = -\frac{18}{7} \qquad x_2 = -\frac{151}{210} \qquad x_3 = \frac{107}{210}$$

In vector form the least squares solution is then,

$$\overline{\mathbf{x}} = \begin{bmatrix} -\frac{18}{7} \\ -\frac{151}{210} \\ \frac{107}{210} \end{bmatrix}$$

We need to address one more issues before we move on to the next section. When we opened this discussion up we said that we were after a solution, denoted $\overline{\mathbf{x}}$, so that $A\overline{\mathbf{x}}$ will be as close to **b** as possible in some way. We then defined,

$$\varepsilon = \mathbf{b} - A\mathbf{x} \qquad\qquad \overline{\varepsilon} = \mathbf{b} - A\overline{\mathbf{x}}$$

and stated that what we meant by as close to **b** as possible was that we wanted to find the $\overline{\mathbf{x}}$ for which,

$$\left\| \overline{\varepsilon} \right\| < \left\| \varepsilon \right\|$$

for all $\mathbf{x} \neq \overline{\mathbf{x}}$.

Okay, this is all fine in terms of mathematically defining what we mean by "as close as possible", but in practical terms just what are we asking for here? Let's go back to Example 3. For this example the general formula for ε is,

$$\varepsilon = \mathbf{b} - A\mathbf{x} = \begin{bmatrix} 64 \\ 21 \\ 110 \\ -35 \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ 1 & 1 \\ -7 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 64 - \left(-3m + b\right) \\ 21 - \left(m + b\right) \\ 110 - \left(-7m + b\right) \\ -35 - \left(5m + b\right) \end{bmatrix} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{bmatrix}$$

So, the components of the error vector, ε, each measure just how close each possible choice of *m* and *b* will get us to the exact answer (which is given by the components of **b**).

We can also think about this in terms of the equation of the line. We've been given a set of points $\left(x_i, y_i\right)$ and we want to determine an *m* and a *b* so that when we plug x_i, the *x* coordinate or our point, into $mx + b$ the error,

$$\varepsilon_i = y_i - \left(mx_i + b\right)$$

is as small as possible (in some way that we're trying to figure out here) for all the points that we've been given. Then if we plug in the points that we've been given we'll see that this formula is nothing more than the components of the error vector.

Now, in the case of our example we were looking for,

$$\overline{\mathbf{x}} = \begin{bmatrix} \overline{m} \\ \overline{b} \end{bmatrix}$$

so that,

$$\left\| \overline{\varepsilon} \right\| = \left\| \mathbf{b} - A\overline{\mathbf{x}} \right\|$$

is as small as possible, or in other words is smaller than all other possible choices of **x**.

We can now answer just what we mean by "as small as possible". First, let's compute the following,

$$\left\| \varepsilon \right\|^2 = \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2$$

The least squares solution, $\overline{\mathbf{x}}$, will be the value of **x** for which,

$$\|\bar{\varepsilon}\|^2 = \bar{\varepsilon}_1^2 + \bar{\varepsilon}_2^2 + \bar{\varepsilon}_3^2 + \bar{\varepsilon}_4^2 < \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = \|\varepsilon\|^2$$

and hence the name "least squares". The solution we're after is the value that will give the least value of the sum of the squares of the errors.

Example 5 Compute the error for the solution from Example 3.

Solution
First, the line that we found using least squares is,

$$y = -12.1x + 29.4$$

We can compute the errors for each of the points by plugging in the given x value into this line and then taking the difference of the result form the equation and the known y value. Here are the error computations for each of the four points in Example 3.

$$\bar{\varepsilon}_1 = 70 - \left(-12.1(-3) + 29.4\right) = 4.3$$
$$\bar{\varepsilon}_2 = 21 - \left(-12.1(1) + 29.4\right) = 3.7$$
$$\bar{\varepsilon}_3 = 110 - \left(-12.1(-7) + 29.4\right) = -4.1$$
$$\bar{\varepsilon}_4 = -35 - \left(-12.1(5) + 29.4\right) = -3.9$$

On a side note, we could have just as easily computed these by doing the following matrix work.

$$\bar{\varepsilon} = \begin{bmatrix} 70 \\ 21 \\ 110 \\ -35 \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ 1 & 1 \\ -7 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} -12.1 \\ 29.4 \end{bmatrix} = \begin{bmatrix} 4.3 \\ 3.7 \\ -4.1 \\ -3.9 \end{bmatrix}$$

The square of the error and the error is then,

$$\|\bar{\varepsilon}\|^2 = (4.3)^2 + (3.7)^2 + (-4.1)^2 + (-3.9)^2 = 64.2 \qquad \Rightarrow \qquad \|\bar{\varepsilon}\| = \sqrt{64.2} = 8.0125$$

Now, according to our discussion above this means that if we choose any other value of m and b and compute the error we will arrive at a value that is larger that 8.0125.

QR-Decomposition

In this section we're going to look at a way to "decompose" or "factor" an $n \times m$ matrix as follows.

Theorem 1 Suppose that A is an $n \times m$ matrix with linearly independent columns then A can be factored as,

$$A = QR$$

where Q is an $n \times m$ matrix with orthonormal columns and R is an invertible $m \times m$ upper triangular matrix.

Proof : The proof here will consist of actually constructing *Q* and *R* and showing that they in fact do multiply to give *A*.

Okay, let's start with *A* and suppose that it's columns are given by $\mathbf{c}_1$, $\mathbf{c}_2$, … , $\mathbf{c}_m$. Also suppose that we perform the Gram-Schmidt process on these vectors and arrive at a set of orthonormal vectors $\mathbf{u}_1$, $\mathbf{u}_2$, … , $\mathbf{u}_m$. Next, define *Q* (yes, the *Q* in the theorem statement) to be the $n\times m$ matrix whose columns are $\mathbf{u}_1$, $\mathbf{u}_2$, … , $\mathbf{u}_m$ and so *Q* will be a matrix with orthonormal columns. We can then write *A* and *Q* as,

$$A=\left[\mathbf{c}_1 \mid \mathbf{c}_2 \mid \cdots \mid \mathbf{c}_m\right] \qquad\qquad Q=\left[\mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_m\right]$$

Next, because each of the $\mathbf{c}_i$'s are in $\operatorname{span}\{\mathbf{u}_1,\mathbf{u}_2,\ldots\mathbf{u}_m\}$ we know from Theorem 2 of the previous section that we can write each $\mathbf{c}_i$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, … , $\mathbf{u}_m$ in the following manner.

$$\begin{aligned}
\mathbf{c}_1 &= \langle \mathbf{c}_1,\mathbf{u}_1\rangle \mathbf{u}_1 + \langle \mathbf{c}_1,\mathbf{u}_2\rangle \mathbf{u}_2 + \cdots + \langle \mathbf{c}_1,\mathbf{u}_m\rangle \mathbf{u}_m \\
\mathbf{c}_2 &= \langle \mathbf{c}_2,\mathbf{u}_1\rangle \mathbf{u}_1 + \langle \mathbf{c}_2,\mathbf{u}_2\rangle \mathbf{u}_2 + \cdots + \langle \mathbf{c}_2,\mathbf{u}_m\rangle \mathbf{u}_m \\
&\ \ \vdots \qquad\qquad\qquad \vdots \\
\mathbf{c}_m &= \langle \mathbf{c}_m,\mathbf{u}_1\rangle \mathbf{u}_1 + \langle \mathbf{c}_m,\mathbf{u}_2\rangle \mathbf{u}_2 + \cdots + \langle \mathbf{c}_m,\mathbf{u}_m\rangle \mathbf{u}_m
\end{aligned}$$

Next, define *R* (and yes, this will eventually be the *R* from the theorem statement) to be the $m\times m$ matrix defined as,

$$R=\begin{bmatrix}
\langle \mathbf{c}_1,\mathbf{u}_1\rangle & \langle \mathbf{c}_2,\mathbf{u}_1\rangle & \cdots & \langle \mathbf{c}_m,\mathbf{u}_1\rangle \\
\langle \mathbf{c}_1,\mathbf{u}_2\rangle & \langle \mathbf{c}_2,\mathbf{u}_2\rangle & \cdots & \langle \mathbf{c}_m,\mathbf{u}_2\rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle \mathbf{c}_1,\mathbf{u}_m\rangle & \langle \mathbf{c}_2,\mathbf{u}_m\rangle & \cdots & \langle \mathbf{c}_m,\mathbf{u}_m\rangle
\end{bmatrix}$$

Now, let's examine the product, *QR*.

$$QR=\left[\mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_m\right]\begin{bmatrix}
\langle \mathbf{c}_1,\mathbf{u}_1\rangle & \langle \mathbf{c}_2,\mathbf{u}_1\rangle & \cdots & \langle \mathbf{c}_m,\mathbf{u}_1\rangle \\
\langle \mathbf{c}_1,\mathbf{u}_2\rangle & \langle \mathbf{c}_2,\mathbf{u}_2\rangle & \cdots & \langle \mathbf{c}_m,\mathbf{u}_2\rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle \mathbf{c}_1,\mathbf{u}_m\rangle & \langle \mathbf{c}_2,\mathbf{u}_m\rangle & \cdots & \langle \mathbf{c}_m,\mathbf{u}_m\rangle
\end{bmatrix}$$

From the section on Matrix Arithmetic we know that the j^{th} column of this product is simply *Q* times the j^{th} column of *R*. However, if you work through a couple of these you'll see that when we multiply *Q* times the j^{th} column of *R* we arrive at the formula for $\mathbf{c}_j$ that we've got above. In other words,

$$QR = \left[\mathbf{u}_1 \;\middle|\; \mathbf{u}_2 \;\middle|\; \cdots \;\middle|\; \mathbf{u}_m \right] \begin{bmatrix} \langle \mathbf{c}_1, \mathbf{u}_1 \rangle & \langle \mathbf{c}_2, \mathbf{u}_1 \rangle & \cdots & \langle \mathbf{c}_m, \mathbf{u}_1 \rangle \\ \langle \mathbf{c}_1, \mathbf{u}_2 \rangle & \langle \mathbf{c}_2, \mathbf{u}_2 \rangle & \cdots & \langle \mathbf{c}_m, \mathbf{u}_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{c}_1, \mathbf{u}_m \rangle & \langle \mathbf{c}_2, \mathbf{u}_m \rangle & \cdots & \langle \mathbf{c}_m, \mathbf{u}_m \rangle \end{bmatrix}$$
$$= \left[\mathbf{c}_1 \;\middle|\; \mathbf{c}_2 \;\middle|\; \cdots \;\middle|\; \mathbf{c}_m \right]$$
$$= A$$

So, we can factor A as a product of Q and R and Q has the correct form. Now all that we need to do is to show that R is an invertible upper triangular matrix and we'll be done. First, from the Gram-Schmidt process we know that u_k is orthogonal to $c_1, c_2, \ldots, c_{k-1}$. This means that all the inner products below the main diagonal must be zero since they are all of the form $\langle \mathbf{c}_i, \mathbf{u}_j \rangle$ with $i < j$.

Now, we know from Theorem 2 from the Special Matrices section that a triangular matrix will be invertible if the main diagonal entries, $\langle \mathbf{c}_i, \mathbf{u}_i \rangle$, are non-zero. This is fairly easy to show. Here is the general formula for $\mathbf{u}_i$ from the Gram-Schmidt process.

$$\mathbf{u}_i = \mathbf{c}_i - \langle \mathbf{c}_i, \mathbf{u}_1 \rangle \mathbf{u}_1 - \langle \mathbf{c}_i, \mathbf{u}_2 \rangle \mathbf{u}_2 - \cdots - \langle \mathbf{c}_i, \mathbf{u}_{i-1} \rangle \mathbf{u}_{i-1}$$

Recall that we're assuming that we found the orthonormal $\mathbf{u}_i$'s and so each of these will have a norm of 1 and so the norms are not needed in the formula. Now, solving this for $\mathbf{c}_i$ gives,

$$\mathbf{c}_i = \mathbf{u}_i + \langle \mathbf{c}_i, \mathbf{u}_1 \rangle \mathbf{u}_1 + \langle \mathbf{c}_i, \mathbf{u}_2 \rangle \mathbf{u}_2 + \cdots + \langle \mathbf{c}_i, \mathbf{u}_{i-1} \rangle \mathbf{u}_{i-1}$$

Let's look at the diagonal entries of R. We'll plug in the formula for $\mathbf{c}_i$ into the inner product and do some rewriting using the properties of the inner product.

$$\begin{aligned} \langle \mathbf{c}_i, \mathbf{u}_i \rangle &= \langle \mathbf{u}_i + \langle \mathbf{c}_i, \mathbf{u}_1 \rangle \mathbf{u}_1 + \langle \mathbf{c}_i, \mathbf{u}_2 \rangle \mathbf{u}_2 + \cdots + \langle \mathbf{c}_i, \mathbf{u}_{i-1} \rangle \mathbf{u}_{i-1}, \mathbf{u}_i \rangle \\ &= \langle \mathbf{u}_i, \mathbf{u}_i \rangle + \langle \mathbf{c}_i, \mathbf{u}_1 \rangle \langle \mathbf{u}_1, \mathbf{u}_i \rangle + \langle \mathbf{c}_i, \mathbf{u}_2 \rangle \langle \mathbf{u}_2, \mathbf{u}_i \rangle + \cdots + \langle \mathbf{c}_i, \mathbf{u}_{i-1} \rangle \langle \mathbf{u}_{i-1}, \mathbf{u}_i \rangle \end{aligned}$$

However the $\mathbf{u}_i$ are orthonormal basis vectors and so we know that

$$\langle \mathbf{u}_j, \mathbf{u}_i \rangle = 0 \qquad j = 1, 2, \ldots i-1 \qquad\qquad \langle \mathbf{u}_i, \mathbf{u}_i \rangle \neq 0$$

Using these we see that the diagonal entries are nothing more than,

$$\langle \mathbf{c}_i, \mathbf{u}_i \rangle = \langle \mathbf{u}_i, \mathbf{u}_i \rangle \neq 0$$

So, the diagonal entries of R are non-zero and hence R must be invertible.

