Structured Irreversibility

Discrete Rewriting and Coherence Realization

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Abstract

We develop Spherepop as a *free symmetric monoidal entropy-decreasing rewriting category* **SP**, prove its initiality in a category **EDSMC** of entropy-decreasing symmetric monoidal categories, and construct a symmetric monoidal functor $F : \mathbf{SP} \to \mathbf{RSVP}$ into a smooth realization category whose morphisms carry entropy-slack witnesses ensuring closure.

The first part builds the discrete algebra: option-spaces with admissibility families, a derived causal preorder, and four irreversible generators—Pop, Refuse, Bind, and Collapse—each defined set-theoretically. We separate sequential composition from the symmetric monoidal product and prove the universal property of Collapse as a coequalizer, the strict symmetric monoidal structure of **SP**, and its initiality in **EDSMC**.

The second part formalizes worldhood via sheaf theory. A presheaf of local event proposals over an admissible cover satisfies the sheaf condition exactly when global coherence holds. The meld operator is shown to be policy-induced sheafification, with decomposition $\mathrm{Meld}_\pi = \mathrm{glue} \circ \mathrm{Col}_\pi$ expressing its universal property.

The third part constructs the realization functor. Objects of **RSVP** are smooth manifolds equipped with coherence potential Φ, velocity field v, and entropy density S, and morphisms (φ, η) satisfy $\varphi^* S' = S + \eta$. We define F on objects via probability simplices and on generators via localized smooth perturbations, prove functoriality, and establish the structural asymmetry: **SP** accumulates constraint (combinatorial, log-bound) while **RSVP** redistributes coherence (differential, field-bound).

We conclude with a worked example, a discrete variational formulation of commitment, information-theoretic interpretations, normalization results, and open problems on faithfulness and compact-closed extensions.

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1 Introduction

1.1 The Inversion

Standard computational frameworks take state as primitive and treat time as an index over state transitions. Spherepop reverses this. The primitive semantic object is not a state but a *history*: a finite sequence of irreversible events that progressively restricts an initial space of possibilities. What we call "state" is a derived view, reconstructed by replay. What we call "meaning" is incurred through the disciplined expenditure of possibility, not stored as a representation.

This inversion has both thermodynamic and categorical motivation. From the side of physics, irreversible computation dissipates degrees of freedom (Landauer, 1961; Bennett, 1973; Green & Altenkirch, 2008): any non-trivial computation in the presence of finite resources must increase entropy somewhere. From the side of algebra, rewriting systems naturally generate free categories whose morphisms are reduction sequences. Spherepop combines these: it treats irreversible rewriting as the fundamental ontological operation and equips the resulting category with an entropy functional.

1.2 Structural Quantities

Before construction, we list the observables and their relationships to prevent notational confusion later.

Quantity	**Type**	**Role**
Optionality $\mathsf{Opt}(\Omega)$	$\mathrm{Ob}(\mathbf{SP}) \to \mathbb{R}_{\geq 0}$	Degrees of freedom, resource
Entropy $\mathsf{H}(\Omega)$	$\mathrm{Ob}(\mathbf{SP}) \to \mathbb{R}_{\geq 0}$	Distributional complexity
Action $\mathcal{S}[\gamma]$	histories $\to \mathbb{R}_{\geq 0}$	Accumulated optionality cost
Commitment π_t	step $\to \mathbb{R}_{\geq 0}$	Conjugate to optionality
Slack η	RSVP morphism data	Continuous entropy production
Coherence Φ	$C^\infty(M)$	Scalar potential in RSVP

Optionality and entropy are distinct observables. Optionality is a pure resource measure ($\mathsf{Opt}(\Omega) \propto \log|\Omega|$ in the finite uniform case). Entropy encodes distributional structure and satisfies subadditivity under tensor. Action is the accumulated optionality cost of a history. Slack is the continuous analogue of action increments.

1.3 Two Structures That Must Be Kept Separate

Composition is history concatenation:

$$(e_2 \circ e_1)(\Omega_0) = e_2(e_1(\Omega_0)).$$

It is strictly associative by construction and encodes temporal order.

Tensor product $\otimes$ is defined on option-spaces and extended to morphisms componentwise:

$$(\Omega_1, \mathcal{A}_1) \otimes (\Omega_2, \mathcal{A}_2) = (\Omega_1 \times \Omega_2,\ \mathcal{A}_1 \otimes \mathcal{A}_2).$$

It encodes parallel independence. It is *not* composition.

Entropy behaves differently under each: sequential composition is entropy-monotone (Axiom 2); tensor product satisfies subadditivity (Proposition 9.3).

1.4 Roadmap

Section 2 defines option-spaces, admissibility, entropy, and optionality. Section 7 constructs **SP** and its generators, with complete independence proofs. Section 8 develops the causal preorder and proves the full universal property of Collapse. Section 9 establishes the strict symmetric monoidal structure and proves initiality of **SP** in **EDSMC**. Section 10 formalizes worldhood and proves the meld theorem. Section 11 constructs **RSVP** with witnessed morphisms. Section 12 defines F and proves its four component lemmas and overall functoriality. Section 14 works through a complete example. Later sections address mechanics, information theory, normalization, further directions, and failure modes.

2 Option-Spaces, Admissibility, and Entropy

2.1 Option-Spaces

Definition 2.1 (Option-space)**.** An *option-space* is a pair $(\Omega, \mathcal{A})$ where:

(i) Ω is a nonempty set of *admissible futures*;

(ii) $\mathcal{A} \subseteq 2^{\Omega}$ is the *admissibility family*, satisfying $\Omega \in \mathcal{A}$ and closure under finite intersection: $U, V \in \mathcal{A} \Rightarrow U \cap V \in \mathcal{A}$.

The family $\mathcal{A}$ encodes structural compatibility constraints imposed by binding events. A subset $U \subseteq \Omega$ lies in $\mathcal{A}$ precisely when it respects all currently active dependency constraints.

Example 2.2. Let $\Omega = \{a, b, c\}$ and $\mathcal{A} = \{\Omega, \{a, b\}, \{a, c\}, \{a\}\}$. Then a is causally necessary for both b and c: no admissible future can contain b or c without also containing a.

2.2 Entropy and Optionality

Definition 2.3 (Entropy functional)**.** A *Spherepop entropy functional* is a map $\mathsf{H} : \mathrm{Ob}(\mathbf{SP}) \to \mathbb{R}_{\geq 0}$ satisfying:

(i) $\mathsf{H}(\Omega) = 0$ iff $|\Omega| = 1$;

(ii) *Monotonicity:* $\Omega' \subseteq \Omega$ implies $\mathsf{H}(\Omega') \leq \mathsf{H}(\Omega)$;

(iii) *Subadditivity:* $\mathsf{H}(\Omega_1 \otimes \Omega_2) \leq \mathsf{H}(\Omega_1) + \mathsf{H}(\Omega_2)$, with equality when Ω_1, Ω_2 are independent.

The canonical instance: for a probability measure p on Ω, $\mathsf{H}(\Omega) = -\sum_{x \in \Omega} p(x) \log p(x)$.

Definition 2.4 (Optionality)**.** The *optionality functional* is a monotone map $\mathsf{Opt} : \mathrm{Ob}(\mathbf{SP}) \to \mathbb{R}_{\geq 0}$ satisfying $\Omega' \subseteq \Omega \Rightarrow \mathsf{Opt}(\Omega') \leq \mathsf{Opt}(\Omega)$ and $\mathsf{Opt}(\{\omega\}) = 0$.

Optionality measures degrees of freedom available to a world. In the finite uniform case one may take $\mathsf{Opt}(\Omega) = \log |\Omega| = \mathsf{H}(\Omega)$; in general they differ.

Summary. The ontological layer is now fixed: option-spaces are the objects; entropy measures distributional complexity; optionality measures resource availability. These are the inputs to the algebraic construction of **SP**.

3 Entropy as a Monoidal Weight

The entropy functional $\mathsf{H} : \mathrm{Ob}(\mathbf{SP}) \to \mathbb{R}_{\geq 0}$ has so far been treated as a monotone observable. We now strengthen its role: entropy induces an enrichment of **SP** over the poset $(\mathbb{R}_{\geq 0}, \leq)$ and yields a strict monoidal weight structure.

3.1 Entropy-Decreasing Enrichment

For objects $\Omega, \Omega' \in \mathrm{Ob}(\mathbf{SP})$, define a preorder on $\mathrm{Hom}_{\mathbf{SP}}(\Omega, \Omega')$ by

$$e \preceq e' \iff \mathsf{H}(e(\Omega)) \geq \mathsf{H}(e'(\Omega)).$$

Compatibility with composition is expressed by the monotonicity rule

$$e_1 \preceq e_1' \text{ and } e_2 \preceq e_2' \implies e_2 \circ e_1 \preceq e_2' \circ e_1'.$$

Proposition 3.1. **SP** *is enriched over the monoidal poset* $(\mathbb{R}_{\geq 0}, \geq, +, 0)$*, with enrichment given by entropy decrease.*

Proof. Define the weight of $e : \Omega \to \Omega'$ to be

$$w(e) := \mathsf{H}(\Omega) - \mathsf{H}(\Omega').$$

Then for composable morphisms $\Omega_0 \xrightarrow{e_1} \Omega_1 \xrightarrow{e_2} \Omega_2$ we have

$$w(e_2 \circ e_1) = \mathsf{H}(\Omega_0) - \mathsf{H}(\Omega_2) = \Big(\mathsf{H}(\Omega_0) - \mathsf{H}(\Omega_1)\Big) + \Big(\mathsf{H}(\Omega_1) - \mathsf{H}(\Omega_2)\Big) = w(e_1) + w(e_2),$$

which is strict additivity. The remaining enrichment axioms follow from the definition of $\preceq$ and Axiom 2. □

This makes **SP** a *strictly additive entropy-weighted category.* In particular, histories admit a canonical factorization by total entropy drop.

3.2 Additive Decomposition of Histories

Any history h admits a decomposition of the schematic form

$$h = \mathrm{Col}^{k_c} \circ \mathrm{Bind}^{k_b} \circ \mathrm{Pop}^{k_p},$$

and the total weight decomposes additively as

$$w(h) = \sum_{i=1}^{k_p} w(\mathrm{Pop}_i) + \sum_{j=1}^{k_c} w(\mathrm{Col}_j),$$

with the understanding that Bind-steps contribute zero Shannon-entropy weight in the current convention.

4 Bi-Graded Resource Functional

The entropy functional H of Section 2 is Shannon-type and measures distributional complexity. As shown in Appendix C, there is a second natural observable, $(\Omega, \mathcal{A}) = \log|\Omega| - \log|\mathrm{Aut}(\Omega, \mathcal{A})|$, which captures *symmetry entropy*: the log-ratio of raw size to structural symmetry. The two functionals behave differently across generators$\mathsf{H}$ is strictly decreased by Pop and Ref and preserved by Bind, while is strictly increased by Bind. This section consolidates them into a single bi-graded resource functional and shows that the result is both a Lawvere enrichment and a natural basis for the factorization system of Section 6.

4.1 The Bi-Graded Ordered Monoid

Definition 4.1 (Bi-graded resource monoid)**.** Let $\mathcal{R} := ([0, \infty] \times [0, \infty], \geq_\times, +)$ where:

- addition is componentwise: $(a, b) + (a', b') := (a + a', b + b')$;
- the partial order is componentwise: $(a, b) \geq_\times (a', b')$ iff $a \geq a'$ and $b \geq b'$.

$(\mathcal{R}, \geq_\times, +, (0, 0))$ is a commutative ordered monoid, hence a symmetric monoidal category when regarded as thin.

Definition 4.2 (Bi-graded resource functional)**.** Define $\mathbf{R} : \mathrm{Ob}(\mathbf{SP}) \to \mathcal{R}$ by

$$\mathbf{R}(\Omega, \mathcal{A}) := \big(\mathsf{H}(\Omega, \mathcal{A}),\ (\Omega, \mathcal{A})\big),$$

where H is the Shannon entropy functional (Definition 2.3) and $(\Omega, \mathcal{A}) := \log|\Omega| - \log|\mathrm{Aut}(\Omega, \mathcal{A})|$ is the symmetry entropy (Appendix C).

4.2 Generator Behavior Under the Bi-Graded Functional

The following table records the direction of change in each component under each generator. Let $\downarrow_s$ denote strict decrease, $\uparrow_s$ strict increase, and $=$ no change.

Generator	H component	component
Pop_U	$\downarrow_s$	depends on automorphisms of $\Omega \setminus U$
Ref_U	$\downarrow_s$	depends on automorphisms of $\Omega \setminus U$
Bind_{ij}	$=$	$\downarrow_s$ (breaks the $i \leftrightarrow j$ automorphism)
$\mathrm{Col}_\sim$	$\downarrow_s$ or $=$	$\downarrow_s$ or $=$

Proposition 4.3 (**R** is resource-monotone)**.** *For every morphism* $f : (\Omega, \mathcal{A}) \to (\Omega', \mathcal{A}')$ *in* **SP***,*

$$\mathbf{R}(\Omega', \mathcal{A}') \leq_\times \mathbf{R}(\Omega, \mathcal{A}),$$

with strict decrease in at least one component for every non-identity morphism.

Proof. For each generator:

- Pop_U, Ref_U: H strictly decreases (Axiom 2). The component may or may not change, but cannot increase past the original value since $|\Omega \setminus U| < |\Omega|$ bounds $\log |\Omega \setminus U|$.
- Bind_{ij}: H is preserved (Axiom 2). The automorphism that swaps $i \leftrightarrow j$ is removed from $\mathrm{Aut}(\Omega, \mathcal{A}')$ (Appendix C), so $|\mathrm{Aut}(\Omega, \mathcal{A}')| < |\mathrm{Aut}(\Omega, \mathcal{A})|$ and strictly decreases.
- $\mathrm{Col}_\sim$: H decreases or stays equal. The coarse-graining reduces $|\Omega/{\sim}| \leq |\Omega|$ and may reduce automorphisms; does not increase.

In each case, **R** is non-increasing componentwise, with a strict decrease in at least one component. □

4.3 Bi-Graded Lawvere Enrichment

The bi-graded functional extends the Lawvere enrichment of Section 5 to two dimensions.

Definition 4.4 (Bi-graded distance)**.** Define $d_\mathbf{R} : \mathrm{Ob}(\mathbf{SP}) \times \mathrm{Ob}(\mathbf{SP}) \to \mathcal{R}$ by

$$d_\mathbf{R}(X, Y) := \inf_{\geq_\times} \Big\{ \mathbf{R}(Y) - \mathbf{R}(X) : \exists f : Y \to X \Big\},$$

where the infimum is taken componentwise and $\mathbf{R}(Y) - \mathbf{R}(X)$ is defined componentwise when all entries are finite.

Proposition 4.5. $(\mathrm{Ob}(\mathbf{SP}), d_\mathbf{R})$ *is a* $\mathcal{R}$*-enriched category (a generalized Lawvere metric space valued in* $\mathcal{R}$*).*

Proof. $d_\mathbf{R}(X, X) = (0, 0)$ (use id_X). Triangle inequality: the argument of Proposition 5.2 applies componentwise. □

4.4 Refined Factorization System

The bi-graded functional sharpens the $\mathcal{E}$–$\mathcal{M}$ factorization of Section 6.

Proposition 4.6 (Bi-graded factorization)**.** *Under* **R***:*

- $\mathcal{E}$ *(the* Bind*-generated morphisms) are exactly the morphisms that move only the component:* $\Delta\mathsf{H} = 0$*,* $\Delta < 0$*.*
- $\mathcal{M}$ *(the* Pop/Ref/Col*-generated morphisms) are exactly the morphisms that move the* H *component:* $\Delta\mathsf{H} < 0$ *(or* $= 0$ *for entropy-neutral* Col*), with* Δ *unconstrained.*

Hence the $\mathcal{E}$–$\mathcal{M}$ *factorization of every morphism in* **SP** *is a factorization into a* symmetry-reducing *step followed by an* option-eliminating *step.*

Proof. Immediate from Proposition 4.3 and the generator analysis above: bindings move only ; pops, refuses, and collapses move H. The factorization theorem of Section 6 (commuting Bind steps past Pop/Col when supports are disjoint) expresses every history in this two-phase form. □

Remark 4.7. The bi-graded resource **R** provides a principled resolution to a subtlety in the entropy-alone treatment: an entropy-neutral Col (one that identifies entropy-equal classes) contributes zero to H but still decreases , so it appears as a non-trivial morphism under **R**. The bi-grading therefore distinguishes between identity morphisms and genuinely non-trivial entropy-neutral collapses, a distinction invisible to the single-component functional.

5 Lawvere-Metric Enrichment of SP

Section 3 showed that entropy differences compose additively along morphism chains. We now consolidate this into a formal enrichment over the Lawvere quantale, giving the entropy grading the status of a genuine categorical metric structure.

5.1 The Lawvere Quantale

Recall that the *Lawvere quantale* is the monoidal poset $\mathbf{L} := ([0,\infty], \geq, +, 0)$, regarded as a thin monoidal category: there is a unique morphism $a \to b$ in **L** if and only if $a \geq b$, composition is addition, and the unit object is 0. An **L**-*enriched* *category* (equivalently, a generalized metric space in the sense of Lawvere (Lawvere, 1970)) is a set of objects X together with a function $d : X \times X \to [0, \infty]$ satisfying

$$d(X, X) = 0 \qquad \text{and} \qquad d(X, Z) \leq d(X, Y) + d(Y, Z).$$

Symmetry is *not* required; this directedness is essential for capturing the asymmetry of irreversible entropy flow.

5.2 The Entropy Distance on $\mathrm{Ob}(\mathbf{SP})$

Definition 5.1 (Entropy distance)**.** Define $d_{\mathsf{H}} : \mathrm{Ob}(\mathbf{SP}) \times \mathrm{Ob}(\mathbf{SP}) \to [0, \infty]$ by

$$d_{\mathsf{H}}(X, Y) := \inf\Big\{\mathsf{H}(Y) - \mathsf{H}(X) \ : \ \exists f : Y \to X \text{ in } \mathbf{SP}\Big\},$$

with the convention $\inf \emptyset = \infty$.

The direction of the morphism $f : Y \to X$ is intentional: by the entropy preorder of Section 19, $X \preceq Y$ iff such an f exists, and Axiom 2 then forces $\mathsf{H}(X) \leq \mathsf{H}(Y)$, making every term in the infimum non-negative.

Proposition 5.2 (d_{H} is a Lawvere metric)**.** $(\mathrm{Ob}(\mathbf{SP}), d_{\mathsf{H}})$ *is a* $\mathbf{L}$*-enriched category. Explicitly:*

(i) $d_{\mathsf{H}}(X, X) = 0$ *for all* X*.*

(ii) $d_{\mathsf{H}}(X, Z) \leq d_{\mathsf{H}}(X, Y) + d_{\mathsf{H}}(Y, Z)$ *for all* X, Y, Z*.*

Proof. **(i).** The identity morphism $\mathrm{id}_X : X \to X$ witnesses $\mathsf{H}(X) - \mathsf{H}(X) = 0$, so the infimum is at most 0; it is at least 0 by Axiom 2.

(ii). Fix $\varepsilon > 0$. Choose $f : Y \to X$ with $\mathsf{H}(Y) - \mathsf{H}(X) < d_{\mathsf{H}}(X, Y) + \varepsilon$, and $g : Z \to Y$ with $\mathsf{H}(Z) - \mathsf{H}(Y) < d_{\mathsf{H}}(Y, Z) + \varepsilon$. The composite $f \circ g : Z \to X$ is a morphism in $\mathbf{SP}$, so

$$d_{\mathsf{H}}(X, Z) \leq \mathsf{H}(Z) - \mathsf{H}(X) = \Big(\mathsf{H}(Z) - \mathsf{H}(Y)\Big) + \Big(\mathsf{H}(Y) - \mathsf{H}(X)\Big) < d_{\mathsf{H}}(X, Y) + d_{\mathsf{H}}(Y, Z) + 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the triangle inequality holds. □

5.3 Collapse to a Clean Formula

When H is strictly functorial (each generator contributes a determined entropy drop and these drops telescope), the infimum in Definition 5.1 is attained and simplifies completely.

Proposition 5.3 (Telescoping collapse)**.** *For any composable history* $\gamma : Y \to X$*, the path length* $\ell_{\mathsf{H}}(\gamma) := \sum_i \delta_{\mathsf{H}}(e_i)$ *equals* $\mathsf{H}(Y) - \mathsf{H}(X)$*. Consequently, if* $X \preceq Y$ *(i.e.* $\exists f : Y \to X$*), then*

$$d_{\mathsf{H}}(X, Y) = \mathsf{H}(Y) - \mathsf{H}(X).$$

If no such morphism exists, $d_{\mathsf{H}}(X, Y) = \infty$*.*

Proof. Telescoping of the entropy functional along the history, as in the proof of the weight-additivity Proposition in Section 3. The formula for d_{H} then follows because every path $Y \to X$ has the same length $\mathsf{H}(Y) - \mathsf{H}(X)$, so the infimum is that common value. □

Remark 5.4. Proposition 5.3 upgrades the Lawvere enrichment to the strongest possible form: d_{H} is the *canonical* Lawvere metric on the entropy-preordered set of objects, with distance equal to the entropy gap when reachable and ∞ otherwise. The asymmetry $d_{\mathsf{H}}(X, Y) \neq d_{\mathsf{H}}(Y, X)$ in general (since $X \preceq Y$ does not imply $Y \preceq X$) reflects the irreversibility of **SP** at the metrized level.

5.4 Thin-Category Formulation

For the purposes of enriched category theory, it is cleaner to record the enrichment on hom-objects rather than on object-pairs. Define the *thin reachability category* $\Pi(\mathbf{SP})$: same objects as **SP**; $\mathrm{Hom}_{\Pi(\mathbf{SP})}(Y, X)$ is a singleton $\{*\}$ if $\mathbf{SP}(Y, X) \neq \emptyset$, and empty otherwise. Equip $\Pi(\mathbf{SP})$ with the **L**-hom-object

$$\underline{\mathrm{Hom}}(Y, X) := d_{\mathsf{H}}(X, Y) \in [0, \infty].$$

Composition in $\Pi(\mathbf{SP})$ sends $(d_1, d_2) \mapsto d_1 + d_2$, which is precisely the **L**-composition. This makes $\Pi(\mathbf{SP})$ a **L**-enriched category in the full sense of Awodey (2010), and d_{H} is its hom-functor.

6 A Canonical Factorization System

The four generators admit a canonical orthogonal factorization system separating *structural introduction* from *option elimination.*

6.1 E–M Factorization

Define two classes of morphisms:

$$\mathcal{E} := \{\text{id, Bind-generated morphisms}\}, \qquad \mathcal{M} := \{\text{Pop, Ref, Col-generated morphisms}\}.$$

Theorem 6.1. *Every morphism h in* **SP** *factors uniquely (up to policy equivalence) as*

$$h = m \circ e$$

with $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

Sketch. Bindings refine admissibility families while leaving the underlying set Ω unchanged, hence they preserve the Shannon-entropy functional under the convention of Axiom 2. Pops and collapses are the only generators that change the underlying option-set and therefore the only generators that can contribute to strict entropy descent. The factorization is obtained by commuting independent Bind steps leftward past Pop/Ref/Col steps whenever their supports are disjoint, using the commutation relations already imposed in Definition 7.5. Uniqueness holds up to the same policy congruence used to identify commuting independent bindings. □

7 The Category SP

7.1 Objects, Morphisms, and Axioms

Objects of **SP** are option-spaces. A morphism $e : (\Omega, \mathcal{A}) \to (\Omega', \mathcal{A}')$ is an *irreversible event*: a function $e : \Omega \to \Omega'$ that is admissibility-preserving ($A' \in \mathcal{A}' \Rightarrow e^{-1}(A') \in \mathcal{A}$) and entropy-monotone.

Axiom 1 (Sequential irreversibility)**.** No non-identity morphism in **SP** admits an inverse.

Axiom 2 (Entropy monotonicity)**.** For every morphism $e : \Omega \to \Omega'$: $\mathsf{H}(\Omega') \leq \mathsf{H}(\Omega)$. Strict decrease holds for Pop and Ref. Bind preserves entropy while reducing symmetry. Col may produce equality or strict decrease depending on whether the quotient collapses entropy-distinct classes.

7.2 Generating Morphisms

Definition 7.1 (Pop)**.** A Pop-event eliminates a nonempty $U \subsetneq \Omega$:

$$\mathrm{Pop}_U : (\Omega, \mathcal{A}) \longrightarrow (\Omega \setminus U,\ \{A \cap (\Omega \setminus U) : A \in \mathcal{A}\}).$$

Entropy decreases strictly: $\mathsf{H}(\Omega \setminus U) < \mathsf{H}(\Omega)$.

Definition 7.2 (Refuse)**.** A Ref-event has the same underlying set-theoretic action as Pop_U but carries an auxiliary tag $\tau \in \mathcal{B}(\Omega)$ recorded by an accounting functor $\mathcal{B} : \mathbf{SP} \to \mathbf{Set}$, distinguishing deliberate non-selection from positive elimination. The underlying morphism of option-spaces coincides with Pop_U.

Definition 7.3 (Bind)**.** A Bind-event tightens the admissibility family without eliminating futures:

$$\mathrm{Bind}_{ij} : (\Omega, \mathcal{A}) \longrightarrow (\Omega,\ \mathcal{A}')$$

where $\mathcal{A}' = \{A \in \mathcal{A} : i \in A \Rightarrow j \in A\} \subsetneq \mathcal{A}$. Entropy is globally preserved: $\mathsf{H}(\Omega, \mathcal{A}') = \mathsf{H}(\Omega, \mathcal{A})$.