■

So, now that we've gotten the proof out of the way let's work an example.

Example 1 Find the *QR*-decomposition for the matrix,

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & 7 \\ 0 & -1 & -1 \end{bmatrix}$$

Solution

The columns from *A* are,

$$\mathbf{c}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \qquad \mathbf{c}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \qquad \mathbf{c}_1 = \begin{bmatrix} 3 \\ 7 \\ -1 \end{bmatrix}$$

We performed Gram-Schmidt on these vectors in Example 3 of the previous section. So, the orthonormal vectors that we'll use for *Q* are,

$$\mathbf{u}_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \\ 0 \end{bmatrix} \qquad \mathbf{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{30}} \\ -\frac{5}{\sqrt{30}} \end{bmatrix} \qquad \mathbf{u}_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

and the matrix *Q* is,

$$Q = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & \frac{2}{\sqrt{6}} \\ 0 & -\frac{5}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

The matrix *R* is,

$$R = \begin{bmatrix} \langle \mathbf{c}_1, \mathbf{u}_1 \rangle & \langle \mathbf{c}_2, \mathbf{u}_1 \rangle & \langle \mathbf{c}_3, \mathbf{u}_1 \rangle \\ 0 & \langle \mathbf{c}_2, \mathbf{u}_2 \rangle & \langle \mathbf{c}_3, \mathbf{u}_2 \rangle \\ 0 & 0 & \langle \mathbf{c}_3, \mathbf{u}_3 \rangle \end{bmatrix} = \begin{bmatrix} \sqrt{5} & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ 0 & \frac{6}{\sqrt{30}} & \frac{22}{\sqrt{30}} \\ 0 & 0 & \frac{16}{\sqrt{6}} \end{bmatrix}$$

So, the *QR*-Decomposition for this matrix is,

$$\begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & 7 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & \frac{2}{\sqrt{6}} \\ 0 & -\frac{5}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \sqrt{5} & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ 0 & \frac{6}{\sqrt{30}} & \frac{22}{\sqrt{30}} \\ 0 & 0 & \frac{16}{\sqrt{6}} \end{bmatrix}$$

We'll leave it to you to verify that this multiplication does in fact give *A*.

There is a nice application of the *QR*-Decomposition to the Least Squares Process that we examined in the previous section. To see this however, we will first need to prove a quick theorem.

Theorem 2 If *Q* is an $n \times m$ matrix with $n \geq m$ then the columns of *Q* are an orthonormal set of vectors in $\mathbb{R}^n$ with the standard Euclidean inner product if and only if $Q^T Q = I_m$.

Note that the only way Q can have orthonormal columns in $\mathbb{R}^n$ is to require that $n \geq m$. Because the columns are vectors in $\mathbb{R}^n$ and we know from Theorem 1 in the Orthonormal Basis section that a set of orthogonal vectors will also be linearly independent. However, from Theorem 2 in the Linear Independence section we know that if $m > n$ the column vectors will be linearly dependent.

Also, because we want to make it clear that we're using the standard Euclidean inner product we will go back to the dot product notation, $\mathbf{u}\bullet\mathbf{v}$, instead of the usual inner product notation, $\langle \mathbf{u}, \mathbf{v} \rangle$.

Proof : Now recall that to prove an "if and only if" theorem we need to assume each part and show that this implies the other part. However, there is some work that we can do that we'll need in both parts so let's do that first.

Let $\mathbf{q}_1$, $\mathbf{q}_2$, … , $\mathbf{q}_m$ be the columns of Q. So,

$$Q = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_m \end{bmatrix}$$

For the transpose we take the columns of Q and turn them into the rows of Q^T. Therefore, the rows of Q^T are $\mathbf{q}_1^T$, $\mathbf{q}_2^T$, … , $\mathbf{q}_m^T$ (the transposes are needed to turn the column vectors, $\mathbf{q}_i$, into row vectors, $\mathbf{q}_i^T$) and,

$$Q^T = \begin{bmatrix} \mathbf{q}_1^{\mathrm{T}} \\ \mathbf{q}_2^{\mathrm{T}} \\ \vdots \\ \mathbf{q}_{\mathrm{m}}^{\mathrm{T}} \end{bmatrix}$$

Now, let's take a look at the product $Q^T Q$. Entries in the product will be rows of Q^T times columns of Q and so the product will be,

$$Q^T Q = \begin{bmatrix} \mathbf{q}_1^T\mathbf{q}_1 & \mathbf{q}_1^T\mathbf{q}_2 & \cdots & \mathbf{q}_1^T\mathbf{q}_m \\ \mathbf{q}_2^T\mathbf{q}_1 & \mathbf{q}_2^T\mathbf{q}_2 & \cdots & \mathbf{q}_2^T\mathbf{q}_m \\ \vdots & \vdots & & \vdots \\ \mathbf{q}_m^T\mathbf{q}_1 & \mathbf{q}_m^T\mathbf{q}_2 & \cdots & \mathbf{q}_m^T\mathbf{q}_m \end{bmatrix}$$

Recalling that $\mathbf{u}\bullet\mathbf{v} = \mathbf{v}^T\mathbf{u}$ we see that we can also write the product as,

$$Q^T Q = \begin{bmatrix} \mathbf{q}_1\bullet\mathbf{q}_1 & \mathbf{q}_1\bullet\mathbf{q}_2 & \cdots & \mathbf{q}_1\bullet\mathbf{q}_m \\ \mathbf{q}_2\bullet\mathbf{q}_1 & \mathbf{q}_2\bullet\mathbf{q}_2 & \cdots & \mathbf{q}_2\bullet\mathbf{q}_m \\ \vdots & \vdots & & \vdots \\ \mathbf{q}_m\bullet\mathbf{q}_1 & \mathbf{q}_m\bullet\mathbf{q}_2 & \cdots & \mathbf{q}_m\bullet\mathbf{q}_m \end{bmatrix}$$

Now, let's actually do the proof.

$(\Rightarrow)$ Assume that the columns of Q are orthogonal and show that this means that we must have $Q^T Q = I_m$.

Since we are assuming that the columns of Q are orthonormal we know that,

$$\mathbf{q}_i \bullet \mathbf{q}_j = 0 \quad i \neq j \qquad\qquad \mathbf{q}_i \bullet \mathbf{q}_i = 1 \qquad i, j = 1, 2, \ldots m$$

Therefore the product is,

$$Q^T Q = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_m$$

So we're done with this part.

$(\Leftarrow)$ Here assume that $Q^T Q = I_m$ and we'll need to show that this means that the columns of Q are orthogonal.

So, we're assuming that,

$$Q^T Q = \begin{bmatrix} \mathbf{q}_1 \bullet \mathbf{q}_1 & \mathbf{q}_1 \bullet \mathbf{q}_2 & \cdots & \mathbf{q}_1 \bullet \mathbf{q}_m \\ \mathbf{q}_2 \bullet \mathbf{q}_1 & \mathbf{q}_2 \bullet \mathbf{q}_2 & \cdots & \mathbf{q}_2 \bullet \mathbf{q}_m \\ \vdots & \vdots & & \vdots \\ \mathbf{q}_m \bullet \mathbf{q}_1 & \mathbf{q}_m \bullet \mathbf{q}_2 & \cdots & \mathbf{q}_m \bullet \mathbf{q}_m \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_m$$

However, simply by setting entries in these two matrices equal we see that,

$$\mathbf{q}_i \bullet \mathbf{q}_j = 0 \quad i \neq j \qquad\qquad \mathbf{q}_i \bullet \mathbf{q}_i = 1 \qquad i, j = 1, 2, \ldots m$$

and this is exactly what it means for the columns to be orthogonal so we're done.

■

The following theorem can be used, on occasion, to significantly reduce the amount of work required for the least squares problem.

Theorem 3 Suppose that A has linearly independent columns. Then the normal system associated with $A\mathbf{x} = \mathbf{b}$ can be written as,

$$R\mathbf{x} = Q^T \mathbf{b}$$

Proof : There really isn't much to do here other than plug formulas in. We'll start with the normal system for $A\mathbf{x} = \mathbf{b}$.

$$A^T A\mathbf{x} = A^T \mathbf{b}$$

Now, A has linearly independent columns we know that it has a QR-Decomposition for A so let's plug the decomposition into the normal system and using properties of transposes we'll rewrite things a little.

$$\left(QR\right)^T QR\mathbf{x} = \left(QR\right)^T \mathbf{b}$$
$$R^T Q^T QR\mathbf{x} = R^T Q^T \mathbf{b}$$

Now, since the columns of Q are orthonormal we know that $Q^T Q = I_m$ by Theorem 2 above. Also, we know that R is an invertible matrix and so know that R^T is also an invertible matrix. So, we'll also multiply both sides by $\left(R^T\right)^{-1}$. Upon doing all this we arrive at,

$$R\mathbf{x} = Q^T\mathbf{b}$$

■

So, just how is this supposed to help us with the Least Squares Problem? We'll since R is upper triangular this will be a very easy system to solve. It can however take some work to get down to this system.

Let's rework the last example from the previous section only this time we'll use the QR-Decomposition method.

Example 2 Find the least squares solution to the following system of equations.

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & -5 & 2 \\ -3 & 1 & -4 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 5 \\ -1 \end{bmatrix}$$

Solution

First, we'll leave it to you to verify that the columns of A are linearly independent. Here are the columns of A.

$$\mathbf{c}_1 = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 1 \end{bmatrix} \qquad \mathbf{c}_2 = \begin{bmatrix} -1 \\ -5 \\ 1 \\ -1 \end{bmatrix} \qquad \mathbf{c}_3 = \begin{bmatrix} 1 \\ 2 \\ -4 \\ 1 \end{bmatrix}$$

Now, we'll need to perform Gram-Schmidt on these to get them into a set of orthonormal vectors. The first step is,

$$\mathbf{u}_1 = \mathbf{c}_1 = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 1 \end{bmatrix}$$

Here's the inner product and norm that we'll need for the second step.

$$\left\langle \mathbf{c}_2, \mathbf{u}_1 \right\rangle = -11 \qquad \left\| \mathbf{u}_1 \right\|^2 = 15$$

The second vector is then,

$$\mathbf{u}_2 = \mathbf{c}_2 - \frac{\left\langle \mathbf{c}_2, \mathbf{u}_1 \right\rangle}{\left\| \mathbf{u}_1 \right\|^2}\mathbf{u}_1 = \begin{bmatrix} -1 \\ -5 \\ 1 \\ -1 \end{bmatrix} - \frac{-11}{15}\begin{bmatrix} 2 \\ 1 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{15} \\ -\frac{64}{15} \\ -\frac{6}{5} \\ -\frac{4}{15} \end{bmatrix}$$

The final step will require the following inner products and norms.

$$\left\langle \mathbf{c}_3, \mathbf{u}_1 \right\rangle = 17 \qquad \left\langle \mathbf{c}_3, \mathbf{u}_2 \right\rangle = -\frac{53}{15} \qquad \left\| \mathbf{u}_1 \right\|^2 = 15 \qquad \left\| \mathbf{u}_2 \right\|^2 = \frac{299}{15}$$

The third, and final orthogonal vector is then,

$$\mathbf{u}_3 = \mathbf{c}_3 - \frac{\left\langle \mathbf{c}_3, \mathbf{u}_1 \right\rangle}{\left\| \mathbf{u}_1 \right\|^2}\mathbf{u}_1 - \frac{\left\langle \mathbf{c}_3, \mathbf{u}_2 \right\rangle}{\left\| \mathbf{u}_2 \right\|^2}\mathbf{u}_2$$

$$= \begin{bmatrix} 1 \\ 2 \\ -4 \\ 1 \end{bmatrix} - \frac{17}{15}\begin{bmatrix} 2 \\ 1 \\ -3 \\ 1 \end{bmatrix} - \frac{-\frac{53}{15}}{\frac{299}{15}}\begin{bmatrix} \frac{7}{15} \\ -\frac{64}{15} \\ -\frac{6}{5} \\ -\frac{4}{15} \end{bmatrix} = \begin{bmatrix} -\frac{354}{299} \\ \frac{33}{299} \\ -\frac{243}{299} \\ -\frac{54}{299} \end{bmatrix}$$

Okay, these are the orthogonal vectors. If we divide each of them by their norms we will get the orthonormal vectors that we need for the decomposition. The norms are,

$$\left\| \mathbf{u}_1 \right\| = \sqrt{15} \qquad \left\| \mathbf{u}_2 \right\| = \frac{\sqrt{4485}}{15} \qquad \left\| \mathbf{u}_3 \right\| = \frac{3\sqrt{20930}}{299}$$

The orthonormal vectors are then,

$$\mathbf{w}_1 = \begin{bmatrix} \frac{2}{\sqrt{15}} \\ \frac{1}{\sqrt{15}} \\ -\frac{3}{\sqrt{15}} \\ \frac{1}{\sqrt{15}} \end{bmatrix} \qquad \mathbf{w}_2 = \begin{bmatrix} \frac{7}{\sqrt{4485}} \\ -\frac{64}{\sqrt{4485}} \\ -\frac{18}{\sqrt{4485}} \\ -\frac{4}{\sqrt{4485}} \end{bmatrix} \qquad \mathbf{w}_3 = \begin{bmatrix} -\frac{118}{\sqrt{20930}} \\ \frac{11}{\sqrt{20930}} \\ -\frac{81}{\sqrt{20930}} \\ -\frac{18}{\sqrt{20930}} \end{bmatrix}$$

We can now write down Q for the decomposition.