Definition 7.4 (Collapse)**.** A Col-event applies a causally admissible equivalence relation $\sim$ (see Definition 8.5) to form the quotient:

$$\mathrm{Col}_\sim : (\Omega, \mathcal{A}) \longrightarrow (\Omega/\!\sim,\ \mathcal{A}/\!\sim).$$

7.3 Free Generation and Independence of Generators

Definition 7.5 (Free generation)**.** Let $\mathcal{G}$ be the directed multigraph with vertices $\mathrm{Ob}(\mathbf{SP})$ and edges labeled by instances of the four generators. $\mathbf{SP}$ is the quotient of the free category $\mathcal{F}(\mathcal{G})$ by the congruence generated by:

(a) entropy monotonicity (Axiom 2);

(b) idempotence of Pop on already-excluded elements: $\mathrm{Pop}_U \circ \mathrm{Pop}_V = \mathrm{Pop}_{U \cup V}$ when $U \cap (\Omega \setminus V) = \emptyset$;

(c) commutativity of independent Bind events: $\mathrm{Bind}_{ij} \circ \mathrm{Bind}_{kl} = \mathrm{Bind}_{kl} \circ \mathrm{Bind}_{ij}$ when $\{i, j\} \cap \{k, l\} = \emptyset$;

(d) the symmetric monoidal laws (Section 9).

No further relations are imposed.

Proposition 7.6 (Independence of generators)**.** *The four generators are independent in the following precise sense:*

(i) Pop *cannot be derived from* Ref*: they differ in the value of the accounting functor $\mathcal{B}$.*

(ii) Ref *cannot be derived from* Pop*: no composition of* Pop *events yields a morphism with a non-trivial tag in $\mathcal{B}$.*

(iii) Bind *is not reducible to* Pop *followed by* Col*:* Bind *preserves $|\Omega|$ while any* Pop *reduces it; a subsequent* Col *could re-identify elements but cannot restore the eliminated element without violating Axiom 1.*

(iv) Col *is not a special case of* Pop*:* Pop *produces a strict subobject;* Col *produces a quotient. These are categorically distinct constructions.*

Proof. For (i) and (ii): the accounting functor $\mathcal{B}$ assigns distinct values to Pop and Ref events by definition. Since $\mathcal{B}$ is a functor, it must agree on composed morphisms with the composed images; but Pop has $\mathcal{B}$ value 0 and Ref has value $\tau \neq 0$, so neither can be derived from a composition involving only the other.

For (iii): let e be any morphism derivable from Pop and Col events starting at $(\Omega, \mathcal{A})$. The Pop step yields $(\Omega \setminus U, \cdot)$ with $|\Omega \setminus U| < |\Omega|$. A subsequent Col reduces $|\Omega \setminus U|$ further or preserves it. The resulting object has $|\Omega''| \leq |\Omega \setminus U| < |\Omega|$. But Bind_{ij} produces $(\Omega, \mathcal{A}')$ with $|\Omega|$ unchanged. Hence Bind morphisms lie outside the image of the subcategory generated by Pop and Col.

For (iv): Pop_U is the inclusion $\Omega \setminus U \hookrightarrow \Omega$ reversed (a restriction), hence an injective-on-objects construction. $\text{Col}_\sim$ is a surjective-on-objects quotient. In a category of structured sets, injective and surjective constructions on objects are distinct. □

Summary. The category **SP** is freely generated by the four independent generators modulo the relations of Definition 7.5. Composition encodes temporal sequence. No generator is reducible to any combination of the others.

7.4 Entropy Slack Witnesses

To ensure strict functoriality of F, we refine RSVP morphisms.

Definition 7.7. An RSVP morphism is a pair (φ, η) where

$$\varphi : M \to M', \qquad \eta \in C^\infty(M, \mathbb{R}_{\geq 0}), \qquad \varphi^* S' = S + \eta.$$

The function η is the *entropy slack witness.* Composition is defined by

$$(\varphi_2, \eta_2) \circ (\varphi_1, \eta_1) = (\varphi_2 \circ \varphi_1,\ \eta_1 + \varphi_1^* \eta_2).$$

Proposition 7.8. *With slack witnesses,* **RSVP** *is strictly closed under composition.*

Proof. Pullback preserves addition, so

$$(\varphi_2 \circ \varphi_1)^* S'' = \varphi_1^*(\varphi_2^* S'') = \varphi_1^*(S' + \eta_2) = \varphi_1^* S' + \varphi_1^* \eta_2 = (S + \eta_1) + \varphi_1^* \eta_2 = S + (\eta_1 + \varphi_1^* \eta_2),$$

which is exactly the defining condition for the composite slack. □

8 Causality and the Universal Property of Collapse

8.1 Causal Preorder

Definition 8.1 (Causal preorder)**.** For $(\Omega, \mathcal{A})$ and $x, y \in \Omega$, define $x \preceq y$ iff every $A \in \mathcal{A}$ containing y also contains x:

$$x \preceq y \iff \forall A \in \mathcal{A} :\ y \in A \Rightarrow x \in A.$$

Write $\downarrow y = \{x \in \Omega : x \preceq y\}$ for the causal past of y.

Lemma 8.2 (Reflexivity and transitivity)**.** $\preceq$ *is a preorder.*

Proof. Reflexivity. For any x: every A containing x contains x. Hence $x \preceq x$.

Transitivity. Suppose $x \preceq y$ and $y \preceq z$. Let $A \in \mathcal{A}$ with $z \in A$. Since $y \preceq z$, we have $y \in A$. Since $x \preceq y$, we have $x \in A$. Hence $x \preceq z$. □

Lemma 8.3 (Antisymmetry under acyclicity)**.** *If the admissibility family $\mathcal{A}$ contains no nontrivial cycles under $\preceq$ (i.e. $x \preceq y$ and $y \preceq x$ implies $x = y$), then $\preceq$ is a partial order.*

Proof. The hypothesis directly gives antisymmetry. Combined with Lemma 8.2, $\preceq$ is a partial order. □

Remark 8.4. Bind_{ij} adds the constraint $i \preceq j$ to $\mathcal{A}$. Acyclicity of the dependency graph (no sequence of Bind events creates a cycle $i \preceq j \preceq i$ with $i \neq j$) is a well-formedness condition on histories. We assume it throughout.

8.2 Causally Admissible Equivalence Relations

Definition 8.5 (Causally admissible equivalence)**.** An equivalence relation $\sim$ on Ω is *causally admissible* if

$$x \sim y \implies {\downarrow}x = {\downarrow}y.$$

This condition ensures that identified futures have identical causal pasts, so collapse does not alter downstream commitment structure.

8.3 Full Universal Property of Collapse

Fix $(\Omega, \mathcal{A})$ and a causally admissible equivalence $\sim_q$. Let $\pi_q : \Omega \twoheadrightarrow \Omega/{\sim_q}$ be the quotient map and $\mathcal{A}_q = \{A \subseteq \Omega/{\sim_q} : \pi_q^{-1}(A) \in \mathcal{A}\}$ the induced admissibility family. Let $\mathbf{SP}_c$ denote the full subcategory of $\mathbf{SP}$ of causal-preorder-preserving morphisms.

Theorem 8.6 (Universal property of causal collapse)**.** *In $\mathbf{SP}_c$, the morphism $\mathrm{Col}_q : (\Omega, \mathcal{A}) \to (\Omega/{\sim_q}, \mathcal{A}_q)$ is a coequalizer of the kernel pair of π_q. Explicitly: define $R_q = \{(x, y) \in \Omega^2 : x \sim_q y\}$ with induced admissibility, and projections $r_1, r_2 : (R_q, \mathcal{A}_{R_q}) \rightrightarrows (\Omega, \mathcal{A})$, $r_1(x, y) = x$, $r_2(x, y) = y$. Then:*

(i) Coequalizing. $\mathrm{Col}_q \circ r_1 = \mathrm{Col}_q \circ r_2$.

(ii) Universal property. *For any $(\Omega'', \mathcal{A}'') \in \mathbf{SP}_c$ and any morphism $f : (\Omega, \mathcal{A}) \to (\Omega'', \mathcal{A}'')$ in $\mathbf{SP}_c$ with $f \circ r_1 = f \circ r_2$, there exists a unique morphism $\bar{f} : (\Omega/{\sim_q}, \mathcal{A}_q) \to (\Omega'', \mathcal{A}'')$ in $\mathbf{SP}_c$ with $\bar{f} \circ \mathrm{Col}_q = f$:*

$$\begin{array}{ccccc} (R_q, \mathcal{A}_{R_q}) & \overset{r_1}{\underset{r_2}{\rightrightarrows}} & (\Omega, \mathcal{A}) & \xrightarrow{\mathrm{Col}_q} & (\Omega/{\sim_q}, \mathcal{A}_q) \\ & & & \searrow^{f} & \Big\downarrow{\scriptstyle \exists!\, \bar{f}} \\ & & & & (\Omega'', \mathcal{A}'') \end{array}$$

(iii) Naturality. *The assignment $f \mapsto \bar{f}$ is natural in $(\Omega'', \mathcal{A}'')$.*

Proof. **Part (i).** For $(x, y) \in R_q$, we have $x \sim_q y$, hence $\pi_q(x) = \pi_q(y)$. Thus $\mathrm{Col}_q(r_1(x, y)) = \pi_q(x) = \pi_q(y) = \mathrm{Col}_q(r_2(x, y))$.

Part (ii). *Existence.* Define $\bar{f}([x]) := f(x)$ where $[x]$ denotes the $\sim_q$-class of x. This is well-defined: if $x \sim_q y$ then $f(r_1(x, y)) = f \circ r_1(x, y) = f \circ r_2(x, y) = f(y)$, so $f(x) = f(y)$.

$\bar{f}$ *is admissibility-preserving.* Let $B \in \mathcal{A}''$. Then $\bar{f}^{-1}(B) = \{[x] : \bar{f}([x]) \in B\} = \{[x] : f(x) \in B\} = \pi_q(f^{-1}(B))$. Since f is admissibility-preserving, $f^{-1}(B) \in \mathcal{A}$. By definition of $\mathcal{A}_q$: $B' \in \mathcal{A}_q$ iff $\pi_q^{-1}(B') \in \mathcal{A}$. We have $\pi_q^{-1}(\bar{f}^{-1}(B)) = f^{-1}(B) \in \mathcal{A}$, hence $\bar{f}^{-1}(B) \in \mathcal{A}_q$.

$\bar{f}$ *is causal-preorder-preserving.* Let $[x] \preceq_q [y]$ in $\Omega/\sim_q$: every $A_q \in \mathcal{A}_q$ containing $[y]$ also contains $[x]$. We must show $\bar{f}([x]) \preceq'' \bar{f}([y])$ in Ω''. Let $B \in \mathcal{A}''$ with $\bar{f}([y]) \in B$, i.e. $f(y) \in B$. Since f is causal-preorder-preserving and $y \in f^{-1}(B) \in \mathcal{A}$, we need $x \in f^{-1}(B)$. Now $f^{-1}(B) \in \mathcal{A}$, so $\pi_q(f^{-1}(B)) \in \mathcal{A}_q$. We have $[y] \in \pi_q(f^{-1}(B))$ and by $[x] \preceq_q [y]$ we get $[x] \in \pi_q(f^{-1}(B))$, i.e. $x \in f^{-1}(B)$, i.e. $f(x) = \bar{f}([x]) \in B$.

The equation $\bar{f} \circ \mathrm{Col}_q = f$ *holds* by construction: $\bar{f}(\pi_q(x)) = \bar{f}([x]) = f(x)$.

Uniqueness. If $\bar{f}'$ also satisfies $\bar{f}' \circ \mathrm{Col}_q = f$, then for any $[x] \in \Omega/\sim_q$: $\bar{f}'([x]) = \bar{f}'(\pi_q(x)) = f(x) = \bar{f}([x])$. Since π_q is surjective, $\bar{f}' = \bar{f}$.

Part (iii). For any morphism $g : (\Omega'', \mathcal{A}'') \to (\Omega''', \mathcal{A}''')$, the natural transformation condition $\overline{g \circ f} = g \circ \bar{f}$ holds because: $\overline{g \circ f}([x]) = g(f(x)) = g(\bar{f}([x])) = (g \circ \bar{f})([x])$. □

Summary. Collapse is not an ad hoc quotient but the unique coequalizing map satisfying the universal property with respect to causal-preorder-preserving morphisms. Every categorical framework that admits entropy-decreasing quotients must factor through Col when the equivalence relation is causally admissible.

9 Symmetric Monoidal Structure and Universal Property of SP

9.1 Tensor Product

Definition 9.1 (Tensor of option-spaces)**.**

$$(\Omega_1, \mathcal{A}_1) \otimes (\Omega_2, \mathcal{A}_2) := (\Omega_1 \times \Omega_2,\ \mathcal{A}_1 \otimes \mathcal{A}_2),$$

where $\mathcal{A}_1 \otimes \mathcal{A}_2 := \{U_1 \times U_2 : U_1 \in \mathcal{A}_1, U_2 \in \mathcal{A}_2\}$. The unit object is $I = (\{*\}, \{\{*\}\})$.

On morphisms: $(e_1 \otimes e_2)(x_1, x_2) = (e_1(x_1), e_2(x_2))$.

Proposition 9.2 (Symmetric monoidal structure)**.** $(\mathbf{SP}, \otimes, I)$ *is a symmetric monoidal category. By Mac Lane's coherence theorem, it may be treated as strict without loss of generality.*

Proof. The tensor product is given on objects by Cartesian product and on morphisms by componentwise application. Associativity and unit constraints arise from the canonical bijections

$$(\Omega_1 \times \Omega_2) \times \Omega_3 \cong \Omega_1 \times (\Omega_2 \times \Omega_3), \qquad \Omega \times \{*\} \cong \Omega \cong \{*\} \times \Omega,$$

which preserve admissibility by functoriality of products.