$$Q = \begin{bmatrix} \frac{2}{\sqrt{15}} & \frac{7}{\sqrt{4485}} & -\frac{118}{\sqrt{20930}} \\ \frac{1}{\sqrt{15}} & -\frac{64}{\sqrt{4485}} & \frac{11}{\sqrt{20930}} \\ -\frac{3}{\sqrt{15}} & -\frac{18}{\sqrt{4485}} & -\frac{81}{\sqrt{20930}} \\ \frac{1}{\sqrt{15}} & -\frac{4}{\sqrt{4485}} & -\frac{18}{\sqrt{20930}} \end{bmatrix}$$

Finally, R is given by,

$$R = \begin{bmatrix} \langle \mathbf{c}_1, \mathbf{u}_1 \rangle & \langle \mathbf{c}_2, \mathbf{u}_1 \rangle & \langle \mathbf{c}_3, \mathbf{u}_1 \rangle \\ 0 & \langle \mathbf{c}_2, \mathbf{u}_2 \rangle & \langle \mathbf{c}_3, \mathbf{u}_2 \rangle \\ 0 & 0 & \langle \mathbf{c}_3, \mathbf{u}_3 \rangle \end{bmatrix} = \begin{bmatrix} \sqrt{15} & -\frac{11}{\sqrt{15}} & \frac{17}{\sqrt{15}} \\ 0 & \frac{299}{\sqrt{4485}} & -\frac{53}{\sqrt{4485}} \\ 0 & 0 & \frac{210}{\sqrt{20930}} \end{bmatrix}$$

Okay, we can now proceed with the Least Squares process. First we'll need Q^T.

$$Q^T = \begin{bmatrix} \frac{2}{\sqrt{15}} & \frac{1}{\sqrt{15}} & -\frac{3}{\sqrt{15}} & \frac{1}{\sqrt{15}} \\ \frac{7}{\sqrt{4485}} & -\frac{64}{\sqrt{4485}} & -\frac{18}{\sqrt{4485}} & -\frac{4}{\sqrt{4485}} \\ -\frac{118}{\sqrt{20930}} & \frac{11}{\sqrt{20930}} & -\frac{81}{\sqrt{20930}} & -\frac{18}{\sqrt{20930}} \end{bmatrix}$$

The normal system can then be written as,

$$\begin{bmatrix} \sqrt{15} & -\frac{11}{\sqrt{15}} & \frac{17}{\sqrt{15}} \\ 0 & \frac{299}{\sqrt{4485}} & -\frac{53}{\sqrt{4485}} \\ 0 & 0 & \frac{210}{\sqrt{20930}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{15}} & \frac{1}{\sqrt{15}} & -\frac{3}{\sqrt{15}} & \frac{1}{\sqrt{15}} \\ \frac{7}{\sqrt{4485}} & -\frac{64}{\sqrt{4485}} & -\frac{18}{\sqrt{4485}} & -\frac{4}{\sqrt{4485}} \\ -\frac{118}{\sqrt{20930}} & \frac{11}{\sqrt{20930}} & -\frac{81}{\sqrt{20930}} & -\frac{18}{\sqrt{20930}} \end{bmatrix} \begin{bmatrix} -4 \\ 2 \\ 5 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{15} & -\frac{11}{\sqrt{15}} & \frac{17}{\sqrt{15}} \\ 0 & \frac{299}{\sqrt{4485}} & -\frac{53}{\sqrt{4485}} \\ 0 & 0 & \frac{210}{\sqrt{20930}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{22}{\sqrt{15}} \\ -\frac{242}{\sqrt{4485}} \\ \frac{107}{\sqrt{20930}} \end{bmatrix}$$

This corresponds to the following system of equations.

$$\begin{aligned} \sqrt{15}\,x_1 - \frac{11}{\sqrt{15}}x_2 + \frac{17}{\sqrt{15}}x_3 &= -\frac{22}{\sqrt{15}} & \Rightarrow \quad & x_1 = -\frac{18}{7} \\ \frac{299}{\sqrt{4485}}x_2 - \frac{53}{\sqrt{4485}}x_3 &= -\frac{242}{\sqrt{4485}} & \Rightarrow \quad & x_2 = -\frac{151}{210} \\ \frac{210}{\sqrt{20930}}x_3 &= \frac{107}{\sqrt{20930}} & \Rightarrow \quad & x_3 = \frac{107}{210} \end{aligned}$$

These are the same values that we received in the previous section.

At this point you are probably asking yourself just why this method is better than the method we used in the previous section. After all, it was a lot of work and some of the numbers were down right awful. The answer is that by hand, this may not be the best way for doing these problems, however, if you are going to program the least squares method into a computer all of the steps here are very easy to program and so this method is a very nice method for programming the Least Squares process.

Orthogonal Matrices

In this section we're going to be talking about a special kind of matrix called an orthogonal matrix. This is also going to be a fairly short section (at least in relation to many of the other sections in this chapter anyway) to close out the chapter. We'll start with the following definition.

Definition 1 Let Q be a square matrix and suppose that

$$Q^{-1} = Q^T$$

then we call Q an **orthogonal** matrix.

Notice that because we need to have an inverse for Q in order for it to be orthogonal we are implicitly assuming that Q is a square matrix here.

Before we see any examples of some orthogonal matrices (and we have already seen at least one orthogonal matrix) let's get a couple of theorems out of the way.

Theorem 1 Suppose that Q is a square matrix then Q is orthogonal if and only if $QQ^T = Q^TQ = I$.

Proof : This is a really simple proof that falls directly from the definition of what it means for a matrix to be orthogonal.

$(\Rightarrow)$ In this direction we'll assume that Q is orthogonal and so we know that $Q^{-1} = Q^T$, but this promptly tells us that,

$$QQ^T = Q^TQ = I$$

$(\Leftarrow)$ In this direction we'll assume that $QQ^T = Q^TQ = I$, since this is exactly what is needed to show that we have an inverse we can see that $Q^{-1} = Q^T$ and so Q is orthogonal.

■

The next theorem gives us an easier check for a matrix being orthogonal.

Theorem 2 Suppose that Q is an $n \times n$ matrix, then the following are all equivalent.

(a) Q is orthogonal.

(b) The columns of Q are an orthonormal set of vectors in $\mathbb{R}^n$ under the standard Euclidean inner product.

(c) The rows of Q are an orthonormal set of vectors in $\mathbb{R}^n$ under the standard Euclidean inner product.

Proof : We've actually done most of this proof already. Normally in this kind of theorem we'd prove a loop of equivalences such as $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$. However, in this case if we prove $(a) \Leftrightarrow (b)$ and $(a) \Leftrightarrow (c)$ we can get the above loop of equivalences by default and it will be much easier to prove the two equivalences as we'll see.

The equivalence $(a) \Leftrightarrow (b)$ is directly given by Theorem 2 from the previous section since that theorem is in fact a more general version of this equivalence.

The proof of the equivalence $(a) \Leftrightarrow (c)$ is nearly identical to the proof of Theorem 2 from the previous section and so we'll leave it to you to fill in the details.

■

Since it is much easier to verify that the columns/rows of a matrix or orthonormal than it is to check $Q^{-1} = Q^T$ in general this theorem will be useful for identifying orthogonal matrices.

As noted above, in order for a matrix to be an orthogonal matrix it must be square. So a matrix that is not square, but does have orthonormal columns will not be orthogonal. Also, note that we did mean to say that the columns are orthonormal. This may seem odd given that we call the matrix "orthogonal" when "orthonormal" would probably be a better name for the matrix, but traditionally this kind of matrix has been called orthogonal and so we'll keep up with tradition.

In the previous section we were finding QR-Decompositions and if you recall the matrix Q had columns that were a set of orthonormal vectors and so if Q is a square matrix then it will also be an orthogonal matrix, while if it isn't square then it won't be an orthogonal matrix.

At this point we should probably do an example or two.

Example 1 Here are the QR-Decompositions that we performed in the previous section.

From Example 1

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & 7 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & \frac{2}{\sqrt{6}} \\ 0 & -\frac{5}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \sqrt{5} & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ 0 & \frac{6}{\sqrt{30}} & \frac{22}{\sqrt{30}} \\ 0 & 0 & \frac{16}{\sqrt{6}} \end{bmatrix} = QR$$

From Example 2

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & -5 & 2 \\ -3 & 1 & -4 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{15}} & \frac{7}{\sqrt{4485}} & -\frac{118}{\sqrt{20930}} \\ \frac{1}{\sqrt{15}} & -\frac{64}{\sqrt{4485}} & \frac{11}{\sqrt{20930}} \\ -\frac{3}{\sqrt{15}} & -\frac{18}{\sqrt{4485}} & -\frac{81}{\sqrt{20930}} \\ \frac{1}{\sqrt{15}} & -\frac{4}{\sqrt{4485}} & -\frac{18}{\sqrt{20930}} \end{bmatrix} \begin{bmatrix} \sqrt{15} & -\frac{11}{\sqrt{15}} & \frac{17}{\sqrt{15}} \\ 0 & \frac{299}{\sqrt{4485}} & -\frac{53}{\sqrt{4485}} \\ 0 & 0 & \frac{210}{\sqrt{20930}} \end{bmatrix} = QR$$

In the first case the matrix Q is,

$$Q = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & \frac{2}{\sqrt{6}} \\ 0 & -\frac{5}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

and by construction this matrix has orthonormal columns and since it is a square matrix it is an orthogonal matrix.

In the second case the matrix Q is,

$$Q = \begin{bmatrix} \frac{2}{\sqrt{15}} & \frac{7}{\sqrt{4485}} & -\frac{118}{\sqrt{20930}} \\ \frac{1}{\sqrt{15}} & -\frac{64}{\sqrt{4485}} & \frac{11}{\sqrt{20930}} \\ -\frac{3}{\sqrt{15}} & -\frac{18}{\sqrt{4485}} & -\frac{81}{\sqrt{20930}} \\ \frac{1}{\sqrt{15}} & -\frac{4}{\sqrt{4485}} & -\frac{18}{\sqrt{20930}} \end{bmatrix}$$

Again, by construction this matrix has orthonormal columns. However, since it is not a square matrix it is NOT an orthogonal matrix.

Example 2 Find value(s) for a, b, and c for which the following matrix will be orthogonal.

$$Q = \begin{bmatrix} 0 & -\frac{2}{3} & a \\ \frac{1}{\sqrt{5}} & \frac{2}{3} & b \\ -\frac{2}{\sqrt{5}} & \frac{1}{3} & c \end{bmatrix}$$

Solution

So, the columns of Q are,

$$\mathbf{q}_1 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix} \qquad \mathbf{q}_2 = \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \qquad \mathbf{q}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

We will leave it to you to verify that $\|\mathbf{q}_1\| = 1$, $\|\mathbf{q}_2\| = 1$ and $\mathbf{q}_1 \bullet \mathbf{q}_2 = 0$ and so all we need to do if find a, b, and c for which we will have $\|\mathbf{q}_3\| = 1$, $\mathbf{q}_1 \bullet \mathbf{q}_3 = 0$ and $\mathbf{q}_2 \bullet \mathbf{q}_3 = 0$.

Let's start with the two dot products and see what we get.

$$\mathbf{q}_1 \bullet \mathbf{q}_3 = \frac{b}{\sqrt{5}} - \frac{2c}{\sqrt{5}} = 0$$

$$\mathbf{q}_2 \bullet \mathbf{q}_3 = -\frac{2}{3}a + \frac{2}{3}b + \frac{1}{3}c = 0$$

From the first dot product we can see that, $b = 2c$. Plugging this into the second dot product gives us, $c = \frac{2}{5}a$. Using the fact that we now know what c is in terms of a and plugging this into $b = 2c$ we can see that $b = \frac{4}{5}a$.

Now, using the above work we now know that in order for the third column to be orthogonal (since we haven't even touched orthonormal yet) it must be in the form,

$$\mathbf{w} = \begin{bmatrix} a \\ \frac{4}{5}a \\ \frac{2}{5}a \end{bmatrix}$$

Finally, we need to make sure that then third column has norm of 1. In other words we need to require that $\|\mathbf{w}\| = 1$, or we can require that $\|\mathbf{w}\|^2 = 1$ since we know that the norm must be a positive quantity here. So, let's compute $\|\mathbf{w}\|^2$, set it equal to one and see what we get.

$$1 = \|\mathbf{w}\|^2 = a^2 + \frac{16}{25}a^2 + \frac{4}{25}a^2 = \frac{21}{25}a^2 \qquad \Rightarrow \qquad a = \pm\frac{5}{\sqrt{21}}$$

This gives us two possible values of a that we can use and this in turn means that we could used either of the following two vectors for $\mathbf{q}_3$

$$\mathbf{q}_3 = \begin{bmatrix} \frac{5}{\sqrt{21}} \\ \frac{4}{\sqrt{21}} \\ \frac{2}{\sqrt{21}} \end{bmatrix} \qquad \text{OR} \qquad \mathbf{q}_3 = \begin{bmatrix} -\frac{5}{\sqrt{21}} \\ -\frac{4}{\sqrt{21}} \\ -\frac{2}{\sqrt{21}} \end{bmatrix}$$

A natural question is why do we care about orthogonal matrices? The following theorem gives some very nice properties of orthogonal matrices.

Theorem 3 If Q is an $n \times n$ matrix then the following are all equivalent.

- **(a)** Q is orthogonal.
- **(b)** $\|Q\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$ in $\mathbb{R}^n$. This is often called preserving norms.
- **(c)** $Q\mathbf{x} \cdot Q\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}$ and all $\mathbf{y}$ in $\mathbb{R}^n$. This is often called preserving dot products.