Symmetry is given by the swap map

$$\sigma_{\Omega_1,\Omega_2}(x_1, x_2) = (x_2, x_1),$$

which preserves admissible sets and is involutive, $\sigma^2 = \mathrm{id}$. Entropy is invariant under factor permutation, so symmetry is entropy-neutral.

Functoriality follows since, for admissibility-preserving morphisms e_i,

$$(e_1 \otimes e_2)^{-1}(U_1 \times U_2) = e_1^{-1}(U_1) \times e_2^{-1}(U_2) \in \mathcal{A}_1 \otimes \mathcal{A}_2.$$

The interchange law

$$(f_2 \circ f_1) \otimes (g_2 \circ g_1) = (f_2 \otimes g_2) \circ (f_1 \otimes g_1)$$

holds pointwise:

$$(x, y) \mapsto \big(f_2(f_1(x)),\, g_2(g_1(y))\big).$$

Since Cartesian product of sets is strictly associative and unital, the structural isomorphisms may be identified with identities, yielding a strict symmetric monoidal structure. □

Proposition 9.3 (Entropy subadditivity)**.**

$$\mathsf{H}(\Omega_1 \otimes \Omega_2) \leq \mathsf{H}(\Omega_1) + \mathsf{H}(\Omega_2),$$

with equality if and only if the joint measure on $\Omega_1 \times \Omega_2$ *is a product measure.*

Proof. Let $p(x, y)$ be a joint probability distribution with marginals $p_1(x)$ and

$p_2(y)$. Then

$$\begin{aligned}
\mathsf{H}(\Omega_1 \otimes \Omega_2) &= -\sum_{x,y} p(x,y) \log p(x,y) \\
&= -\sum_{x,y} p(x,y) \log p_1(x) - \sum_{x,y} p(x,y) \log \frac{p(x,y)}{p_1(x)} \\
&= \mathsf{H}(\Omega_1) + \mathsf{H}(\Omega_2 \mid \Omega_1).
\end{aligned}$$

Since conditional entropy satisfies

$$\mathsf{H}(\Omega_2 \mid \Omega_1) \leq \mathsf{H}(\Omega_2),$$

we obtain the stated inequality. Equality holds precisely when $p(x, y) = p_1(x)p_2(y)$, i.e., when Ω_1 and Ω_2 are independent. □

9.2 The Category EDSMC and Initiality of SP

Definition 9.4 (EDSMC)**.** The category **EDSMC** of *entropy-decreasing symmetric monoidal categories* has:

- *Objects:* tuples $(\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}}, \mathsf{H}_{\mathcal{C}}, P, R, B, C)$ where $(\mathcal{C}, \otimes_{\mathcal{C}}, I_{\mathcal{C}})$ is a strict symmetric monoidal category, $\mathsf{H}_{\mathcal{C}} : \mathrm{Ob}(\mathcal{C}) \to \mathbb{R}_{\geq 0}$ is monotone along all morphisms, and P, R, B, C are families of morphisms in $\mathcal{C}$ satisfying entropy axioms analogous to those of $\mathrm{Pop}, \mathrm{Ref}, \mathrm{Bind}, \mathrm{Col}$ in **SP**.

- *Morphisms:* strict symmetric monoidal functors $\Phi : \mathcal{C} \to \mathcal{D}$ that preserve H (i.e. $\mathsf{H}_{\mathcal{D}}(\Phi(X)) = \mathsf{H}_{\mathcal{C}}(X)$ for all X) and map generators to generators ($\Phi(P_{\mathcal{C}}) \subseteq P_{\mathcal{D}}$, etc.).

Theorem 9.5 (Initiality of **SP**)**.** **SP** *is the initial object in* **EDSMC***: for every* $(\mathcal{C}, \ldots) \in$ **EDSMC***, there exists a unique morphism* $F_{\mathcal{C}} : \mathbf{SP} \to \mathcal{C}$ *in* **EDSMC***.*

Proof. Existence. We construct $F_{\mathcal{C}}$ explicitly. On objects: send an option-space $(\Omega, \mathcal{A})$ to the image of the initial construction in $\mathcal{C}$. Since **SP** is freely generated (Definition 7.5), a morphism out of **SP** is determined by its values on generators, subject to the relations in Definition 7.5. Define $F_{\mathcal{C}}(\mathrm{Pop}_U) = P_U^{(\mathcal{C})}$ (the corresponding P-family morphism in $\mathcal{C}$), and similarly for $\mathrm{Ref}, \mathrm{Bind}, \mathrm{Col}$. The relations (a)–(d) of Definition 7.5 hold in $\mathcal{C}$ by hypothesis (since $\mathcal{C} \in$ **EDSMC**), so $F_{\mathcal{C}}$ is well-defined on all morphisms. $F_{\mathcal{C}}$ is strict symmetric monoidal because **SP**'s monoidal structure is free and $\mathcal{C}$'s monoidal structure satisfies the same strict laws. Entropy preservation: $\mathsf{H}_{\mathcal{C}}(F_{\mathcal{C}}(\Omega)) = \mathsf{H}(\Omega)$ by construction (the H-functional in $\mathcal{C}$ is required to agree with the abstract entropy assigned to each generator's domain and codomain).

Uniqueness. Any morphism $G : \mathbf{SP} \to \mathcal{C}$ in **EDSMC** must send generators to generators and preserve composition. Since **SP** is freely generated, G is completely determined by its values on generators. The generator values are uniquely forced by the type constraints: $G(\mathrm{Pop}_U)$ must be a P-family morphism at $G(\Omega, \mathcal{A})$, and there is exactly one such morphism for each elimination U by the hypothesis that $\mathcal{C}$'s generators satisfy the same axioms as **SP**'s. Hence $G = F_{\mathcal{C}}$. □

Remark 9.6. Theorem 9.5 upgrades **SP** from "a constructed category" to "the initial object in the class of entropy-decreasing systems." Every such system receives a canonical map from **SP**, and this map is unique. The construction of the realization functor $F : \mathbf{SP} \to \mathbf{RSVP}$ (Section 12) is a special case of this universal property.

Summary. **SP** carries a strict symmetric monoidal structure independent of its sequential composition, with entropy subadditivity under tensor and strict decrease under composition. It is the initial object in **EDSMC**: the free entropy-decreasing rewriting category.

Theorem 9.7 (Structural asymmetry)**.** *There exists no functor $G : \mathbf{RSVP} \to \mathbf{SP}$ such that $G \circ F$ is naturally isomorphic to* $\mathrm{id}_{\mathbf{SP}}$*. Equivalently, F admits no right-inverse up to natural isomorphism, and in particular F is not part of an equivalence of categories.*

Sketch. The functor F forgets geometric degrees of freedom by passing from smooth field data to discrete elimination histories. Distinct smooth coupling structures can induce the same discrete history and the same entropy-weight profile, so F is not faithful on the relevant substructures. A natural right-inverse would, by composition, reconstruct the forgotten geometric information from the discrete data in a way stable under morphisms, contradicting the existence of nontrivial diffeomorphism classes and coupling deformations that preserve the induced entropy integrals while varying the smooth structure. □

10 Meld and Sheaf-Theoretic Worldhood

10.1 Presheaf of Local Proposals

Let $(\Omega, \mathcal{A})$ be an option-space and $\mathcal{U} = \{U_i\}_{i \in I}$ a cover of Ω by admissible subsets. The *site* $\mathcal{U}$ has the induced admissibility structure as its topology (finite intersections are admissible).

Definition 10.1 (Presheaf of proposals)**.** Define a presheaf $\mathcal{T} : \mathcal{U}^{\mathrm{op}} \to \mathbf{Set}$ by:

$$\mathcal{T}(U) = \{\text{admissible event proposals with support in } U\},$$

with restriction maps $\rho_{UV} : \mathcal{T}(U) \to \mathcal{T}(V)$ for $V \subseteq U$ given by restricting a proposal to the smaller scope.

A family $\{\delta_i \in \mathcal{T}(U_i)\}_{i \in I}$ is *compatible on overlaps* if $\rho_{U_i, U_i \cap U_j}(\delta_i) = \rho_{U_j, U_i \cap U_j}(\delta_j)$ for all $i, j \in I$.

Definition 10.2 (Worldhood)**.** $(\Omega, \mathcal{A})$ *exhibits worldhood* with respect to $\mathcal{U}$ if $\mathcal{T}$ is a sheaf: every compatible family $\{\delta_i\}$ determines a unique global section $\Delta \in \mathcal{T}(\Omega)$ with $\rho_{\Omega, U_i}(\Delta) = \delta_i$.

Worldhood is the mathematical content of the claim that locally coherent commitments can be uniquely assembled into a globally coherent history. Before sufficient commitment, $\mathcal{T}$ is a presheaf (local sections may fail to glue). Each Pop, Bind, or Refuse event tightens constraints and moves $\mathcal{T}$ closer to the sheaf condition.

10.2 Meld as Policy-Induced Sheafification

Fix a merge policy π, which specifies an equivalence relation $\equiv_\pi$ on proposal data. Let $R_\pi \rightrightarrows \mathcal{T}$ be the corresponding presheaf equivalence relation. Denote by $q_\pi : \mathcal{T} \to \mathcal{T}/{\equiv_\pi}$ the pointwise quotient presheaf.

Assume the site is subcanonical (which holds in the admissible-cover setting, since representable presheaves are sheaves). Let $a : \mathbf{PSh} \to \mathbf{Sh}$ be sheafification, left adjoint to the inclusion $i : \mathbf{Sh} \hookrightarrow \mathbf{PSh}$.

Definition 10.3 (Policy sheafification)**.** Define $a_\pi(\mathcal{T}) := a(\mathcal{T}/{\equiv_\pi})$ and let $\eta_\pi : \mathcal{T} \to i\, a_\pi(\mathcal{T})$ be the composite of the quotient q_π with the unit η of the adjunction.

Theorem 10.4 (Universal property of policy-induced sheafification)**.** *For every sheaf $\mathcal{S} \in \mathbf{Sh}(\mathcal{U})$, composition with η_π gives a natural bijection:*

$$\mathrm{Hom}_{\mathbf{Sh}}(a_\pi(\mathcal{T}), \mathcal{S}) \;\cong\; \{f \in \mathrm{Hom}_{\mathbf{PSh}}(\mathcal{T}, i\mathcal{S}) : f \textit{ is } \pi\textit{-invariant}\},$$

where π-invariant means f coequalizes $R_\pi \rightrightarrows \mathcal{T}$ (constant on $\equiv_\pi$-classes in each section, stably under restriction). Equivalently, $a_\pi(\mathcal{T})$ is initial among sheaves receiving a π-invariant map from $\mathcal{T}$.

Proof. Step 1. By the universal property of coequalizers in **PSh**: giving a π-invariant presheaf morphism $f : \mathcal{T} \to i\mathcal{S}$ is equivalent to giving a unique presheaf morphism $f' : \mathcal{T}/{\equiv_\pi} \to i\mathcal{S}$ with $f' \circ q_\pi = f$.

Step 2. By the sheafification adjunction $a \dashv i$: $\mathrm{Hom}_{\mathbf{Sh}}(a(\mathcal{P}), \mathcal{S}) \cong \mathrm{Hom}_{\mathbf{PSh}}(\mathcal{P}, i\mathcal{S})$, naturally in $\mathcal{P}$ and $\mathcal{S}$. Apply with $\mathcal{P} = \mathcal{T}/{\equiv_\pi}$: $\mathrm{Hom}_{\mathbf{Sh}}(a_\pi(\mathcal{T}), \mathcal{S}) \cong \mathrm{Hom}_{\mathbf{PSh}}(\mathcal{T}/{\equiv_\pi}, i\mathcal{S})$.

Step 3. Composing the bijections of Steps 1 and 2: $\mathrm{Hom}_{\mathbf{Sh}}(a_\pi(\mathcal{T}), \mathcal{S}) \cong$ π-invariant maps $\mathcal{T} \to i\mathcal{S}$. The composite unit $\eta_\pi = \eta \circ q_\pi$ is the universal π-invariant map.

Naturality in $\mathcal{S}$ is inherited from naturality of coequalizer factorization and of the adjunction bijection.

Uniqueness of factorization. Given any sheaf morphism $\bar{f} : a_\pi(\mathcal{T}) \to \mathcal{S}$, the condition $i(\bar{f}) \circ \eta_\pi = f$ determines $\bar{f}$ uniquely by the adjunction (the unit $\eta_{\mathcal{T}/\equiv_\pi}$ is initial). □

Definition 10.5 (Meld)**.** Given histories H_1, H_2 from a common prefix Ω_0 and a merge policy π, the meld $\mathrm{Meld}_\pi(H_1, H_2)$ is the irreversible event in **SP** that commits to the policy-sheafified global section $\Delta^\pi \in a_\pi(\mathcal{T})(\Omega)$ as the authoritative continuation. The operation decomposes canonically:

$$\mathrm{Meld}_\pi \;=\; \mathrm{glue} \circ \mathrm{Col}_\pi,$$

where Col_π removes the overlap obstructions (the collapse step applying $\equiv_\pi$) and glue produces the unique global section (the sheafification step).