Proof : We'll prove this set of statements in the order : $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$

$(a) \Rightarrow (b)$: We'll start off by assuming that Q is orthogonal and let's write down the norm.

$$\|Q\mathbf{x}\| = (Q\mathbf{x} \cdot Q\mathbf{x})^{\frac{1}{2}}$$

However, we know that we can write the dot product as,

$$\|Q\mathbf{x}\| = (\mathbf{x} \cdot Q^T Q\mathbf{x})^{\frac{1}{2}}$$

Now we can use the fact that Q is orthogonal to write $Q^T Q = I$. Using this gives,

$$\|Q\mathbf{x}\| = (\mathbf{x}\bullet\mathbf{x})^{\frac{1}{2}} = \|\mathbf{x}\|$$

which is what we were after.

$(b) \Rightarrow (c)$: We'll assume that $\|Q\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Let's assume that $\mathbf{x}$ and $\mathbf{y}$ are any two vectors in $\mathbb{R}^n$. Then using Theorem 8 from the section on Euclidean n-space we have,

$$Q\mathbf{x}\bullet Q\mathbf{y} = \frac{1}{4}\|Q\mathbf{x}+Q\mathbf{y}\|^2 - \frac{1}{4}\|Q\mathbf{x}-Q\mathbf{y}\|^2 == \frac{1}{4}\|Q(\mathbf{x}+\mathbf{y})\|^2 - \frac{1}{4}\|Q(\mathbf{x}-\mathbf{y})\|^2$$

Next both $\mathbf{x}+\mathbf{y}$ and $\mathbf{x}-\mathbf{y}$ are in $\mathbb{R}^n$ and so by assumption and a use of Theorem 8 again we have,

$$Q\mathbf{x}\bullet Q\mathbf{y} = \frac{1}{4}\|\mathbf{x}+\mathbf{y}\|^2 - \frac{1}{4}\|\mathbf{x}-\mathbf{y}\|^2 = \mathbf{x}\bullet\mathbf{y}$$

which is what we were after in this case.

$(c) \Rightarrow (a)$: In this case we'll assume that $Q\mathbf{x}\bullet Q\mathbf{y} = \mathbf{x}\bullet\mathbf{y}$ for all $\mathbf{x}$ and all $\mathbf{y}$ in $\mathbb{R}^n$. As we did in the first part of this proof we'll rewrite the dot product on the left.

$$\mathbf{x}\bullet Q^TQ\mathbf{y} = \mathbf{x}\bullet\mathbf{y}$$

Now, rearrange things a little and we can arrive at the following,

$$\mathbf{x}\bullet Q^TQ\mathbf{y} - \mathbf{x}\bullet\mathbf{y} = 0$$
$$\mathbf{x}\bullet(Q^TQ\mathbf{y} - \mathbf{y}) = 0$$
$$\mathbf{x}\bullet(Q^TQ - I)\mathbf{y} = 0$$

Now, this must hold for a $\mathbf{x}$ in $\mathbb{R}^n$ and so let $\mathbf{x} = (Q^TQ - I)\mathbf{y}$. This then gives,

$$(Q^TQ - I)\mathbf{y}\bullet(Q^TQ - I)\mathbf{y} = 0$$

Theorem 2(e) from the Euclidean n-space section tells us that we must then have,

$$(Q^TQ - I)\mathbf{y} = \mathbf{0}$$

and this must be true for all $\mathbf{y}$ in $\mathbb{R}^n$. That can only happen if the coefficient matrix of this system is the zero matrix or,

$$Q^TQ - I = \mathbf{0} \qquad \Rightarrow \qquad Q^TQ = I$$

Finally, by Theorem 1 above Q must be orthogonal.

■

The second and third statement for this theorem are very useful since they tell us that we can add or take out an orthogonal matrix from a norm or a dot product at will and we'll preserve the result.

As a final theorem for this section here are a couple of other nice properties of orthogonal matrices.

Theorem 4 Suppose that A and B are two orthogonal $n \times n$ matrices then,

(a) A^{-1} is an orthogonal matrix.
(b) AB is an orthogonal matrix.
(c) Either $\det(A) = 1$ or $\det(A) = -1$

Proof : The proof of all three parts follow pretty much from basic properties of orthogonal matrices.
(a) Since A is orthogonal then its column vectors form an orthogonal (in fact they are orthonormal, but we only need orthogonal for this) set of vectors. Now, by the definition of orthogonal matrices, we have $A^{-1} = A^T$. But this means that the rows of A^{-1} are nothing more than the columns of A and so are an orthogonal set of vectors and so by Theorem 2 above A^{-1} is an orthogonal matrix.

(b) In this case let's start with the following norm.

$$\|AB\mathbf{x}\| = \|A(B\mathbf{x})\|$$

where $\mathbf{x}$ is any vector from $\mathbb{R}^n$. But A is orthogonal and so by Theorem 3 above must preserve norms. In other words we must have,

$$\|AB\mathbf{x}\| = \|A(B\mathbf{x})\| = \|B\mathbf{x}\|$$

Now we can use the fact that B is also orthogonal and so will preserve norms a well. This gives,

$$\|AB\mathbf{x}\| = \|B\mathbf{x}\| = \|\mathbf{x}\|$$

Therefore, the product AB also preserves norms and hence by Theorem 3 must be orthogonal.

(c) In this case we'll start with fact that since A is orthogonal we know that $AA^T = I$ and let's take the determinant of both sides,

$$\det(AA^T) = \det(I) = 1$$

Next use Theorem 3 and Theorem 6 from the Properties of Determinants section to rewrite this as,

$$\begin{aligned} \det(A)\det(A^T) &= 1 \\ \det(A)\det(A) &= 1 \\ [\det(A)]^2 &= 1 \qquad \Rightarrow \qquad \det(A) = \pm 1 \end{aligned}$$

So, we get the result.

■

Eigenvalues and Eigenvectors

Introduction

This is going to be a very short chapter. The main topic of this chapter will be the Eigenvalues and Eigenvectors section. In this section we will be looking at special situations where given a square matrix A and a vector $\mathbf{x}$ the product $A\mathbf{x}$ will be the same as the scalar multiplication $\lambda\mathbf{x}$ for some scalar, λ. This idea has important applications in many areas of math and science and so we put it into a chapter of its own.

We'll also have a quick review of determinants since those will be required in order to due the work in the Eigenvalues and Eigenvectors section. We'll also take a look at an application that uses eigenvalues.

Here is a listing of the topics in this chapter.

Review of Determinants – In this section we'll do a quick review of determinants.

Eigenvalues and Eigenvectors – Here we will take a look at the main section in this chapter. We'll be looking at the concept of Eigenvalues and Eigenvectors.

Diagonalization – We'll be looking at diagonalizable matrices in this section.

Review of Determinants

In this section we are going to do a quick review of determinants and we'll be concentrating almost exclusively on how to compute them. For a more in depth look at determinants you should check out the second chapter which is devoted to determinants and their properties. Also, we'll acknowledge that the examples in this section are all examples that were worked in the second chapter.

We'll start off with a quick "working" definition of a determinant. See The Determinant Function from the second chapter for the exact definition of a determinant. What we're going to give here will be sufficient for what we're going to be doing in this chapter.

So, given a square matrix, A, the **determinant** of A, denoted by $\det(A)$, is a function that associated with A a number. That's it. That's what a determinant does. It takes a matrix and associates a number with that matrix. There is also some alternate notation that we should acknowledge because we'll be using it quite a bit. The alternate notation is, $\det(A) = |A|$.

We now need to discuss how to compute determinants. There are many ways of computing determinants, but most of the general methods can lead to some fairly long computations. We will see one general method towards the end of this section, but there are some nice quick formulas that can help with some special cases so we'll start with those. We'll be working mostly with matrices in this chapter that fit into these special cases.

We will start with the formulas for 2×2 and 3×3 matrices.

Definition 1 If $A=\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$ then the determinant of A is,

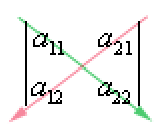
$$\det(A)=\begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} = a_{11}a_{22}-a_{12}a_{21}$$

Definition 2 If $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then the determinant of A is,

$$\begin{aligned}\det(A) &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32}-a_{13}a_{22}a_{31}\end{aligned}$$

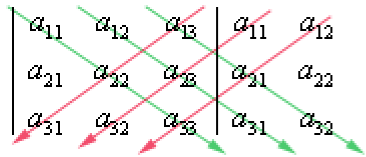
Okay, we said that these were "nice" and "quick" formulas and the formula for the 2×2 matrix is fairly nice and quick, but the formula for the 3×3 matrix is neither nice nor quick. Luckily there are some nice little "tricks" that can help us to write down both formulas.

We'll start with the following determinant of a 2×2 matrix and we'll sketch in two diagonals as shown



Note that if you multiply along the green diagonal you will get the first product in formula for 2×2 matrices and if you multiply along the red diagonal you will get the second product in the formula. Also, notice that the red diagonal, running from right to left, was the product that was subtracted off, while the green diagonal, running from left to right, gave the product that was added.

We can do something similar for 3×3 matrices, but there is a difference. First, we need to tack a copy of the leftmost two columns onto the right side of the determinant. We then have three diagonals that run from left to right (shown in green below) and three diagonals that run from right to left (shown in red below).



As will the 2×2 case, if we multiply along the green diagonals we get the products in the formula that are added in the formula and if we multiply long the red diagonals we get the products in the formula that are subtracted in the formula.

Here are a couple of quick examples.

Example 8 Compute the determinant of each of the following matrices.

(a) $A=\begin{bmatrix} 3 & 2 \\ -9 & 5 \end{bmatrix}$

(b) $B=\begin{bmatrix} 3 & 5 & 4 \\ -2 & -1 & 8 \\ -11 & 1 & 7 \end{bmatrix}$

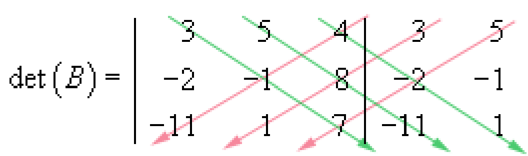
(c) $C=\begin{bmatrix} 2 & -6 & 2 \\ 2 & -8 & 3 \\ -3 & 1 & 1 \end{bmatrix}$

Solution

(a) We don't really need to sketch in the diagonals for 2×2 matrices. The determinant is simply the product of the diagonal running left to right minus the product of the diagonal running from right to left. So, here is the determinant for this matrix. The only thing we need to worry about is paying attention to minus signs. It is easy to make a mistake with minus signs in these computations if you aren't paying attention.

$$\det(A)=(3)(5)-(2)(-9)=33$$

(b) Okay, with this one we'll copy the two columns over and sketch in the diagonals to make sure we've got the idea of these down.



Now, just remember to add products along the left to right diagonals and subtract products along the right to left diagonals.

$$\begin{aligned}\det(B)&=(3)(-1)(7)+(5)(8)(-11)+(4)(-2)(1)-(5)(-2)(7)-\\&\qquad\qquad\qquad(3)(8)(1)-(4)(-1)(-11)\\&=-467\end{aligned}$$

(c) We'll do this one with a little less detail. We'll copy the columns but not bother to actually sketch in the diagonals this time.

$$\det(C) = \begin{vmatrix} 2 & -6 & 2 \\ 2 & -8 & 3 \\ -3 & 1 & 1 \end{vmatrix} \begin{matrix} 2 & -6 \\ 2 & -8 \\ -3 & 1 \end{matrix}$$

$$= (2)(-8)(1) + (-6)(3)(-3) + (2)(2)(1) - (-6)(2)(1) - (2)(3)(1) - (2)(-8)(-3)$$

$$= 0$$

As we can see from this example the determinant for a matrix can be positive, negative or zero. Likewise, as we will see towards the end of this review we are going to be especially interested in when the determinant of a matrix is zero. Because of this we have the following definition.

Definition 3 Suppose *A* is a square matrix.

(a) If $\det(A) = 0$ we call *A* a **singular** matrix.

(b) If $\det(A) \neq 0$ we call *A* a **non-singular** matrix.

So, in Example 1 above, both *A* and *B* are non-singular while *C* is singular.

Before we proceed we should point out that while there are formulas for larger matrices (see the Determinant Function section for details on how to write them down) there are not any easy tricks with diagonals to write them down as we had for 2×2 and 3×3 matrices.

With the statement above made we should note that there is a simple formula for general matrices of certain kinds. The following theorem gives this formula.

Theorem 1 Suppose that *A* is an $n \times n$ triangular matrix with diagonal entries a_{11}, a_{22}, … , a_{nn} the determinant of *A* is,

$$\det(A) = a_{11}a_{22}\cdots a_{nn}$$

This theorem will be valid regardless of whether the triangular matrix is an upper triangular matrix or a lower triangular matrix. Also, because a diagonal matrix can also be considered to be a triangular matrix Theorem 1 is also valid for diagonal matrices.

Here are a couple of quick examples of this.

Example 9 Compute the determinant of each of the following matrices.

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & 0 \\ 2 & -1 \end{bmatrix} \qquad C = \begin{bmatrix} 10 & 5 & 1 & 3 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & 6 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Solution
Here are these determinants.

$$\begin{aligned} \det(A) &= (5)(-3)(4) = -60 \\ \det(B) &= (6)(-1) = -6 \\ \det(C) &= (10)(0)(6)(5) = 0 \end{aligned}$$

There are several methods for finding determinants in general. One of them is the Method of Cofactors. What follows is a very brief overview of this method. For a more detailed discussion of this method see the Method of Cofactors in the Determinants Chapter.

We'll start with a couple of definitions first.

Definition 4 If A is a square matrix then the **minor of** a_{ij}, denoted by M_{ij}, is the determinant of the submatrix that results from removing the i^{th} row and j^{th} column of A.

Definition 5 If A is a square matrix then the **cofactor of** a_{ij}, denoted by C_{ij}, is the number $(-1)^{i+j} M_{ij}$.

Here is a quick example showing some minor and cofactor computations.