Remark 10.6. Declaring the global section Δ^π authoritative is not additional mathematical content within the sheaf category; it is an event in **SP** (specifically a Col_π event). The mathematics of Theorem 10.4 shows what the post-meld option-space looks like; the commitment to that option-space is the Spherepop-side contribution.

Summary. Worldhood is the sheaf condition on local event proposals: it is achieved when commitments are sufficient for local data to glue globally. Meld is the categorical operation that enforces this by policy-induced sheafification. The theorem gives it a precise universal property: it is the initial sheaf receiving a π-invariant map from the presheaf of proposals.

11 The Category RSVP

11.1 Objects

Definition 11.1 (RSVP state)**.** An *RSVP state* is a quadruple (M, Φ, v, S) where M is a smooth, connected, oriented Riemannian manifold; $\Phi \in C^\infty(M)$ is the *scalar coherence potential*; $v \in \mathfrak{X}(M)$ is a smooth vector field; and $S \in C^\infty(M, \mathbb{R}_{\geq 0})$ is the *entropy density.* The fields may satisfy the entropy-transport equation $\partial_t S = \nabla \cdot (\kappa \nabla \Phi)$ for $\kappa > 0$, but this is a structural constraint on the dynamics, not a defining condition on objects.

11.2 Morphisms with Slack Witnesses

Definition 11.2 (RSVP morphism)**.** A morphism $(M, \Phi, v, S) \to (M', \Phi', v', S')$ in **RSVP** is a pair (φ, η) where $\varphi : M \to M'$ is smooth and $\eta \in C^\infty(M, \mathbb{R}_{\geq 0})$ is the *entropy-slack witness* satisfying $\varphi^* S' = S + \eta$ pointwise.

Lemma 11.3 (Identity morphisms)**.** $(\mathrm{id}_M, 0) : (M, \Phi, v, S) \to (M, \Phi, v, S)$ *is a morphism in* **RSVP***.*

Proof. $\mathrm{id}_M^* S = S = S + 0$. □

Lemma 11.4 (Closure under composition)**.** *If* $(\varphi_1, \eta_1) : (M_1, \ldots) \to (M_2, \ldots)$ *and* $(\varphi_2, \eta_2) : (M_2, \ldots) \to (M_3, \ldots)$ *are morphisms in* **RSVP***, then*

$$(\varphi_2 \circ \varphi_1,\ \eta_1 + \varphi_1^* \eta_2) : (M_1, \ldots) \to (M_3, \ldots)$$

is a morphism in **RSVP***.*

Proof. $(\varphi_2 \circ \varphi_1)^* S_3 = \varphi_1^*(\varphi_2^* S_3) = \varphi_1^*(S_2 + \eta_2) = \varphi_1^* S_2 + \varphi_1^* \eta_2 = (S_1 + \eta_1) + \varphi_1^* \eta_2 = S_1 + (\eta_1 + \varphi_1^* \eta_2)$. Non-negativity: $\eta_1 \geq 0$, $\eta_2 \geq 0$, pullback preserves non-negativity. □

Lemma 11.5 (Associativity of composition)**.** *Composition in* **RSVP** *is associative.*

Proof. $(\varphi_3 \circ (\varphi_2 \circ \varphi_1),\ \eta_1 + \varphi_1^* \eta_2 + \varphi_1^* \varphi_2^* \eta_3) = ((\varphi_3 \circ \varphi_2) \circ \varphi_1,\ \eta_1 + \varphi_1^*(\eta_2 + \varphi_2^* \eta_3))$ since smooth map composition is associative and φ_1^* is additive. □

Proposition 11.6 (**RSVP** is a category)**.** **RSVP** *with composition defined by Lemma 11.4 is a well-defined category.*

Proof. Identity morphisms exist by Lemma 11.3. Composition is closed (Lemma 11.4) and associative (Lemma 11.5). Left and right unit laws: $(\mathrm{id}, 0) \circ (\varphi, \eta) = (\varphi, \eta + \varphi^* 0) = (\varphi, \eta)$; $(\varphi, \eta) \circ (\mathrm{id}, 0) = (\varphi, 0 + \varphi^*|_{\mathrm{id}} \eta) = (\varphi, \eta)$. □

Summary. **RSVP** is a well-defined category with objects being smooth entropy-carrying manifolds and morphisms being smooth maps with explicit non-negative entropy-slack witnesses. The slack data resolve the earlier ambiguity about "preservation up to boundary terms" by making the entropy production at each morphism explicit.

12 The Realization Functor $F : \mathbf{SP} \to \mathbf{RSVP}$

12.1 On Objects

Definition 12.1 (Geometric realization on objects)**.** For an option-space $(\Omega, \mathcal{A})$, define

$$F(\Omega, \mathcal{A}) := (\Delta(\Omega),\ \Phi_\Omega,\ v_\Omega,\ S_\Omega),$$

where $\Delta(\Omega)$ is the *probability simplex* $\{p \in \mathbb{R}^\Omega_{\geq 0} : \sum_x p(x) = 1\}$; $S_\Omega(p) = -\sum_{x \in \Omega} p(x) \log p(x)$ is the Shannon entropy density at the interior of $\Delta(\Omega)$; Φ_Ω and v_Ω are smooth interpolants of the admissibility structure, defined by a fixed smooth embedding of $\mathcal{A}$ into $C^\infty(\Delta(\Omega))$.

12.2 On Generating Morphisms

***F* on Pop.** Let $\mathrm{Pop}_U : (\Omega, \mathcal{A}) \to (\Omega \setminus U, \cdot)$.

Definition 12.2 (Pop realization)**.** Define $F(\mathrm{Pop}_U) = (\iota_U, \eta_U)$ where $\iota_U : \Delta(\Omega \setminus U) \hookrightarrow \Delta(\Omega)$ is the face inclusion (restriction to the face corresponding to $\Omega \setminus U$) and $\eta_U \in C^\infty(\Delta(\Omega \setminus U), \mathbb{R}_{\geq 0})$ is a bump function supported near $\partial\Delta(U)$, normalized so that $\iota_U^* S_\Omega = S_{\Omega \setminus U} + \eta_U$.

The bump η_U records the entropy difference between the full simplex and the restricted face. It is the *boundary-sharpening slack*: the entropy production localized at the boundary of the eliminated region.

***F* on Refuse.** $F(\mathrm{Ref}_U) = F(\mathrm{Pop}_U)$ as a smooth map, with the same slack η_U. The accounting tag is preserved by $\mathcal{B} : \mathbf{SP} \to \mathbf{Set}$ and does not affect the field data.

***F* on Bind.** Let $\mathrm{Bind}_{ij} : (\Omega, \mathcal{A}) \to (\Omega, \mathcal{A}')$.

Definition 12.3 (Bind realization)**.** Define $F(\mathrm{Bind}_{ij}) = (\mathrm{id}_{\Delta(\Omega)}, 0)$ with the modification $v_\Omega \mapsto v_\Omega + \delta v_{ij}$, where δv_{ij} is a smooth directed vector field increment coupling the regions $\Delta(\{i\})$ and $\Delta(\{j\})$ in $\Delta(\Omega)$.

Entropy slack is zero ($\eta = 0$) since Bind preserves entropy globally. The coupling δv_{ij} breaks the rotational symmetry between the i- and j-directions of the simplex, realizing the admissibility reduction $\mathcal{A}' \subsetneq \mathcal{A}$ as a directional bias in the flow.

F on Collapse. Let $\mathrm{Col}_\sim : (\Omega, \mathcal{A}) \to (\Omega/\sim, \mathcal{A}_\sim)$.

Definition 12.4 (Collapse realization)**.** Define $F(\mathrm{Col}_\sim) = (\varphi_\sim, \eta_\sim)$ where $\varphi_\sim : \Delta(\Omega) \to \Delta(\Omega/\sim)$ is the coarse-graining map (marginalizing over equivalence classes: $\varphi_\sim(p)([x]) := \sum_{y\sim x} p(y)$) and $\eta_\sim \in C^\infty(\Delta(\Omega), \mathbb{R}_{\geq 0})$ is the fine-to-coarse entropy difference $\eta_\sim(p) = S_\Omega(p) - \varphi_\sim^* S_{\Omega/\sim}(p) \geq 0$ (non-negative by Jensen's inequality applied to the logarithm).

This coarse-graining is a Wilsonian renormalization step: fine-grained distinctions within each equivalence class of $\sim$ are integrated out. The slack $\eta_\sim$ records the entropy of the intra-class distribution.

12.3 Functoriality

Lemma 12.5 (F preserves identities)**.** $F(\mathrm{id}_{(\Omega,\mathcal{A})}) = (\mathrm{id}_{\Delta(\Omega)}, 0) = \mathrm{id}_{F(\Omega,\mathcal{A})}$.

Proof. The identity history is the empty sequence. Its realization is $(\mathrm{id}, 0)$ by Definition 12.1: the simplex is unchanged, no slack is produced. □

Lemma 12.6 (F preserves generator composition)**.** *For composable generators e_1 and e_2: $F(e_2 \circ e_1) = F(e_2) \circ F(e_1)$ with slacks composed additively per Lemma 11.4.*

Proof. We check each combination. The key case is $\mathrm{Pop}_V \circ \mathrm{Pop}_U$ (elimination of U then $V \subseteq \Omega \setminus U$): $F(\mathrm{Pop}_V) \circ F(\mathrm{Pop}_U) = (\iota_V, \eta_V) \circ (\iota_U, \eta_U) = (\iota_V \circ \iota_U, \eta_U + \iota_U^* \eta_V)$. On the other side, $F(\mathrm{Pop}_{U\cup V}) = (\iota_{U\cup V}, \eta_{U\cup V})$ where $\iota_{U\cup V} = \iota_V \circ \iota_U$ (composition of face inclusions) and $\eta_{U\cup V}(p) = \eta_U(p) + \eta_V(\iota_U(p)) = \eta_U + \iota_U^* \eta_V$. The cases Bind $\circ$ Pop, Col $\circ$ Pop, and combinations thereof follow analogously from the locality of each perturbation and the functoriality of pullback. □

Lemma 12.7 (Slack composition is associative)**.** *For three composable morphisms $(\varphi_1, \eta_1), (\varphi_2, \eta_2), (\varphi_3, \eta_3)$: $(\eta_1 + \varphi_1^* \eta_2) + \varphi_1^* \varphi_2^* \eta_3 = \eta_1 + \varphi_1^*(\eta_2 + \varphi_2^* \eta_3)$.*

Proof. By distributivity of φ_1^* over addition and functoriality: $\varphi_1^*(\eta_2 + \varphi_2^* \eta_3) = \varphi_1^* \eta_2 + \varphi_1^* \varphi_2^* \eta_3$. □

Lemma 12.8 (Tensor compatibility)**.** $F((\Omega_1, \mathcal{A}_1) \otimes (\Omega_2, \mathcal{A}_2)) \cong F(\Omega_1, \mathcal{A}_1) \otimes_{\mathbf{RSVP}} F(\Omega_2, \mathcal{A}_2)$*, where the tensor in* **RSVP** *is the product of smooth manifolds with product fields.*

Proof. $\Delta(\Omega_1 \times \Omega_2) \cong \Delta(\Omega_1) \times \Delta(\Omega_2)$ as smooth manifolds (a probability measure on a product is a joint measure; the simplex over the product is the product of simplices). Shannon entropy of the product is the sum of marginal entropies in the independent case, matching subadditivity. □

Theorem 12.9 (Functoriality of F)**.** *$F : \mathbf{SP} \to \mathbf{RSVP}$ is a well-defined symmetric monoidal functor. Specifically:*

(a) F preserves identities (Lemma 12.5);

(b) F preserves composition, with slacks composed additively (Lemma 12.6);

(c) Slack composition is associative (Lemma 12.7);

(d) F is symmetric monoidal (Lemma 12.8);

(e) Entropy compatibility: $\int_{\Delta(\Omega)} S_\Omega \, d\mathrm{vol} = \mathsf{H}(\Omega)$ up to normalization, so entropy decrease in **SP** *corresponds to net positive slack in* **RSVP***.*

Proof. Parts (a)–(d) follow from the four lemmas above. For (e): by construction, $S_\Omega(p) = -\sum_x p(x) \log p(x)$. Integrating over the uniform measure on $\Delta(\Omega)$ yields $\mathsf{H}(\Omega)$ (up to the normalization factor of the simplex volume, which is absorbed into the definition of the volume form). Entropy decrease $\mathsf{H}(\Omega') < \mathsf{H}(\Omega)$ after Pop_U corresponds to $\eta_U > 0$ being the positive slack that records the entropy difference between S_Ω and $\iota_U^* S_\Omega$. □

Commutative diagram.

$$\begin{array}{ccc} (\Omega, \mathcal{A}) & \xrightarrow{\quad e \quad} & (\Omega', \mathcal{A}') \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F} \\ (\Delta(\Omega), \Phi, v, S) & \xrightarrow[(\varphi,\eta)]{} & (\Delta(\Omega'), \Phi', v', S') \end{array}$$

13 Initiality of SP: Mac Lane Presentation

We refactor Theorem 9.5 (initiality of **SP** in **EDSMC**) in the style of Mac Lane's treatment of free monoidal categories (Mac Lane, 1998). The benefit is a clean separation between the *term model* (syntax), the *congruence* (axioms), and the *evaluation functor* (semantics), making uniqueness immediate without appeal to "exactly one generator per type."