Example 10 For the following matrix compute the cofactors C_{12}, C_{24}, and C_{32}.

$$A = \begin{bmatrix} 4 & 0 & 10 & 4 \\ -1 & 2 & 3 & 9 \\ 5 & -5 & -1 & 6 \\ 3 & 7 & 1 & -2 \end{bmatrix}$$

Solution
In order to compute the cofactors we'll first need the minor associated with each cofactor. Remember that in order to compute the minor we will remove the i^{th} row and j^{th} column of A.

So, to compute M_{12} (which we'll need for C_{12}) we'll need to compute the determinate of the matrix we get by removing the 1st row and 2nd column of A. Here is that work.

$$\begin{bmatrix} 4 & 0 & 10 & 4 \\ -1 & 2 & 3 & 9 \\ 5 & -5 & -1 & 6 \\ 3 & 7 & 1 & -2 \end{bmatrix} \quad \Rightarrow \quad M_{12} = \begin{vmatrix} -1 & 3 & 9 \\ 5 & -1 & 6 \\ 3 & 1 & -2 \end{vmatrix} = 160$$

We've marked out the row and column that we eliminated and we'll leave it to you to verify the determinant computation. Now we can get the cofactor.

$$C_{12} = (-1)^{1+2} M_{12} = (-1)^{3}(160) = -160$$

Let's now move onto the second cofactor. Here is the work for the minor.

$$\begin{bmatrix} 4 & 0 & 10 & 4 \\ -1 & 2 & 3 & 9 \\ 5 & -5 & -1 & 6 \\ 3 & 7 & 1 & -2 \end{bmatrix} \quad \Rightarrow \quad M_{24} = \begin{vmatrix} 4 & 0 & 10 \\ 5 & -5 & -1 \\ 3 & 7 & 1 \end{vmatrix} = 508$$

The cofactor in this case is,

$$C_{24} = (-1)^{2+4} M_{24} = (-1)^{6}(508) = 508$$

Here is the work for the final cofactor.

$$\begin{bmatrix} 4 & 0 & 10 & 4 \\ -1 & 2 & 3 & 9 \\ 5 & -5 & -1 & 6 \\ 3 & 7 & 1 & -2 \end{bmatrix} \quad \Rightarrow \quad M_{32} = \begin{vmatrix} 4 & 10 & 4 \\ -1 & 3 & 9 \\ 3 & 1 & -2 \end{vmatrix} = 150$$

$$C_{32} = (-1)^{3+2} M_{32} = (-1)^{5}(150) = -150$$

Notice that the cofactor for a given entry is really just the minor for the same entry with a "+1" or a "-1" in front of it. The following "table" shows whether or not there should be a "+1" or a "-1" in front of a minor for a given cofactor.

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

To use the table for the cofactor $C_{i\,j}$ we simply go to the i^{th} row and j^{th} column in the table above and if there is a "+" there we leave the minor alone and if there is a "-" there we will tack a "-1" onto the appropriate minor. So, for C_{34} we go to the 3rd row and 4th column and see that we have a minus sign and so we know that $C_{34} = -M_{34}$.

Here is how we can use cofactors to compute the determinant of any matrix.

Theorem 2 If A is an $n\times n$ matrix.

(a) Choose any row, say row i, then,

$$\det\left(A\right)=a_{i1}C_{i1}+a_{i2}C_{i2}+\cdots a_{in}C_{in}$$

(b) Choose any column, say column j, then,

$$\det\left(A\right)=a_{1j}C_{1j}+a_{2j}C_{2j}+\cdots+a_{nj}C_{nj}$$

Here is a quick example of how to use this theorem.

Example 11 For the following matrix compute the determinant using the given cofactor expansions.

$$A=\begin{bmatrix} 4 & 2 & 1 \\ -2 & -6 & 3 \\ -7 & 5 & 0 \end{bmatrix}$$

(a) Expand along the first row.
(b) Expand along the third row.
(c) Expand along the second column.

Solution

First, notice that according to the theorem we should get the same result in all three parts.

(a) Here is the cofactor expansion in terms of symbols for this part.

$$\det\left(A\right)=a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}$$

Now, let's plug in for all the quantities. We will just plug in for the entries. For the cofactors we'll write down the minor and a "+1" or a "-1" depending on which sign each minor needs. We'll determine these signs by going to our "sign matrix" above starting at the first entry in the particular row/column we're expanding along and then as we move along that row or column we'll write down the appropriate sign.

Here is the work for this expansion.

$$\begin{aligned}\det\left(A\right)&=\left(4\right)\left(+1\right)\begin{vmatrix} -6 & 3 \\ 5 & 0 \end{vmatrix}+\left(2\right)\left(-1\right)\begin{vmatrix} -2 & 3 \\ -7 & 0 \end{vmatrix}+\left(1\right)\left(+1\right)\begin{vmatrix} -2 & -6 \\ -7 & 5 \end{vmatrix}\\ &=4\left(-15\right)-2\left(21\right)+\left(1\right)\left(-52\right)\\ &=-154\end{aligned}$$

We'll leave it to you to verify the 2×2 determinant computations.

(b) We'll do this one without all the explanations.

$$\begin{aligned}\det(A) &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\ &(-7)(+1)\begin{vmatrix} 2 & 1 \\ -6 & 3 \end{vmatrix} + (5)(-1)\begin{vmatrix} 4 & 1 \\ -2 & 3 \end{vmatrix} + (0)(+1)\begin{vmatrix} 4 & 2 \\ -2 & -6 \end{vmatrix} \\ &= -7(12) - 5(14) + (0)(-20) \\ &= -154\end{aligned}$$

So, the same answer as the first part which is good since that was supposed to happen.

Notice that the signs for the cofactors in this case were the same as the signs in the first case. This is because the first and third row of our "sign matrix" are identical. Also, notice that we didn't really need to compute the third cofactor since the third entry was zero. We did it here just to get one more example of a cofactor into the notes.

(c) Let's take a look at the final expansion. In this one we're going down a column and notice that from our "sign matrix" that this time we'll be starting the cofactor signs off with a "-1" unlike the first two expansions.

$$\begin{aligned}\det(A) &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \\ &(2)(-1)\begin{vmatrix} -2 & 3 \\ -7 & 0 \end{vmatrix} + (-6)(+1)\begin{vmatrix} 4 & 1 \\ -7 & 0 \end{vmatrix} + (5)(-1)\begin{vmatrix} 4 & 1 \\ -2 & 3 \end{vmatrix} \\ &= -2(21) - 6(7) - 5(14) \\ &= -154\end{aligned}$$

Again, the same as the first two as we expected.

As this example has show it doesn't matter which row or column we expand along we will always get the same result.

In this example we performed a cofactor expansion on a 3×3 since we could easily check the results using the process we discussed above. Let's work one more example only this time we'll find the determinant of a 4×4 matrix and so we'll not have any choice but to use a cofactor expansion.

Example 12 Using a cofactor expansion compute the determinant of,

$$A = \begin{bmatrix} 5 & -2 & 2 & 7 \\ 1 & 0 & 0 & 3 \\ -3 & 1 & 5 & 0 \\ 3 & -1 & -9 & 4 \end{bmatrix}$$

Solution

Since the row or column to use for the cofactor expansion was not given in the problem statement we get to choose which one we want to use. From the previous example we know that it won't matter which row or column we choose.. However, having said that notice that if there is a zero entry we won't need to compute the cofactor/minor for that entry since it will just multiply out to zero.

So, it looks like the second row would be a good choice for the expansion since it has two zeroes in it. Here is the expansion for this row. As with the previous expansions we'll explicitly give the "+1" or "-1" for the cofactors and the minors as well so you can see where everything in the expansion is coming from.

$$\det(A) = (1)(-1)\begin{vmatrix} -2 & 2 & 7 \\ 1 & 5 & 0 \\ -1 & 9 & 4 \end{vmatrix} + (0)(+1)M_{22} + (0)(-1)M_{23} + (3)(+1)\begin{vmatrix} 5 & -2 & 2 \\ -3 & 1 & 5 \\ 3 & -1 & -9 \end{vmatrix}$$

We didn't bother to write down the minors M_{22} and M_{23} because of the zero entry. How we choose to compute the determinants for the first and last entry is up to us at this point. We could use a cofactor expansion on each of them or we could use the technique we saw above. Either way will get the same answer and we'll leave it to you to verify these determinants.

The determinant for this matrix is,

$$\det(A) = -(-76) + 3(4) = 88$$

We'll close this review off with a significantly shortened version of Theorem 9 from Properties of Determinants section. We won't need most of the theorem, but there are two bits of it that we'll need so here they are. Also, there are two ways in which the theorem can be stated now that we've stripped out the other pieces and so we'll give both ways of stating it here.

Theorem 3 If A is an $n \times n$ matrix then

(a) The only solution to the system $A\mathbf{x} = 0$ is the trivial solution (*i.e.* $\mathbf{x} = 0$) if and only if $\det(A) \neq 0$.

(b) The system $A\mathbf{x} = 0$ will have a non-trivial solution (*i.e.* $\mathbf{x} \neq 0$) if and only if $\det(A) = 0$.

Note that these two statements really are equivalent. Also, recall that when we say "if and only if" in a theorem statement we mean that the statement works in both directions. For example, let's take a look at the second part of this theorem. This statement says that if $A\mathbf{x} = 0$ has non-trivial solutions then we know that we'll also have $\det(A) = 0$. On the other hand, it also says that if $\det(A) = 0$ then we'll also know that the system will have non-trivial solutions.

This theorem will be key to allowing us to work problems in the next section.

This is then the review of determinants. Again, if you need a more detailed look at either determinants or their properties you should go back and take a look at the Determinant chapter.

Eigenvalues and Eigenvectors

As noted in the introduction to this chapter we're going to start with a square matrix A and try to determine vectors $\mathbf{x}$ and scalars λ so that we will have,

$$A\mathbf{x} = \lambda\mathbf{x}$$

In other words, when this happens, multiplying $\mathbf{x}$ by the matrix A will be equivalent of multiplying $\mathbf{x}$ by the scalar λ. This will not be possible for all vectors, $\mathbf{x}$, nor will it be possible for all scalars λ. The goal of this section is to determine the vectors and scalars for which this will happen.

So, let's start off with the following definition.

Definition 1 Suppose that A is an $n \times n$ matrix. Also suppose that $\mathbf{x}$ is a non-zero vector from $\mathbb{R}^n$ and that λ is any scalar (this can be zero) so that,

$$A\mathbf{x} = \lambda\mathbf{x}$$

We then call $\mathbf{x}$ an **eigenvector** of A and λ an **eigenvalue** of A.

We will often call $\mathbf{x}$ the **eigenvector corresponding to** or **associated with** λ and we will often call λ the **eigenvalue corresponding to** or **associated with x**.

Note that eigenvalues and eigenvectors will always occur in pairs. You can't have an eigenvalue without an eigenvector and you can't have an eigenvector without an eigenvalue.

Example 1 Suppose $A = \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix}$ then $\mathbf{x} = \begin{bmatrix} -8 \\ 1 \end{bmatrix}$ is an eigenvector with corresponding eigenvalue 4 because

$$A\mathbf{x} = \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} -8 \\ 1 \end{bmatrix} = \begin{bmatrix} -32 \\ 4 \end{bmatrix} = 4\begin{bmatrix} -8 \\ 1 \end{bmatrix} = \lambda\mathbf{x}$$

Okay, what we need to do is figure out just how we can determine the eigenvalues and eigenvectors for a given matrix. This is actually easier to do that it might at first appear to be. We'll start with finding the eigenvalues for a matrix and once we have those we'll be able to find the eigenvalues corresponding to each eigenvalue.

Let's start with $A\mathbf{x} = \lambda\mathbf{x}$ and rewrite it as follows,

$$A\mathbf{x} = \lambda I\mathbf{x}$$

Note that all we did was insert the identity matrix into the right side. Doing this will allow us to further rewrite this equation as follows,

$$\lambda I\mathbf{x} - A\mathbf{x} = \mathbf{0}$$
$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

Now, if λ is going to be an eigenvalue of *A*, this system must have a non-zero solution, **x**, since we know that eigenvectors associated with λ cannot be the zero vector. However, Theorem 3 from the previous section or more generally Theorem 8 from the Fundamental Subspaces section tells us that this system will have a non-zero solution if and only if

$$\det(\lambda I - A) = 0$$

So, eigenvalues will be scalars, λ, for which the matrix $\lambda I - A$ will be singular, *i.e.* $\det(\lambda I - A) = 0$. Let's get a couple more definitions out of the way and then we'll work some examples of finding eigenvalues.

Definition 2 Suppose *A* is an $n \times n$ matrix then, $\det(\lambda I - A) = 0$ is called the **characteristic equation** of *A*. When computed it will be an n^{th} degree polynomial in λ of the form,

$$p(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

called the **characteristic polynomial** of *A*.

Note that the coefficient of λ^n is 1 (one) and that is NOT a typo in the definition. This also guarantees that this polynomial will be of degree *n*. Also, from the Fundamental Theorem of Algebra we now know that there will be exactly *n* eigenvalues (possibly including repeats) for an $n \times n$ matrix *A*. Note that because the Fundamental Theorem of Algebra does allow for the possibility of repeated eigenvalues there will be at most *n* distinct eigenvalues for an $n \times n$ matrix. Because an eigenvalue can repeat itself in the list of all eigenvalues we'd like a way to differentiate between eigenvalues that repeat and those that don't repeat. The following definition will do this for us.

Definition 3 Suppose *A* is an $n \times n$ matrix and that $\lambda_1, \lambda_2, \ldots, \lambda_n$ is the complete list of all the eigenvalues of *A* including repeats. If λ occurs exactly once in this list then we call λ a **simple eigenvalue**. If λ occurs $k \geq 2$ times in the list we say that λ has **multiplicity of *k***.

Okay, let's find some eigenvalues we'll start with some 2×2 matrices.

Example 2 Find all the eigenvalues for the given matrices.

(a) $A = \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix}$

(b) $A = \begin{bmatrix} -4 & 2 \\ 3 & -5 \end{bmatrix}$

(c) $A = \begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix}$

Solution

(a) We'll do this one with a little more detail than we'll do the other two. First we'll need the matrix $\lambda I - A$.

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} \lambda - 6 & -16 \\ 1 & \lambda + 4 \end{bmatrix}$$

Next we need the determinant of this matrix, which gives us the characteristic polynomial.

$$\det\left(\lambda I - A\right) = \left(\lambda - 6\right)\left(\lambda + 4\right) - \left(-16\right) = \lambda^2 - 2\lambda - 8$$

Now, set this equal to zero and solve for the eigenvalues.

$$\lambda^2 - 2\lambda - 8 = \left(\lambda - 4\right)\left(\lambda + 2\right) = 0 \qquad \Rightarrow \qquad \lambda_1 = -2,\ \lambda_2 = 4$$

So, we have two eigenvalues and since they occur only once in the list they are both simple eigenvalues.

(b) Here is the matrix $\lambda I - A$ and its characteristic polynomial.

$$\lambda I - A = \begin{bmatrix} \lambda + 4 & -2 \\ -3 & \lambda + 5 \end{bmatrix} \qquad \det\left(\lambda I - A\right) = \lambda^2 + 9\lambda + 14$$

We'll leave it to you to verify both of these. Now, set the characteristic polynomial equal to zero and solve for the eigenvalues.

$$\lambda^2 + 9\lambda + 14 = \left(\lambda + 7\right)\left(\lambda + 2\right) = 0 \qquad \Rightarrow \qquad \lambda_1 = -7,\ \lambda_2 = -2$$

Again, we get two simple eigenvalues.

(c) Here is the matrix $\lambda I - A$ and its characteristic polynomial.

$$\lambda I - A = \begin{bmatrix} \lambda - 7 & 1 \\ -4 & \lambda - 3 \end{bmatrix} \qquad \det\left(\lambda I - A\right) = \lambda^2 - 10\lambda + 25$$

We'll leave it to you to verify both of these. Now, set the characteristic polynomial equal to zero and solve for the eigenvalues.

$$\lambda^2 - 10\lambda + 25 = \left(\lambda - 5\right)^2 = 0 \qquad \Rightarrow \qquad \lambda_{1,2} = 5$$

In this case we have an eigenvalue of multiplicity two. Sometimes we call this kind of eigenvalue a double eigenvalue. Notice as well that we used the notation $\lambda_{1,2}$ to denote the fact that this was a double eigenvalue.

Now, let's take a look at some 3×3 matrices.

Example 3 Find all the eigenvalues for the given matrices.

(a) $A = \begin{bmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$

(c) $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

(d) $A = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Solution

(a) As with the previous example we'll do this one in a little more detail than the remaining two parts. First, we'll need $\lambda_{1,2}$,

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda - 4 & 0 & -1 \\ 1 & \lambda + 6 & 2 \\ -5 & 0 & \lambda \end{bmatrix}$$

Now, let's take the determinant of this matrix and get the characteristic polynomial for *A*. We'll use the "trick" that we reviewed in the previous section to take the determinant. You could also use cofactors if you prefer that method. The result will be the same.

$$\begin{aligned} \det\left(\lambda I - A\right) &= \left| \begin{array}{ccc|cc} \lambda - 4 & 0 & -1 & \lambda - 4 & 0 \\ 1 & \lambda + 6 & 2 & 1 & \lambda + 6 \\ -5 & 0 & \lambda & -5 & 0 \end{array} \right. \\ &= \lambda\left(\lambda - 4\right)\left(\lambda + 6\right) - 5\left(\lambda + 6\right) \\ &= \lambda^3 + 2\lambda^2 - 29\lambda - 30 \end{aligned}$$

Next, set this equal to zero.

$$\lambda^3 + 2\lambda^2 - 29\lambda - 30 = 0$$

Now, most of us aren't that great at find the roots of a cubic polynomial. Luckily there is a way to at least get us started. It won't always work, but if it does it can greatly reduce the amount of work that we need to do.

Suppose we're trying to find the roots of an equation of the form,

$$\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 = 0$$

where the c_i are all integers. If there are integer solutions to this (and there may NOT be) then we know that they must be divisors of c_0. This won't give us any integer solutions, but it will allow us to write down a list of possible integer solutions. The list will be all possible divisors of c_0.

In this case the list of possible integer solutions is all possible divisors of -30.

$$\pm 1,\ \pm 2,\ \pm 3,\ \pm 5,\ \pm 6,\ \pm 10,\ \pm 15,\ \pm 30$$

Now, that may seem like a lot of solutions that we'll need to check. However, it isn't quite that bad. Start with the smaller possible solutions and plug them in until you find one (*i.e.* until the polynomial is zero for one of them) and then stop. In this case the smallest one in the list that works is -1. This means that

$$\lambda - (-1) = \lambda + 1$$

must be a factor in the characteristic polynomial. In other words, we can write the characteristic polynomial as,

$$\lambda^3 + 2\lambda^2 - 29\lambda - 30 = (\lambda + 1)\, q(\lambda)$$

where $q(\lambda)$ is a quadratic polynomial. We find $q(\lambda)$ by performing long division on the characteristic polynomial. Doing this in this case gives,

$$\lambda^3 + 2\lambda^2 - 29\lambda - 30 = (\lambda + 1)\left(\lambda^2 + \lambda - 30\right)$$

At this point all we need to do is find the solutions to the quadratic and nicely enough for us that factors in this case. So, putting all this together gives,

$$(\lambda + 1)(\lambda + 6)(\lambda - 5) = 0 \qquad \Rightarrow \qquad \lambda_1 = -1,\ \lambda_2 = -6,\ \lambda_3 = 5$$

So, this matrix has three simple eigenvalues.

(b) Here is $\lambda I - A$ and the characteristic polynomial for this matrix.

$$\lambda I - A = \begin{bmatrix} \lambda - 6 & -3 & 8 \\ 0 & \lambda + 2 & 0 \\ -1 & 0 & \lambda + 3 \end{bmatrix} \qquad \det(\lambda I - A) = \lambda^3 - \lambda^2 - 16\lambda - 20$$

Now, in this case the list of possible integer solutions to the characteristic polynomial are,

$$\pm 1,\ \pm 2,\ \pm 4,\ \pm 5,\ \pm 10,\ \pm 20$$

Again, if we start with the smallest integers in the list we'll find that -2 is the first integer solution. Therefore, $\lambda - (-2) = \lambda + 2$ must be a factor of the characteristic polynomial. Factoring this out of the characteristic polynomial gives,

$$\lambda^3 - \lambda^2 - 16\lambda - 20 = (\lambda + 2)\left(\lambda^2 - 3\lambda - 10\right)$$

Finally, factoring the quadratic and setting equal to zero gives us,

$$(\lambda + 2)^2 (\lambda - 5) = 0 \qquad \Rightarrow \qquad \lambda_{1,2} = -2,\ \lambda_3 = 5$$

So, we have one double eigenvalue ($\lambda_{1,2} = -2$) and one simple eigenvalue ($\lambda_3 = 5$).

(c) Here is $\lambda I - A$ and the characteristic polynomial for this matrix.

$$\lambda I - A = \begin{bmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{bmatrix} \qquad \det(\lambda I - A) = \lambda^3 - 3\lambda - 2$$

We have a very small list of possible integer solutions for this characteristic polynomial.

$$\pm 1, \pm 2$$

The smallest integer that works in this case is -1 and we'll leave it to you to verify that the complete factored form is characteristic polynomial is,

$$\lambda^3 - 3\lambda - 2 = (\lambda + 1)^2 (\lambda - 2)$$

and so we can see that we've got two eigenvalues $\lambda_{1,2} = -1$ (a multiplicity 2 eigenvalue) and $\lambda_3 = 2$ (a simple eigenvalue).

(d) Here is $\lambda I - A$ and the characteristic polynomial for this matrix.

$$\lambda I - A = \begin{bmatrix} \lambda - 4 & 0 & 1 \\ 0 & \lambda - 3 & 0 \\ -1 & 0 & \lambda - 2 \end{bmatrix} \qquad \det(\lambda I - A) = \lambda^3 - 9\lambda^2 + 27\lambda - 27$$

Okay, in this case the list of possible integer solutions is,

$$\pm 1, \pm 3, \pm 9, \pm 27$$

The smallest integer that will work in this case is 3. We'll leave it to you to verify that the factored form of the characteristic polynomial is,

$$\lambda^3 - 9\lambda^2 + 27\lambda - 27 = (\lambda - 3)^3$$

and so we can see that if we set this equal to zero and solve we will have one eigenvalue of multiplicity 3 (sometimes called a triple eigenvalue),

$$\lambda_{1,2,3} = 3$$

As you can see the work for these get progressively more difficult as we increase the size of the matrix, for a general matrix larger than 3×3 we'd need to use the method of cofactors to determine the characteristic polynomial.

There is one case kind of matrix however that we can pick the eigenvalues right off the matrix it self without doing any work and the size won't matter.

Theorem 1 Suppose A is an $n \times n$ triangular matrix then the eigenvalues will be the diagonal entries, a_{11}, a_{22}, … , a_{nn}.

Proof : We'll give the proof for an upper triangular matrix, and leave it to you to verify the proof for a lower triangular and a diagonal matrix. We'll start with,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Now, we can write down $\lambda I - A$,

$$\lambda I - A = \begin{bmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ 0 & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda - a_{nn} \end{bmatrix}$$

Now, this is still an upper triangular matrix and we know that the determinant of a triangular matrix is just the product of its main diagonal entries. The characteristic polynomial is then,

$$\det\left(\lambda I - A\right) = \left(\lambda - a_{11}\right)\left(\lambda - a_{22}\right)\cdots\left(\lambda - a_{nn}\right)$$

Setting this equal to zero and solving gives the eigenvalues of,

$$\lambda_1 = a_{11} \qquad \lambda_2 = a_{22} \qquad \cdots \qquad \lambda_n = a_{nn}$$

■

Example 4 Find the eigenvalues of the following matrix.

$$A = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 \\ 9 & -4 & 0 & 0 & 0 \\ -2 & 0 & 11 & 0 & 0 \\ 1 & -1 & 3 & 0 & 0 \\ 0 & 1 & -7 & 4 & 8 \end{bmatrix}$$

Solution

Since this is a lower triangular matrix we can use the previous theorem to write down the eigenvalues. It will simply be the main diagonal entries. The eigenvalues are,

$$\lambda_1 = 6 \qquad \lambda_2 = -4 \qquad \lambda_3 = 11 \qquad \lambda_4 = 0 \qquad \lambda_5 = 8$$

We can now find the eigenvalues for a matrix. We next need to address the issue of finding their corresponding eigenvectors. Recall given an eigenvalue, λ, the eigenvector(s) of *A* that correspond to λ will be the vectors $\mathbf{x}$ such that,

$$A\mathbf{x} = \lambda\mathbf{x} \qquad \text{OR} \qquad \left(\lambda I - A\right)\mathbf{x} = \mathbf{0}$$

Also, recall that λ was chosen so that $\lambda I - A$ was a singular matrix. This in turn guaranteed that we would have a non-zero solution to the equation above. Note that in doing this we don't just guarantee a non-zero solution, but we also guarantee that we'll have infinitely many solutions to the system. We have one quick definition that we need to take care of before we start working problems.

Definition 4 The set of all solutions to $(\lambda I - A)\mathbf{x} = \mathbf{0}$ is called the **eigenspace** of A corresponding to λ.

Note that there will be one eigenspace of *A* for each distinct eigenvalue and so there will be anywhere from 1 to *n* eigenspaces for an $n \times n$ matrix depending upon the number of distinct eigenvalues that the matrix has. Also, notice that we're really just finding the null space for a system and we've looked at that in several sections in the previous chapter.

Let's take a look at some eigenspaces for some of the matrices we found eigenvalues for above.

Example 5 For each of the following matrices determine the eigenvectors corresponding to each eigenvalue and determine a basis for the eigenspace of the matrix corresponding to each eigenvalue.

(a) $A = \begin{bmatrix} 6 & 16 \\ -1 & -4 \end{bmatrix}$

(b) $A = \begin{bmatrix} 7 & -1 \\ 4 & 3 \end{bmatrix}$

Solution

We determined the eigenvalues for each of these in Example 2 above so refer to that example for the details in finding them. For each eigenvalue we will need to solve the system,

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

to determine the general form of the eigenvector. Once we have that we can use the general form of the eigenvector to find a basis for the eigenspace.

(a) We know that the eigenvalues for this matrix are $\lambda_1 = -2$ and $\lambda_2 = 4$.

Let's first find the eigenvector(s) and eigenspace for $\lambda_1 = -2$. Referring to Example 2 for the formula for $\lambda I - A$ and plugging $\lambda_1 = -2$ into this we can see that the system we need to solve is,

$$\begin{bmatrix} -8 & -16 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We'll leave it to you to verify that the solution to this system is,

$$x_1 = -2t \qquad\qquad x_2 = t$$

Therefore, the general eigenvector corresponding to $\lambda_1 = -2$ is of the form,

$$\mathbf{x} = \begin{bmatrix} -2t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The eigenspaces is all vectors of this form and so we can see that a basis for the eigenspace corresponding to $\lambda_1 = -2$ is,

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Now, let's find the eigenvector(s) and eigenspace for $\lambda_2 = 4$. Plugging $\lambda_2 = 4$ into the formula for $\lambda I - A$ from Example 2 gives the following system we need to solve,

$$\begin{bmatrix} -2 & -16 \\ 1 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution to this system is (you should verify this),

$$x_1 = -8t \qquad\qquad x_2 = t$$

The general eigenvector and a basis for the eigenspace corresponding to $\lambda_2 = 4$ is then,

$$\mathbf{x} = \begin{bmatrix} -8t \\ t \end{bmatrix} = t\begin{bmatrix} -8 \\ 1 \end{bmatrix} \qquad \& \qquad \mathbf{v}_2 = \begin{bmatrix} -8 \\ 1 \end{bmatrix}$$

Note that if we wanted our hands on specific eigenvalues for each eigenvector the basis vector for each eigenspace would work. So, if we do that we could use the following eigenvectors (and their corresponding eigenvalues) if we'd like.

$$\lambda_1 = -2 \quad \mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad \lambda_2 = 4 \quad \mathbf{v}_2 = \begin{bmatrix} -8 \\ 1 \end{bmatrix}$$

Note as well that these eigenvectors are linearly independent vectors.