13.1 The Term Model $\mathcal{T}$

Definition 13.1 (Term model)**.** Let Ob be the class of all option-spaces. Define the *term model* $\mathcal{T}$ as follows.

- *Objects.* Formal tensor-products of elements of Ob under a binary operation $\otimes$ and a unit symbol I: the free commutative monoid generated by Ob.
- *Morphisms.* Formal expressions built from:
  - identity symbols id_X for each object X;
  - generator symbols Pop_U, Ref_U for each nonempty $U \subsetneq \Omega$;
  - generator symbols Bind_{ij} for each valid dependency pair;
  - generator symbols $\mathsf{Col}_\sim$ for each causally admissible $\sim$;

  closed under composition $\circ$ and tensor $\otimes$ with sources and targets determined by the set-theoretic typing already given in Definitions 7.1–7.4.

$\mathcal{T}$ is a strict symmetric monoidal category (the free such category on these generators and types, before imposing any axioms beyond typing).

13.2 The Congruence $\equiv$

Definition 13.2 (Presentation congruence)**.** Let $\equiv$ be the smallest congruence on morphisms of $\mathcal{T}$ containing:

(a) the strict symmetric monoidal axioms (associativity, unit, symmetry, interchange law);

(b) *Pop-idempotence:* $\mathsf{Pop}_U \circ \mathsf{Pop}_V \equiv \mathsf{Pop}_{U\cup V}$ when $U \cap (\Omega \setminus V) = \emptyset$;

(c) *Independent-bind commutativity:* $\mathsf{Bind}_{ij} \circ \mathsf{Bind}_{kl} \equiv \mathsf{Bind}_{kl} \circ \mathsf{Bind}_{ij}$ when $\{i, j\} \cap \{k, l\} = \emptyset$;

(d) *Entropy monotonicity:* any morphism f with $\mathsf{H}(\operatorname{cod} f) > \mathsf{H}(\operatorname{dom} f)$ is identified with the empty class (i.e., such expressions are inadmissible and removed from $\mathcal{T}$).

Proposition 13.3. **SP** $\cong \mathcal{T}/{\equiv}$ *as strict symmetric monoidal categories.*

Proof. The quotient $\mathcal{T}/{\equiv}$ has the same objects and generating morphisms as the category defined in Section 7, modulo exactly the relations of Definition 7.5. The identification is the identity on generators. □

13.3 Evaluation Functors and Initiality

Definition 13.4 (EDSMC (revised))**.** An object of **EDSMC** is a strict symmetric monoidal category $\mathcal{C}$ together with distinguished families of morphisms $\mathbf{P}, \mathbf{R}, \mathbf{B}, \mathbf{C}$ interpreting the four generator symbols, and a monotone functional $\mathsf{H}_{\mathcal{C}} : \mathrm{Ob}(\mathcal{C}) \to \mathbb{R}_{\geq 0}$, such that the relations (a)–(d) of Definition 13.2 hold (with generator families substituted for generator symbols). A morphism in **EDSMC** is a strict symmetric monoidal functor that maps each generator family to the corresponding generator family and satisfies $\mathsf{H}_{\mathcal{C}}(X) = \mathsf{H}_{\mathcal{D}}(\Phi(X))$ for all X.

Theorem 13.5 (Initiality, Mac Lane presentation)**.** $\mathbf{SP} = \mathcal{T}/\equiv$ *is the initial object in* **EDSMC***: for every* $\mathcal{C} \in \mathbf{EDSMC}$ *there is a unique morphism* $\Phi_{\mathcal{C}} : \mathbf{SP} \to \mathcal{C}$ *in* **EDSMC**.

Proof. Existence. Define $\Phi_{\mathcal{C}}$ on generators by

$$\Phi_{\mathcal{C}}(\mathsf{Pop}_U) := P_U^{(\mathcal{C})}, \quad \Phi_{\mathcal{C}}(\mathsf{Ref}_U) := R_U^{(\mathcal{C})}, \quad \Phi_{\mathcal{C}}(\mathsf{Bind}_{ij}) := B_{ij}^{(\mathcal{C})}, \quad \Phi_{\mathcal{C}}(\mathsf{Col}_{\sim}) := C_{\sim}^{(\mathcal{C})}.$$

Extend by strict monoidality. Well-definedness on $\mathcal{T}/\equiv$: the relations (a)–(d) hold in $\mathcal{C}$ by hypothesis, so $\Phi_{\mathcal{C}}$ respects $\equiv$.

Uniqueness. Any morphism $\Psi : \mathbf{SP} \to \mathcal{C}$ in **EDSMC** must send each generator family to the corresponding family in $\mathcal{C}$ (by the definition of **EDSMC**-morphisms) and be strictly monoidal. Since **SP** is generated by the four families under composition and tensor, Ψ is uniquely determined by these values. Hence $\Psi = \Phi_{\mathcal{C}}$. □

Remark 13.6. Theorem 13.5 supersedes Theorem 9.5 with a cleaner proof structure. Uniqueness no longer requires the "exactly one generator per elimination" argument; it follows immediately from the fact that **EDSMC**-morphisms are required to preserve generator families, and **SP** is freely generated by those families. The realization functor $F : \mathbf{SP} \to \mathbf{RSVP}$ (Section 12) is the special case $\mathcal{C} = \mathbf{RSVP}$ with generator families $\mathbf{P} = \{(\iota_U, \eta_U)\}$, $\mathbf{B} = \{(\mathrm{id}, \delta v_{ij})\}$, $\mathbf{C} = \{(\varphi_{\sim}, \eta_{\sim})\}$.

Structural asymmetry theorem.

Theorem 13.7 (Structural asymmetry)**.** **SP** *and* **RSVP** *are related by* F *but are not equivalent categories. The asymmetry is:*

(i) **SP** *is discrete and combinatorial;* **RSVP** *is smooth and differential.*

(ii) **SP** accumulates constraint*: each morphism adds an irreversible record; no morphism in* **SP** *increases entropy.*

(iii) **RSVP** redistributes coherence*: the entropy-transport equation* $\partial_t S = \nabla \cdot (\kappa \nabla \Phi)$ *diffuses entropy across* M*; local concentrations propagate globally.*

(iv) F *is not an equivalence of categories: there exist morphisms in* **RSVP** *that are not in the image of* F *(smooth maps that do not correspond to any finite composition of the four generators).*

Proof. (i) **SP** has discrete objects (finite option-spaces); **RSVP** has smooth manifolds. (ii) By Axiom 2, all morphisms in **SP** are entropy-monotone non-increasing. (iii) The PDE $\partial_t S = \nabla \cdot (\kappa \nabla \Phi)$ is parabolic; it redistributes entropy according to the gradient of Φ, potentially increasing S locally (where $\nabla \cdot (\kappa \nabla \Phi) > 0$) while decreasing it elsewhere. This is not monotone. (iv) The image of F consists of restriction maps, vector field additions, and coarse-graining maps of the prescribed forms. A generic smooth map $\varphi : M \to M'$ need not have this structure. □

14 Worked Example: A Four-Event History

We trace a complete history through **SP** and **RSVP**.

14.1 Setup

Let $\Omega_0 = \{a, b, c, d\}$ with uniform probability and trivial admissibility $\mathcal{A}_0 = 2^{\Omega_0}$. Initial entropy: $\mathsf{H}(\Omega_0) = \log 4 = 2 \log 2$. Initial optionality: $\mathsf{Opt}(\Omega_0) = 2 \log 2$. Under F: the probability simplex $\Delta(\Omega_0)$ is a regular 3-simplex with vertices e_a, e_b, e_c, e_d. Initial entropy density $S(p) = -\sum_x p(x) \log p(x)$ achieves maximum $\log 4$ at the centroid.

14.2 Event 1: Pop

$e_1 = \mathrm{Pop}_{\{c,d\}} : (\Omega_0, \mathcal{A}_0) \to (\Omega_1, \mathcal{A}_1)$ where $\Omega_1 = \{a, b\}$ and $\mathcal{A}_1 = \{\{a, b\}, \{a\}, \{b\}, \emptyset\}$.

In SP. $\mathsf{H}(\Omega_1) = \log 2$. Entropy decrease: $\Delta\mathsf{H} = \log 4 - \log 2 = \log 2$. Optionality: $\mathsf{Opt}(\Omega_1) = \log 2$. Action increment: $L_1 = \log 2$.

In RSVP. $F(e_1) = (\iota_{\{c,d\}}, \eta_{\{c,d\}})$. The simplex $\Delta(\Omega_1)$ is the edge $[e_a, e_b]$ of the original tetrahedron. The slack function $\eta_{\{c,d\}}$ is a smooth bump on $[e_a, e_b]$ that is positive near the boundary points e_a and e_b and equals $S_{\Omega_0}(\iota(p)) - S_{\Omega_1}(p)$ pointwise, which is the entropy of the eliminated mass on $\{c, d\}$. Geometrically:

the coherence potential Φ sharpens at the boundary of the eliminated region; S decreases from its maximum $\log 4$ toward $\log 2$.

14.3 Event 2: Bind

$e_2 = \mathrm{Bind}_{ab} : (\Omega_1, \mathcal{A}_1) \to (\Omega_2, \mathcal{A}_2)$ with $\mathcal{A}_2 = \{\{a,b\}, \{a\}, \emptyset\}$ (every admissible future containing b must contain a).

In SP. $\mathsf{H}(\Omega_2) = \log 2 = \mathsf{H}(\Omega_1)$: entropy preserved. Optionality unchanged: $\mathsf{Opt}(\Omega_2) = \log 2$. Action increment: $L_2 = 0$. But the causal preorder is now $a \preceq b$ (causally, a is necessary for b). The symmetry group of $(\Omega_2, \mathcal{A}_2)$ has been reduced: the two elements a, b are no longer equivalent under admissibility-preserving automorphisms.

In RSVP. $F(e_2) = (\mathrm{id}_{[e_a, e_b]}, 0)$ with $v \mapsto v + \delta v_{ab}$, where δv_{ab} is a smooth vector field on $[e_a, e_b]$ directed from e_a toward e_b. This represents the directed coupling: probability flow is biased toward b (the dependent element) conditional on a (the prerequisite). The entropy density S is unchanged; the coherence structure v is modified.

14.4 Event 3: Renaming (not a Collapse)

We consider $e_3 = \mathrm{Col}_{a \sim b}$ under the equivalence $a \sim b$. Causal admissibility fails: under the causal preorder induced by Event 2, we have $\downarrow a = \{a\}$ while $\downarrow b = \{a, b\}$, hence $\downarrow a \neq \downarrow b$ and Definition 8.5 forbids this collapse.

Remark 14.1. This is a genuine failure mode: collapsing after binding can violate causal admissibility, because binding changes causal pasts. The allowed repairs within the current generator set are to apply the intended collapse before the binding step, or to choose a different causally admissible collapse policy. If one wishes to model an explicit "authority relabeling" that introduces fresh names, that operation is not a quotient and should be added as a separate primitive (or treated as an isomorphism) rather than being folded into Col.

Accordingly, we take e_3 to be a harmless relabeling isomorphism $\rho : \Omega_2 \to \Omega_3$ with $\Omega_2 = \{a, b\}$ and $\Omega_3 = \{a, [b]\}$, defined by $\rho(a) = a$ and $\rho(b) = [b]$, and with admissibility transported along ρ so that $\mathcal{A}_3 = \{\rho(A) : A \in \mathcal{A}_2\}$.

In SP. Entropy and optionality are unchanged: $\mathsf{H}(\Omega_3) = \mathsf{H}(\Omega_2)$ and $L_3 = 0$. The step exists only to make the later notation explicit; it performs no quotienting and should not be confused with Col.

In RSVP. $F(\rho)$ is the identity on the underlying simplex up to the canonical identification induced by the relabeling of vertices, and the slack remains zero.

14.5 Event 4: Pop to Terminal

$e_4 = \mathrm{Pop}_{\{[b]\}} : (\Omega_3, \mathcal{A}_3) \to (\Omega_4, \mathcal{A}_4)$ where $\Omega_4 = \{a\}$.

In SP. $\mathsf{H}(\Omega_4) = 0$. Entropy decrease: $\Delta\mathsf{H} = \log 2$. Action increment: $L_4 = \log 2$. The world is now fully committed: a single future a remains. Total action: $\mathcal{S}[\gamma] = L_1 + L_2 + L_3 + L_4 = \log 2 + 0 + 0 + \log 2 = \log 4 = \mathsf{H}(\Omega_0)$.

In RSVP. $F(e_4) = (\iota_{\{[b]\}}, \eta_{\{[b]\}})$ where ι maps the vertex e_a to the edge endpoint, and η records the final entropy production $\log 2$. The coherence potential Φ concentrates at the vertex e_a; entropy density $S \to 0$ at the vertex. This is the *coherence attractor*: the trajectory has converged to a point.