(b) From Example 2 we know that $\lambda_{1,2} = 5$ is a double eigenvalue and so there will be a single eigenspace to compute for this matrix. Using the formula for $\lambda I - A$ from Example 2 and plugging $\lambda_{1,2} = 5$ into this gives the following system that we'll need to solve for the eigenvector and eigenspace.

$$\begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution to this system is,

$$x_1 = \frac{1}{2}t \qquad\qquad x_2 = t$$

The general eigenvector and a basis for the eigenspace corresponding $\lambda_{1,2} = 5$ is then,

$$\mathbf{x} = \begin{bmatrix} \frac{1}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \qquad \& \qquad \mathbf{v}_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

In this case we get only a single eigenvector and so a good eigenvalue/eigenvector pair is,

$$\lambda_{1,2} = 5 \qquad\qquad \mathbf{v}_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

We didn't look at the second matrix from Example 2 in the previous example. You should try and determine the eigenspace(s) for that matrix. The work will follow what we did in the first part of the previous example since there are two simple eigenvalues.

Now, let's determine the eigenspaces for the matrices in Example 3

Example 6 Determine the eigenvectors corresponding to each eigenvalue and a basis for the eigenspace corresponding to each eigenvalue for each of the matrices from Example 3 above.

Solution

The work finding the eigenvalues for each of these is shown in Example 3 above. Also, we'll be doing this with less detail than those in the previous example. In each part we'll use the formula for $\lambda I - A$ found in Example 3 and plug in each eigenvalue to find the system that we need to solve for the eigenvector(s) and eigenspace.

(a) The eigenvalues for this matrix are $\lambda_1 = -1$, $\lambda_2 = -6$ and $\lambda_3 = 5$ so we'll have three eigenspaces to find.

Starting with $\lambda_1 = -1$ we'll need to find the solution to the following system,

$$\begin{bmatrix} -5 & 0 & -1 \\ 1 & 5 & 2 \\ -5 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad x_1 = -\frac{1}{5}t \qquad x_2 = -\frac{9}{25}t \qquad x_3 = t$$

The general eigenvector and a basis for the eigenspace corresponding to $\lambda_1 = -1$ is then,

$$\mathbf{x} = \begin{bmatrix} -\frac{1}{5}t \\ -\frac{9}{25}t \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{1}{5} \\ -\frac{9}{25} \\ 1 \end{bmatrix} \qquad \& \qquad \mathbf{v}_1 = \begin{bmatrix} -\frac{1}{5} \\ -\frac{9}{25} \\ 1 \end{bmatrix}$$

Now, let's take a look at $\lambda_2 = -6$. Here is the system we need to solve,

$$\begin{bmatrix} -10 & 0 & -1 \\ 1 & 0 & 2 \\ -5 & 0 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \qquad x_1 = 0 \qquad x_2 = t \qquad x_3 = 0$$

Here is the general eigenvector and a basis for the eigenspace corresponding to $\lambda_2 = -6$.

$$\mathbf{x} = \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \& \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Finally, here is the system for $\lambda_3 = 5$.

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 11 & 2 \\ -5 & 0 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad x_1 = t \qquad x_2 = -\frac{3}{11}t \qquad x_3 = t$$

The general eigenvector and a basis for the eigenspace corresponding to $\lambda_3 = 5$ is then,

$$\mathbf{x} = \begin{bmatrix} t \\ -\frac{3}{11}t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ -\frac{3}{11} \\ 1 \end{bmatrix} \qquad \& \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -\frac{3}{11} \\ 1 \end{bmatrix}$$

Now, we as with the previous example, let's write down a specific set of eigenvalue/eigenvector pairs for this matrix just in case we happen to need them for some reason. We can get specific eigenvectors using the basis vectors for the eigenspace as we did in the previous example.

$$\lambda_1 = -1 \quad \mathbf{v}_1 = \begin{bmatrix} -\frac{1}{5} \\ -\frac{9}{25} \\ 1 \end{bmatrix}, \quad \lambda_2 = -6 \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \lambda_3 = 5 \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -\frac{3}{11} \\ 1 \end{bmatrix}$$

You might want to verify that these three vectors are linearly independent vectors.

(b) The eigenvalues for this matrix are $\lambda_{1,2} = -2$ and $\lambda_3 = 5$ so it looks like we'll have two eigenspaces to find for this matrix.

We'll start with $\lambda_{1,2} = -2$. Here is the system that we need to solve and it's solution.

$$\begin{bmatrix} -8 & -3 & 8 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad x_1 = t \qquad x_2 = 0 \qquad x_3 = t$$

The general eigenvector and a basis for the eigenspace corresponding $\lambda_{1,2} = -2$ is then,

$$\mathbf{x} = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \& \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Note that even though we have a double eigenvalue we get a single basis vector here.

Next, the system for $\lambda_3 = 5$ that we need to solve and it's solution is,

$$\begin{bmatrix} -1 & -3 & 8 \\ 0 & 7 & 0 \\ -1 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad x_1 = 8t \qquad x_2 = 0 \qquad x_3 = t$$

The general eigenvector and a basis for the eigenspace corresponding to $\lambda_3 = 5$ is,

$$\mathbf{x} = \begin{bmatrix} 8t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix} \qquad \& \qquad \mathbf{v}_2 = \begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix}$$

A set of eigenvalue/eigenvector pairs for this matrix is,

$$\lambda_{1,2} = -2 \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \lambda_3 = 5 \quad \mathbf{v}_2 = \begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix}$$

Unlike the previous part we only have two eigenvectors here even though we have three eigenvalues (if you include repeats anyway).

(c) As with the previous part we've got two eigenvalues, $\lambda_{1,2} = -1$ and $\lambda_3 = 2$ and so we'll again have two eigenspaces to find here.

We'll start with $\lambda_{1,2} = -1$. Here is the system we need to solve,

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad x_1 = -s - t \qquad x_2 = s \qquad x_3 = t$$

The general eigenvector corresponding to $\lambda_{1,2} = -1$ is then,

$$\mathbf{x} = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Now, the eigenspace is spanned by the two vectors above and since they are linearly independent we can see that a basis for the eigenspace corresponding to $\lambda_{1,2} = -1$ is,

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Here is the system for $\lambda_3 = 2$ that we need to solve,

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad x_1 = t \qquad x_2 = t \qquad x_3 = t$$

The general eigenvector and a basis for the eigenspace corresponding to $\lambda_3 = 2$ is,

$$\mathbf{x} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \& \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Okay, in this case if we want to write down a set of eigenvalue/eigenvector pairs we've got a slightly different situation that we've seen to this point. In the previous example we had an eigenvalue of multiplicity two but only got a single eigenvector for it. In this case because the eigenspace for our multiplicity two eigenvalue has a dimension of two we can use each basis vector as a separate eigenvector and so the eigenvalue/eigenvector pairs this time are,

$$\lambda_1 = -1 \quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \lambda_2 = -1 \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \lambda_3 = 2 \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Note that we listed the eigenvalue of "-1" twice, once for each eigenvector. You should verify that these are all linearly independent.

(d) In this case we had a single eigenvalue, $\lambda_{1,2,3} = 3$ so we'll have a single eigenspace to find. Here is the system and its solution for this eigenvalue.

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad x_1 = t \qquad x_2 = s \qquad x_3 = t$$

The general eigenvector corresponding to $\lambda_{1,2,3} = 3$ is then,

$$\mathbf{x} = \begin{bmatrix} t \\ s \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

As with the previous example we can see that the eigenspace is spanned by the two vectors above and since they are linearly independent we can see that a basis for the eigenspace corresponding to $\lambda_{1,2,3} = 3$ is,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Note that the two vectors above would also make a nice pair of eigenvectors for the single eigenvalue in this case.

Okay, let's go back and take a look at the eigenvector/eigenvalue pairs for a second. First, there are reasons for wanting these as we'll see in the next section. On occasion we really do want specific eigenvectors and we generally want them to be linearly independent as well as we'll see. Also, we saw two examples of eigenvalues of multiplicity 2. In one case we got a single eigenvector and in the other we got two linearly independent eigenvectors. This will always be the case, if λ is an eigenvalue of multiplicity *k* then there will be anywhere from 1 to *k* linearly independent eigenvectors depending upon the dimension of the eigenspace.

How there is one type of eigenvalue that we've been avoiding to this point and you may have noticed that already. Let's take a look at the following example to see what we've been avoiding to this point.

Example 7 Find all the eigenvalues of

$$A = \begin{bmatrix} 6 & 5 \\ -8 & -6 \end{bmatrix}$$

Solution

First we'll need the matrix $\lambda I - A$ and then we'll use that to find the characteristic equation.

$$\lambda I - A = \begin{bmatrix} \lambda - 6 & -5 \\ 8 & \lambda + 6 \end{bmatrix}$$

$$\det\left(\lambda I - A\right) = \left(\lambda - 6\right)\left(\lambda + 6\right) + 40 = \lambda^2 + 4$$

From this we can see that the eigenvalues will be complex. In fact they will be,

$$\lambda_1 = 2i \qquad \lambda_2 = -2i$$

So we got a complex eigenvalue. We've not seen any of these to this point and this will be the only one we'll look at here. We have avoided complex eigenvalues to this point on very important reason. Let's recall just what an eigenvalue is. An eigenvalue is a scalar such that,

$$\begin{bmatrix} 6 & 5 \\ -8 & -6 \end{bmatrix} \mathbf{x} = \left(2i\right)\mathbf{x}$$

Can you see the problem? In order to talk about the scalar multiplication of the right we need to be in a complex vector space! Up to this point we've been working exclusively with real vector spaces and remember that in a real vector space the scalars are all real numbers. So, in order to work with complex eigenvalues we would need to be working in a complex vector space and we haven't looked at those yet. So, since we haven't looked at complex vector spaces yet we will be working only with matrices that have real eigenvalues.

Note that this doesn't mean that complex eigenvalues are not important. There are some very important applications to complex eigenvalues in various areas of math, engineering and the sciences. Of course, there are also times where we would ignore complex eigenvalues.

We will leave this section out with a couple of nice theorems.

Theorem 2 Suppose that λ is an eigenvalue of the matrix A with corresponding eigenvector $\mathbf{x}$. Then if k is a positive integer λ^k is an eigenvalue of the matrix A^k with corresponding eigenvector $\mathbf{x}$.

Proof : The proof here is pretty simple.

$$\begin{aligned} A^k\mathbf{x} &= A^{k-1}\left(A\mathbf{x}\right) \\ &= A^{k-1}\left(\lambda\mathbf{x}\right) \\ &= \lambda A^{k-2}\left(A\mathbf{x}\right) \\ &= \lambda A^{k-2}\left(\lambda\mathbf{x}\right) \\ &\vdots \\ &= \lambda^{k-1}\left(A\mathbf{x}\right) \\ &= \lambda^{k-1}\left(\lambda\mathbf{x}\right) \\ &= \lambda^k\mathbf{x} \end{aligned}$$

So, from this we can see that λ^k is an eigenvalue of A^k with corresponding eigenvector $\mathbf{x}$.

■

Theorem 3 Suppose A is an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (possibly including repeats). Then,

(a) The determinant of A is $\det\left(A\right) = \lambda_1\lambda_2\cdots\lambda_n$.

(b) The trace of A is $\operatorname{tr}\left(A\right) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

We'll prove part **(a)** here. The proof of part **(b)** involves some fairly messy algebra and so we won't show it here.

Proof of (a) : First, recall that eigenvalues of A are roots of the characteristic polynomial and hence we can write the characteristic polynomial as,

$$\det\left(\lambda I - A\right) = \left(\lambda - \lambda_1\right)\left(\lambda - \lambda_2\right)\cdots\left(\lambda - \lambda_n\right)$$

Now, plug in $\lambda = 0$ into this and we get,

$$\det\left(-A\right) = \left(-\lambda_1\right)\left(-\lambda_2\right)\cdots\left(-\lambda_n\right) = \left(-1\right)^n \lambda_1\lambda_2\cdots\lambda_n$$

Finally, from Theorem 1 of the Properties of Determinant section we know that

$$\det(-A) = (-1)^n \det(A)$$

So, plugging this in gives,

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$$

■

Diagonalization

In this section we're going to take a look at a special kind of matrix. We'll start out with the following definition.

Definition 1 Suppose that *A* is a square matrix and further suppose that there exists an invertible matrix *P* (of the same size as *A* of course) such that $P^{-1}AP$ is a diagonal matrix. In such a case we call *A* **diagonalizable** and say that *P* **diagonalizes** *A*.

The following theorem will not only tell us when a matrix is diagonalizable, but the proof will tell us how to construct *P* when *A* is diagonalizable.

Theorem 1 Suppose that *A* is an $n \times n$ matrix, then the following are equivalent.

(a) *A* is diagonalizable.

(b) *A* has *n* linearly independent eigenvectors.