14.6 Summary of the History

Step	Event	H	L_t	RSVP image
0	—	$\log 4$	—	Centroid of tetrahedron
1	$\mathrm{Pop}_{\{c,d\}}$	$\log 2$	$\log 2$	Restriction to edge $[e_a, e_b]$
2	Bind_{ab}	$\log 2$	0	Directed flow $a \to b$
3	$\mathrm{Col}_{\sim_{\mathrm{triv}}}$	$\log 2$	0	Identity (trivial collapse)
4	$\mathrm{Pop}_{\{[b]\}}$	0	$\log 2$	Convergence to vertex e_a
Total action			$\log 4$	

The total action equals the initial entropy, as expected: a history that fully commits a world of entropy $\log 4$ must pay action $\log 4$.

15 Discrete Mechanics of Commitment

15.1 Lagrangian Formulation

Each event $e_t : X_{t-1} \to X_t$ induces a local cost $L_t = \Delta\mathsf{Opt}_t + \lambda\,\Delta C_t$ where $\Delta\mathsf{Opt}_t = \mathsf{Opt}(X_{t-1}) - \mathsf{Opt}(X_t)$ and ΔC_t records auxiliary accounting costs. The action of a history is $\mathcal{S}[\gamma] = \sum_{t=1}^{n} L_t$.

Proposition 15.1. *Every history with at least one* Pop *or* Ref *event has* $\mathcal{S}[\gamma] > 0$.

Proof. Pop and Ref strictly reduce Opt, contributing $\Delta\mathsf{Opt}_t > 0$. Since all terms are non-negative, $\mathcal{S}[\gamma] > 0$. □

15.2 Collapse Reduces Action

If Col identifies k elements, action reduces by $\log k$ in the Shannon realization: $\mathcal{S} \mapsto \mathcal{S} - \log k$.

This matches the RSVP side: the slack produced by $F(\mathrm{Col}_\sim)$ equals the fine-structure entropy integrated out, which is $\log k$ for a uniform distribution over k identified elements.

15.3 Hamiltonian and Remaining Freedom

Define commitment $\pi_t = -\partial L_t / \partial\mathsf{Opt}$ and Hamiltonian $H_t = \pi_t \mathsf{Opt}(X_t) - L_t$. Then H_t decays monotonically toward zero as commitment accumulates, measuring the remaining flexibility of the system. A fully committed world has $\mathsf{Opt} = 0$ and $H = 0$.

16 Continuum Limit of Commitment

Let $\Delta t \to 0$ and assume optionality is smooth in t. Define

$$\Omega(t + \Delta t) - \Omega(t) = -\alpha \int_{\partial M_t} \nabla\Phi \cdot \mathbf{n}\, dA\, \Delta t.$$

Then

$$\frac{d\Omega}{dt} = -\alpha \int_{\partial M_t} \nabla\Phi \cdot \mathbf{n}\, dA.$$

Using the divergence theorem:

$$\frac{d\Omega}{dt} = -\alpha \int_{M_t} \Delta\Phi\, dV.$$

Comparing with entropy transport ($\partial_t S = \nabla \cdot (\kappa \nabla \Phi)$, $\alpha = \kappa$): optionality decrease is the integral of entropy flux across the boundary—the discrete pop is the boundary flux of coherence.

17 Information-Theoretic Interpretation

Each generator corresponds to a standard information-theoretic operation:

$$\begin{aligned}
\text{Pop}: &\quad \mathsf{H}_{t+1} = \mathsf{H}_t - \Delta, \quad \Delta > 0 \\
\text{Bind}: &\quad I(X;Y) \uparrow \ \text{(mutual information coupling increases)} \\
\text{Col}: &\quad \mathrm{K}(\Omega/\!\sim) \le \mathrm{K}(\Omega) \ \text{(Kolmogorov complexity decreases)} \\
F: &\quad \partial_t S = \nabla \cdot (\kappa \nabla \Phi) \ \text{(entropy-transport PDE)}
\end{aligned}$$

The second relation captures why Bind is entropy-neutral at the distribution level but produces information-theoretic work: it increases the mutual information between the bound elements without changing marginal entropies, encoding a dependency that was not there before.

18 Normalization

Theorem 18.1 (Strong normalization, deterministic fragment)**.** *Every history in* **SP** *without Choice events, and in which every* Bind *step strictly refines admissibility and every* Col *step is nontrivial, terminates.*

Proof. Define the termination measure $\mu(\Omega, \mathcal{A}) = |\Omega| + |\mathcal{A}|$. Each Pop strictly decreases $|\Omega|$. Each Bind strictly decreases $|\mathcal{A}|$ by hypothesis (strict refinement). Each nontrivial Col strictly decreases $|\Omega|$. Hence μ strictly decreases at each step, and since μ is a natural number bounded below, the history must be finite. □

Remark 18.2 (Probabilistic normalization)*.* In the presence of Choice events, almost-sure normalization is conjectured: every well-typed closed probabilistic term terminates with probability 1. The formal proof requires a probabilistic termination measure satisfying the supermartingale condition $\mathbb{E}[\mu_{t+1}] \le \mu_t - \delta$ for some $\delta > 0$. This is plausible under bounded entropy conditions but remains open.

19 Structured Irreversibility

The defining feature of **SP** is not entropy monotonicity alone, but the structural absence of inverses together with resource descent. This section isolates the precise categorical content of this condition.

Definition 19.1 (Entropy preorder)**.** Let $(\mathbf{SP}, \mathsf{H})$ be as defined previously. Define a preorder $\preceq$ on $\mathrm{Ob}(\mathbf{SP})$ by

$$X \preceq Y \iff \exists f : Y \to X.$$

Proposition 19.2. *If* $X \preceq Y$*, then* $\mathsf{H}(X) \leq \mathsf{H}(Y)$*.*

Proof. Immediate from entropy monotonicity along morphisms. □

Thus entropy induces a monotone map from the preorder $(\mathrm{Ob}(\mathbf{SP}), \preceq)$ into $(\mathbb{R}_{\geq 0}, \leq)$.

Definition 19.3 (Structured irreversibility)**.** A symmetric monoidal category $(\mathcal{C}, \otimes, I)$ equipped with $R : \mathrm{Ob}(\mathcal{C}) \to \mathbb{R}_{\geq 0}$ is *structurally irreversible* if:

1. For all non-identity f, there exists no g with $g \circ f = \mathrm{id}$ or $f \circ g = \mathrm{id}$.
2. R is monotone along morphisms.
3. $R(X) = 0$ implies X is minimal in the preorder.

Theorem 19.4. **SP** *is structurally irreversible.*

Proof. (1) is Axiom 1. (2) is Axiom 2. (3) follows from Definition 2.3(i). □

Hence irreversibility in **SP** is not an emergent phenomenon but a categorical constraint. The resource functional does not merely measure behavior; it defines directionality.

20 Irreversibility Across Scales

Let $F : \mathbf{SP} \to \mathbf{RSVP}$ be the symmetric monoidal functor constructed in Section 12. This section formalizes how structural irreversibility is preserved under F.

Proposition 20.1 (Entropy preservation under F). *For any $X \in \mathrm{Ob}(\mathbf{SP})$,*

$$\mathsf{H}(X) = \int_{\Delta(X)} S_X \, d\mu$$

up to normalization.

Proof. By construction of $F(X) = (\Delta(X), S_X)$ with S_X the Shannon entropy density. Integration over the simplex yields the discrete entropy. □

Proposition 20.2 (Slack monotonicity). *For any morphism f in* **SP**,

$$F(f) = (\varphi, \eta)$$

satisfies $\eta \geq 0$ and

$$\int \eta \, d\mu = \mathsf{H}(X) - \mathsf{H}(Y).$$

Proof. By construction of slack witnesses in Section 12. □

Thus entropy descent in **SP** corresponds to non-negative entropy production in **RSVP**.

Theorem 20.3 (Scale invariance of irreversibility). *If f is non-invertible in* **SP**, *then $F(f)$ is non-invertible in* **RSVP**.

Proof. Suppose (φ, η) had an inverse. Then slack would have to vanish and φ be a diffeomorphism. This contradicts strict entropy descent for Pop and Ref. □

Therefore, irreversibility is preserved functorially across discrete and continuous scales.

21 Persistence Without Substance

Identity in **SP** is defined entirely by morphism chains.

Definition 21.1 (Log equivalence). Two histories γ, γ' are equivalent under policy q if

$$q(\gamma(\Omega_0)) = q(\gamma'(\Omega_0)).$$

Proposition 21.2. *Objects of* **SP** *are equivalence classes of histories modulo q.*

Proof. By Axiom **??** and Axiom **??**. □

Hence object identity is induced, not primitive.

Under F, identity corresponds to simplex face inclusion:

Proposition 21.3. *If X_n arises from history γ, then*

$$F(X_n) \subseteq \Delta(\Omega_0)$$

is a face determined uniquely by γ.

Thus persistence corresponds to geometric stabilization rather than substrate invariance.

22 Minimal Commitment

Let D be a target equivalence class. Define

$$\Gamma(D) = \{\gamma : \gamma(\Omega_0) \in D\}.$$

Definition 22.1 (Minimal commitment)**.** A history $\gamma \in \Gamma(D)$ is minimal if

$$S[\gamma] \leq S[\gamma']$$

for all $\gamma' \in \Gamma(D)$.

Proposition 22.2. *Minimal commitment histories maximize sheaf gluing prior to collapse.*

Proof. Collapse reduces action by $\log|\ker q|$. Gluing preserves entropy. Hence minimal action is achieved by postponing collapse until necessary. □

23 Finitude

Theorem 23.1 (Finite descent)**.** *Any strictly entropy-decreasing sequence in* **SP** *terminates.*

Proof. Entropy is bounded below by zero and decreases strictly under Pop/Ref. Hence no infinite strictly decreasing chain exists. □

Corollary 23.2. *The deterministic fragment of* **SP** *is strongly normalizing.*

24 Variational Structure of Irreversibility

Let $\gamma = (e_1, \dots, e_n)$ be a history in **SP**. Recall the action functional:

$$S[\gamma] = \sum_{i=1}^{n} \Delta\mathsf{Opt}_i$$

where $\Delta\mathsf{Opt}_i = \mathsf{H}(\Omega_{i-1}) - \mathsf{H}(\Omega_i)$.

Proposition 24.1. $S[\gamma] = \mathsf{H}(\Omega_0) - \mathsf{H}(\Omega_n)$.

Proof. Telescoping sum. □

Thus the action depends only on endpoints.

Definition 24.2 (Extremal history)**.** A history γ between X and Y is extremal if

$$S[\gamma] = \min_{\gamma' : X \to Y} S[\gamma'].$$

Theorem 24.3 (Endpoint minimality)**.** *A history is extremal iff it performs no redundant collapse before maximal gluing under admissibility constraints.*

Proof. Collapse reduces entropy but may introduce unnecessary identification. Since action depends only on entropy difference, any collapse that does not change H must correspond to non-essential equivalence. Such collapses can be postponed without altering endpoints. □

Hence irreversibility admits a variational characterization: the system selects histories minimizing entropy expenditure subject to structural constraints.

Under the realization functor F, this becomes:

$$S[\gamma] = \int_M \eta_\gamma \, d\mu.$$

Thus discrete action equals integrated entropy slack in **RSVP**.

25 Adjoint Structure of Discrete–Continuous Realization

Let $F : \mathbf{SP} \to \mathbf{RSVP}$ be the realization functor. Define a coarse-graining functor

$$G : \mathbf{RSVP} \to \mathbf{SP}$$

by mapping (M, Φ, v, S) to the partition of M into coherence basins under gradient flow of Φ.

Definition 25.1. $G(M, \Phi, v, S)$ is the option-space whose elements are stable attractor components of the flow induced by v.

Proposition 25.2. *G is entropy non-increasing.*

Theorem 25.3 (Adjoint approximation)**.** *There exists a natural transformation*

$$\eta : \mathrm{id}_{\mathbf{SP}} \Rightarrow G \circ F$$

which is initial among entropy-preserving embeddings.

Sketch. Each Ω embeds into $\Delta(\Omega)$. Coherence basins correspond to vertices. The unit map identifies discrete elements with Dirac measures. Universality follows from the simplex universal property. □

Thus discrete option-spaces are initial among coherence fields whose basins reproduce their combinatorics.

While full adjunction may require restriction to a subcategory of **RSVP**, the construction demonstrates that the discrete–continuous duality is structurally tight.

26 Classification of Entropy-Monotone Irreversible Categories

Let $\mathcal{C}$ be a symmetric monoidal irreversible category equipped with a monotone resource functional R.

Definition 26.1. $\mathcal{C}$ is freely generated by a set $\mathcal{G}$ of primitives if every morphism is a finite tensor-composition of elements of $\mathcal{G}$ subject only to resource monotonicity constraints.

Theorem 26.2 (Initiality of **SP**)**.** *Let $\mathcal{C}$ be a resource-monotone irreversible symmetric monoidal category generated by four primitives satisfying:*

1. *Strict entropy descent,*
2. *Admissibility refinement without entropy change,*
3. *Causal-preserving quotient,*

4. Annotated elimination.

Then there exists a unique symmetric monoidal functor

$$H : \mathbf{SP} \to \mathcal{C}$$

preserving generators and resource functional.

Proof. By free generation of **SP** under entropy axioms and universality of free symmetric monoidal categories. □

Hence **SP** is the initial object in the category of entropy-monotone irreversible symmetric monoidal categories generated by four primitives.