Proof : We'll start by proving that $(a) \Rightarrow (b)$. So, assume that *A* is diagonalizable and so we know that an invertible matrix *P* exists so that $P^{-1}AP$ is a diagonal matrix. Now, let $\mathbf{p}_1$, $\mathbf{p}_2$, … , $\mathbf{p}_n$ be the columns of *P* and suppose that *D* is the diagonal matrix we get from $P^{-1}AP$, *i.e.* $D = P^{-1}AP$. So, both *P* and *D* have the following forms,

$$P = \left[\, \mathbf{p}_1 \mid \mathbf{p}_2 \mid \cdots \mid \mathbf{p}_n \,\right] \qquad D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Also note that because *P* is an invertible matrix Theorem 8 from the Fundamental Subspaces section tells us that the columns of *P* will form a basis for $\mathbb{R}^n$ and hence must be linearly independent. Therefore, $\mathbf{p}_1$, $\mathbf{p}_2$, … , $\mathbf{p}_n$ are a set of linearly independent columns vectors.

Now, if we rewrite $D = P^{-1}AP$ we arrive at $AP = PD$ or,

$$A\left[\mathbf{p}_1 \mid \mathbf{p}_2 \mid \cdots \mid \mathbf{p}_n\right] = \left[\mathbf{p}_1 \mid \mathbf{p}_2 \mid \cdots \mid \mathbf{p}_n\right]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Theorem 1 from the Matrix Arithmetic section tell us that the j^{th} column of PD is $P[j^{\text{th}}$ column of $D]$ and so the j^{th} column of PD is nothing more than $\lambda_j \mathbf{p}_j$. The same theorem tells us that j^{th} column of AP is $A[j^{\text{th}}$ column of $P]$ or $A\mathbf{p}_j$.

Now, since we have $AP = PD$ the columns of both sides must be equal and so we must have,

$$A\mathbf{p}_1 = \lambda_1 \mathbf{p}_1 \qquad A\mathbf{p}_2 = \lambda_2 \mathbf{p}_2 \qquad \cdots \qquad A\mathbf{p}_n = \lambda_n \mathbf{p}_n$$

So, the diagonal entries from D, λ_1, λ_2, … , λ_n are the eigenvalues of A and their corresponding eigenvectors are the columns of P, $\mathbf{p}_1$, $\mathbf{p}_2$, … , $\mathbf{p}_n$. Also as we noted above these are a set of linearly independent vectors which is what we were asked to prove.

We now need to prove $(b) \Rightarrow (a)$ and we've done most of the work for this in the previous part. Let's start by assuming that the eigenvalues of A are λ_1, λ_2, … , λ_n and that their associated eigenvectors are $\mathbf{p}_1$, $\mathbf{p}_2$, … , $\mathbf{p}_n$ are linearly independent.

Now, form a matrix P whose columns are $\mathbf{p}_1$, $\mathbf{p}_2$, … , $\mathbf{p}_n$. So, P has the form,

$$P = \left[\mathbf{p}_1 \mid \mathbf{p}_2 \mid \cdots \mid \mathbf{p}_n\right]$$

Now, as we noted above the columns of AP are given by

$$A\mathbf{p}_1 \qquad A\mathbf{p}_2 \quad \cdots \quad A\mathbf{p}_n$$

However, using the fact that $\mathbf{p}_1$, $\mathbf{p}_2$, … , $\mathbf{p}_n$ are the eigenvectors of A each of these columns can be written as,

$$A\mathbf{p}_1 = \lambda_1 \mathbf{p}_1 \qquad A\mathbf{p}_2 = \lambda_2 \mathbf{p}_2 \qquad \cdots \qquad A\mathbf{p}_n = \lambda_n \mathbf{p}_n$$

Therefore, AP can be written as,

$$AP = \left[A\mathbf{p}_1 \mid A\mathbf{p}_2 \mid \cdots \mid A\mathbf{p}_n\right] = \left[\lambda_1 \mathbf{p}_1 \mid \lambda_2 \mathbf{p}_2 \mid \cdots \mid \lambda_n \mathbf{p}_n\right]$$

However, as we saw above, the matrix on the right can be written as PD where D the following diagonal matrix,

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

So, we've managed to show that by defining *P* as above we have $AP = PD$. Finally, since the columns of *P* are *n* linearly independent vectors in $\mathbb{R}^n$ we know that they will form a basis for $\mathbb{R}^n$ and so by Theorem 8 from the Fundamental Subspaces section we know that *P* must be invertible and hence we have,

$$P^{-1}AP = D$$

where *D* is an invertible matrix. Therefore *A* in diagonalizable.

■

Let's take a look at a couple of examples.

Example 1 Find a matrix *P* that will diagonalize each of the following matrices.

(a) $A = \begin{bmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Solution

Okay, provided we can find 3 linearly independent eigenvectors for each of these we'll have a pretty easy time of this since we know that that the columns of *P* will then be these three eigenvectors.

Nicely enough for us, we did exactly this in the Example 6 of the previous section. At the time is probably seemed like there was no reason for writing down specific eigenvectors for each eigenvalue, but we did it for the problems in this section. So, in each case we'll just go back to Example 6 and pull the eigenvectors from that example and form up *P*.

(a) This was part (a) from Example 6 and so *P* is,

$$P = \begin{bmatrix} -\frac{1}{5} & 0 & 1 \\ -\frac{9}{25} & 1 & -\frac{3}{11} \\ 1 & 0 & 1 \end{bmatrix}$$

We'll leave it to you to verify that we get,

$$P^{-1}AP = \begin{bmatrix} -\frac{5}{6} & 0 & \frac{5}{6} \\ -\frac{4}{55} & 1 & \frac{19}{55} \\ \frac{5}{6} & 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{5} & 0 & 1 \\ -\frac{9}{25} & 1 & -\frac{3}{11} \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

(b) This was part (c) from Example 6 so *P* is,

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Again, we'll leave it to you to verify that,

$$P^{-1}AP = \begin{bmatrix} -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Example 2 Neither of the following matrices are diagonalizable.

(a) $A = \begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Solution

To see that neither of these are diagonalizable simply go back to Example 6 in the previous section to see that neither matrix has 3 linearly independent eigenvectors. In both cases we have only two linearly independent eigenvectors and so neither matrix is diagonalizable.

For reference purposes. Part (a) of this example matches part(b) of Example 6 and part (b) of this example matches part (d) of Example 6.

We didn't actually do any of the work here for these problems so let's summarize up how we need to go about finding *P*, provided it exists of course. We first find the eigenvalues for the matrix *A* and then for each eigenvalue find a basis for the eigenspace corresponding to that eigenvalue. The set of basis vectors will then serve as a set of linearly independent eigenvectors for the eigenvalue. If, after we've done this work for all the eigenvalues we have a set of *n* eigenvectors then *A* is diagonalizable and we use the eigenvectors to form *P*. If we don't have a set of *n* eigenvectors then *A* is not diagonalizable.

Actually, we should be careful here. In the above statement we assumed that if we had *n* eigenvectors that they would be linearly independent. We should always verify this of course. There is also one case were we can guarantee that we'll have *n* linearly independent eigenvectors.

Theorem 2 If $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k$ are eigenvectors of *A* corresponding to the *k* distinct eigenvalues $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_k$ then they form a linearly independent set of vectors.

Proof : We'll prove this by assuming that $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k$ are in fact linearly dependent and from this we'll get a contradiction and we we'll see that $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k$ must be linearly independent.

So, assume that $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k$ form a linearly dependent set. Now, since these are eigenvectors we know that they are all non-zero vectors. This means that the set $\left\{\mathbf{v}_1\right\}$ must be a linearly independent set. So, we know that there must be a linearly independent subset of $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k$. So, let *p* be the largest integer such that $\left\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_p\right\}$ is a linearly independent set. Note that we must have $1 \le p < k$ because we are assuming that $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k$ are linearly dependent. Therefore we know that if we take the next vector $\mathbf{v}_{p+1}$ and add it to our linearly independent vectors, $\left\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_p\right\}$, the set $\left\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_p,\mathbf{v}_{p+1}\right\}$ will be a linearly dependent set.

So, if we know that $\left\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_p,\mathbf{v}_{p+1}\right\}$ is a linearly dependent set we know that there are scalars $c_1,c_2,\ldots,c_p,c_{p+1}$, not all zero so that,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p + c_{p+1}\mathbf{v}_{p+1} = \mathbf{0} \qquad \textbf{(1)}$$

Now, multiply this by *A* to get,

$$c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + \cdots + c_pA\mathbf{v}_p + c_{p+1}A\mathbf{v}_{p+1} = \mathbf{0}$$

We know that the $\mathbf{v}_i$ are eigenvectors of *A* corresponding to the eigenvalue λ_i and so we know that $A\mathbf{v}_1 = \lambda_i\mathbf{v}_i$. Using this gives us,

$$c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + \cdots + c_p\lambda_p\mathbf{v}_p + c_{p+1}\lambda_{p+1}\mathbf{v}_{p+1} = \mathbf{0} \qquad \textbf{(2)}$$

Next, multiply both sides of (1) by λ_{p+1} to get,

$$c_1\lambda_{p+1}\mathbf{v}_1 + c_2\lambda_{p+1}\mathbf{v}_2 + \cdots + c_p\lambda_{p+1}\mathbf{v}_p + c_{p+1}\lambda_{p+1}\mathbf{v}_{p+1} = \mathbf{0}$$

and subtract this from (2). Doing this gives,

$$c_1\left(\lambda_1 - \lambda_{p+1}\right)\mathbf{v}_1 + c_2\left(\lambda_2 - \lambda_{p+1}\right)\mathbf{v}_2 + \cdots + c_p\left(\lambda_p - \lambda_{p+1}\right)\mathbf{v}_p + c_{p+1}\left(\lambda_{p+1} - \lambda_{p+1}\right)\mathbf{v}_{p+1} = \mathbf{0}$$

$$c_1\left(\lambda_1 - \lambda_{p+1}\right)\mathbf{v}_1 + c_2\left(\lambda_2 - \lambda_{p+1}\right)\mathbf{v}_2 + \cdots + c_p\left(\lambda_p - \lambda_{p+1}\right)\mathbf{v}_p = \mathbf{0}$$

Now, recall that we assumed that $\left\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\right\}$ were a linearly independent set and so the coefficients here must all be zero. Or,

$$c_1\left(\lambda_1-\lambda_{p+1}\right)=0 \qquad c_2\left(\lambda_2-\lambda_{p+1}\right)=0 \qquad \cdots \qquad c_p\left(\lambda_p-\lambda_{p+1}\right)=0$$

However the eigenvalues are distinct and so the only way all these can be zero is if,

$$c_1=0 \quad c_2=0 \quad \cdots \qquad c_p=0$$

Plugging these values into (1) gives us

$$c_{p+1}\mathbf{v}_{p+1}=\mathbf{0}$$

but, $\mathbf{v}_{p+1}$ is an eigenvector and hence is not the zero vector and so we must have

$$c_{p+1}=\mathbf{0}$$

So, what have we shown to this point? We'll we've just seen that the only possible solution to

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+\cdots+c_p\mathbf{v}_p+c_{p+1}\mathbf{v}_{p+1}=\mathbf{0}$$

is

$$c_1=0 \quad c_2=0 \quad \cdots \qquad c_p=0 \quad c_{p+1}=0$$

This however would mean that the set $\left\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p, \mathbf{v}_{p+1}\right\}$ is linearly independent and we assumed that at least some of the scalars were not zero. Therefore, this contradicts the fact that we assumed that this set was linearly dependent. Therefore our original assumption that $\mathbf{v}_1$, $\mathbf{v}_2$, … , $\mathbf{v}_k$ form a linearly dependent set must be wrong.

We can then see that $\mathbf{v}_1$, $\mathbf{v}_2$, … , $\mathbf{v}_k$ form a linearly independent set.

■

We can use this theorem to quickly identify some diagonalizable matrices.

Theorem 3 Suppose that A is an $n \times n$ matrix and that A has n distinct eigenvalues, then A is diagonalizable.

Proof : By Theorem 2 we know that the eigenvectors corresponding to each of the eigenvectors are a linearly independent set and then by Theorem 1 above we know that A will be diagonalizable.

■

We'll close this section out with a nice theorem about powers diagonalizable matrices and the inverse of an invertible diagonalizable matrix.

Theorem 4 Suppose that A is a diagonalizable matrix and that $P^{-1}AP = D$ then,

(a) If k is any positive integer we have,

$$A^k = PD^kP^{-1}$$

(b) If all the diagonal entries of D are non-zero then A is invertible and,

$$A^{-1} = PD^{-1}P^{-1}$$

Proof :

(a) We'll give the proof for $k=2$ and leave it to you to generalize the proof for larger values of k. Let's start with the following.

$$D^2 = \left(P^{-1}AP\right)^2 = \left(P^{-1}AP\right)\left(P^{-1}AP\right) = P^{-1}A\left(PP^{-1}\right)AP = P^{-1}A\left(I\right)AP = P^{-1}A^2P$$

So, we can see that,

$$D^2 = P^{-1}A^2P$$

We can finish this off by multiply the left of this equation by P and the right by P^{-1} to arrive at,

$$A^2 = PD^2P^{-1}$$

(b) First, we know that if the main diagonal entries of a diagonal matrix are non-zero then the diagonal matrix is invertible. Now, all that we need to show that,

$$A\left(PD^{-1}P^{-1}\right) = I$$

This is easy enough to do. All we need to do is plug in the fact that from part **(a)**, using $k=1$, we have,

$$A = PDP^{-1}$$

So, let's do the following.

$$A\left(PD^{-1}P^{-1}\right) = PDP^{-1}\left(PD^{-1}P^{-1}\right) = PD\left(P^{-1}P\right)D^{-1}P^{-1} = P\left(DD^{-1}\right)P^{-1} = PP^{-1} = I$$

So, we're done.

■