27 The Adjunction $F \dashv G$ via the Simplex-Realization Subcategory

Section 25 introduced a coarse-graining functor $G : \mathbf{RSVP} \to \mathbf{SP}$ defined by coherence basins of gradient flow, and asserted an adjoint approximation. The obstruction to a clean adjunction on all of **RSVP** is that basin-selection is not functorial under general smooth maps. This section repairs the statement by restricting to a reflective subcategory $\mathbf{RSVP}_{\text{simp}} \subseteq \mathbf{RSVP}$ on which G is defined precisely, and proves $G \circ F = \text{id}_{\mathbf{SP}}$ on the nose.

27.1 The Simplex-Realization Subcategory

Definition 27.1 (Realization fields)**.** For each option-space $(\Omega, \mathcal{A})$, let $\mathcal{K}(\Omega, \mathcal{A})$ denote the set of pairs (Φ, v) in $C^\infty(\Delta(\Omega)) \times \mathfrak{X}(\Delta(\Omega))$ of the form produced by the object-construction of F (Definition 12.1): $\Phi = \Phi_\Omega$ and $v = v_\Omega + \sum_{(i,j)\in B} \delta v_{ij}$ for some finite binding set $B \subseteq \Omega^2$. Call such pairs *admissible realization fields.*

Definition 27.2 (Simplex-realization subcategory)**.** Define $\mathbf{RSVP}_{\text{simp}}$ to be the full subcategory of **RSVP** spanned by objects of the form $(\Delta(\Omega), \Phi, v, S_\Omega)$ where $(\Phi, v) \in \mathcal{K}(\Omega, \mathcal{A})$ for some option-space $(\Omega, \mathcal{A})$, with S_Ω the Shannon entropy density. Morphisms in $\mathbf{RSVP}_{\text{simp}}$ are those in **RSVP** generated under composition and tensor by:

(a) face inclusions (ι_U, η_U) realizing Pop_U events;

(b) coarse-graining maps $(\varphi_\sim, \eta_\sim)$ realizing $\text{Col}_\sim$ events;

(c) identity-on-manifold maps $(\mathrm{id}, 0)$ with a single allowed v-increment δv_{ij} realizing Bind_{ij} events.

By construction, F lands in $\mathbf{RSVP}_{\mathrm{simp}}$: the image of every generating morphism of $\mathbf{SP}$ is one of the generators (a), (b), (c) above.

27.2 The Discretization Functor G

Definition 27.3 (Discretization functor)**.** Define $G : \mathbf{RSVP}_{\mathrm{simp}} \to \mathbf{SP}$ as follows.

- *On objects.* Send $(\Delta(\Omega), \Phi, v, S_\Omega)$ to $(\Omega, \mathcal{A})$, where $\mathcal{A}$ is read off from the binding structure recorded in v: specifically, $i \preceq j$ in $\mathcal{A}$ iff the field v contains a δv_{ij} increment.
- *On generating morphisms.* Set $G(\iota_U, \eta_U) := \mathrm{Pop}_U$, $G(\varphi_\sim, \eta_\sim) := \mathrm{Col}_\sim$, and $G(\mathrm{id}, 0 \text{ with } \delta v_{ij}) := \mathrm{Bind}_{ij}$.
- *On composites.* Extend by strict monoidal functoriality.

Proposition 27.4 (G is a well-defined strict symmetric monoidal functor)**.** *$G : \mathbf{RSVP}_{\mathrm{simp}} \to \mathbf{SP}$ is a well-defined strict symmetric monoidal functor.*

Proof. On objects, G is well-defined because each object of $\mathbf{RSVP}_{\mathrm{simp}}$ carries a unique $(\Omega, \mathcal{A})$ by definition. On morphisms, G sends generators to generators by construction, and the defining relations of $\mathbf{SP}$ (Definition 7.5) are satisfied in $\mathbf{RSVP}_{\mathrm{simp}}$: pop-idempotence corresponds to face-inclusion composition; bind-commutativity follows from the commutativity of independent v-increments. Strict monoidality: $G(\Delta(\Omega_1) \times \Delta(\Omega_2)) = G(\Delta(\Omega_1 \times \Omega_2)) = (\Omega_1 \times \Omega_2, \cdot) = (\Omega_1, \cdot) \otimes (\Omega_2, \cdot) = G(\Delta(\Omega_1)) \otimes G(\Delta(\Omega_2))$. □

27.3 The Adjunction

Theorem 27.5 ($G \circ F = \mathrm{id}_{\mathbf{SP}}$ and the unit-counit pair)**.** *(i) $G \circ F = \mathrm{id}_{\mathbf{SP}}$ strictly (not merely up to natural isomorphism).*

(ii) There is a natural transformation $\varepsilon : F \circ G \Rightarrow \mathrm{id}_{\mathbf{RSVP}_{\mathrm{simp}}}$ whose components are the identity morphisms on $\mathbf{RSVP}_{\mathrm{simp}}$-objects, making F left adjoint to G.

(iii) The adjunction unit $\eta : \mathrm{id}_{\mathbf{SP}} \Rightarrow G \circ F$ is the identity natural transformation.

Proof. **(i).** On objects: $G(F(\Omega, \mathcal{A})) = G(\Delta(\Omega), \Phi_\Omega, v_\Omega, S_\Omega) = (\Omega, \mathcal{A})$ by Definition 27.3. On generators: $G(F(\mathrm{Pop}_U)) = G(\iota_U, \eta_U) = \mathrm{Pop}_U$, and similarly for Bind and Col. Since both are strict monoidal functors and **SP** is freely generated, this forces $G \circ F = \mathrm{id}_{\mathbf{SP}}$.

(ii). For each object $M = (\Delta(\Omega), \Phi, v, S_\Omega) \in \mathbf{RSVP}_{\mathrm{simp}}$, define $\varepsilon_M := \mathrm{id}_M$. This is a morphism in $\mathbf{RSVP}_{\mathrm{simp}}$ (the identity). Naturality: for any morphism $h : M \to M'$ in $\mathbf{RSVP}_{\mathrm{simp}}$, the square $\varepsilon_{M'} \circ FG(h) = h \circ \varepsilon_M$ reduces to $\mathrm{id}_{M'} \circ h = h \circ \mathrm{id}_M$, which holds trivially.

(iii). The adjunction unit is $\eta_{(\Omega,\mathcal{A})} := \mathrm{id}_{(\Omega,\mathcal{A})}$, which is consistent with $G \circ F = \mathrm{id}_{\mathbf{SP}}$. □

Remark 27.6. The adjunction is tautological by design: $\mathbf{RSVP}_{\mathrm{simp}}$ is defined as the image-presentation of F, so G inverts F on the nose. The non-trivial content is (a) that the restriction $\mathbf{RSVP}_{\mathrm{simp}} \subseteq \mathbf{RSVP}$ is closed under the operations used in Section 12, which follows from Definition 27.2, and (b) that the asymmetry theorem (Theorem 13.7) is preserved: the full **RSVP** strictly contains $\mathbf{RSVP}_{\mathrm{simp}}$ (there exist smooth maps and smooth entropy densities not of simplex form), so no extension of G to all of **RSVP** making $G \circ F = \mathrm{id}_{\mathbf{SP}}$ can be both total and faithful.

28 Mathematical Status and Failure Modes

28.1 Fully Formal

The following are rigorous within standard mathematics: (1) The category **SP** and its generating relations (Section 7); (2) The causal preorder and the universal property of Col (Theorem 8.6); (3) The strict symmetric monoidal structure (Proposition 9.2); (4) Initiality of **SP** in **EDSMC** (Theorem 9.5); (5) The meld theorem (Theorem 10.4); (6) The **RSVP** category with witnessed morphisms (Proposition 11.6); (7) Functoriality of F (Theorem 12.9); (8) Strong normalization of the deterministic fragment (Theorem 18.1).

28.2 Modeling-Dependent

The following require additional choices: (1) The probability simplex $\Delta(\Omega)$ as the geometric realization of option-spaces (alternatives: Wasserstein spaces, Hilbert spaces over Ω); (2) The specific entropy-transport PDE governing **RSVP** dynamics; (3) The explicit form of δv_{ij} in $F(\mathrm{Bind})$.

28.3 Failure Modes

(i) *Collapse violating causality.* As demonstrated in Section 14, applying $\mathrm{Col}_{a\sim b}$ after Bind_{ab} (which establishes $a \preceq b$ with $\downarrow a \neq \downarrow b$) violates causal admissibility. The system rejects such collapses in $\mathbf{SP}_c$.

(ii) *Bind creating cycles.* A sequence $\mathrm{Bind}_{ab} \circ \mathrm{Bind}_{ba}$ creates $a \preceq b \preceq a$ with $a \neq b$, violating Lemma 8.3. The well-formedness condition (acyclicity of the dependency graph) must be enforced externally or by a confluence condition.

(iii) *Entropy functional failure.* If p is not a probability measure on Ω (e.g., unnormalized), the Shannon entropy need not satisfy monotonicity under subsets. The framework requires that the entropy functional be consistently defined on all option-spaces with a fixed normalization.

(iv) *F failing to be faithful.* If two distinct binding constraints Bind_{ij} and $\mathrm{Bind}_{ij'}$ produce the same vector field modification $\delta v_{ij} = \delta v_{ij'}$ (because the coupling depends only on the simplex geometry, not on the labels), then F is not faithful on Bind events. This requires the coupling to be injective in the label pair (i, j), which is a non-trivial constraint on the realization.

28.4 Open Problems

1. **Faithfulness of F.** Full characterization of when F is faithful, particularly for Bind events.

2. **Entropy functional for** Bind**.** Construction of a functional making Axiom 2 strictly non-trivial for Bind (e.g., symmetry entropy $\mathsf{H}_{\mathrm{sym}} = \log |\Omega| - \log |\mathrm{Aut}(\Omega, \mathcal{A})|$).

3. **Compact-closed completion.** Whether **SP** admits a traced symmetric monoidal completion (the obstruction to full compact-closure is that $\eta : I \to X^* \otimes X$ would create optionality, violating Axiom 1).

4. **Almost-sure normalization** in the probabilistic fragment.

5. **∞-categorical enhancement** replacing **SP** with an $(\infty, 1)$-category where 2-morphisms are equivalences between histories.

6. **Derived Enhancement.** Can **SP** be enhanced to an $(\infty, 1)$-category where collapse corresponds to homotopy coequalizers rather than strict quotients?

7. **Entropy–Symmetry Duality.** Construct an entropy functional on admissibility lattices via groupoid cardinality:

$$\mathsf{H}_{\text{sym}}(\Omega) = \log |\text{Aut}(\Omega)|.$$

8. **Measure-Theoretic Completion.** Replace $\Delta(\Omega)$ with the Wasserstein space $\mathcal{P}_2(\Omega)$ and study whether F lifts to a functor into gradient-flow categories.

9. **Faithfulness Criterion.** Characterize minimal geometric data on $\Delta(\Omega)$ making F faithful. Conjecture: a non-degenerate Riemannian metric with curvature-sensitive slack is sufficient.

10. **Compact Completion and Trace.** Determine whether adding formal time-reversal objects yields a traced monoidal category whose fixed points correspond to cyclic commitment structures.

29 Dual Descriptions and Scale Separation

The relationship between **SP** and **RSVP** is a scaling limit, not a symmetric duality.

SP	**RSVP**
Rewriting system	Thermodynamic limit
Discrete, combinatorial	Smooth, differential
Log-bound, append-only	Field-bound, diffusive
Constraint accumulation	Coherence redistribution
Col = coequalizer	$F(\text{Col})$ = renormalization
Action monotone increasing	PDE, parabolic

Informally: as event step-size $\epsilon \to 0$ and event frequency $\to \infty$, the discrete action $\mathcal{S}[\gamma]$ converges to a continuous action functional whose Euler–Lagrange equation is the entropy-transport PDE. The functor F is the finite-step approximation. A rigorous derivation requires specifying convergence in a Wasserstein topology on $\Delta(\Omega)$; this is recorded in Appendix E.

30 Further Directions

1. **∞-categorical enhancement.** Replace **SP** with an $(\infty, 1)$- category in which 2-morphisms are policy-equivalences between histories and higher morphisms encode coherence conditions. Meld would then correspond to a homotopy pushout.

2. **Stochastic extension.** Incorporate a Markov chain on **SP** (transition probabilities on generators) and connect to the Independent Channels Lemma.

3. **Renormalization duality.** Formalize the duality between fine-grained **SP** categories and coarse-grained **RSVP** dynamics as a Wilsonian effective theory.

4. **Thermodynamic bounds.** Derive Landauer-type bounds on the minimum action required to realize a given collapse, using the discrete mechanics of Section 15.

5. **Numerical simulation.** Discretize the entropy-transport PDE on $\Delta(\Omega)$ and simulate convergence to coherence attractors for specific histories.

31 Philosophical Remark

Irreversibility in Spherepop is not an empirical claim about thermodynamic systems. It is Axiom 1: the absence of inverses is constitutive of the framework. The connection to Landauer's principle and Bennett's logical reversibility is structural analogy, not reduction. This insulates the formalism from physical objections while preserving the thermodynamic intuition.

The phrase "joy of Spherepop" names a phenomenological consequence of this structure: in a universe of infinite options, nothing is decided. It is only by popping futures, binding promises, and collapsing distinctions that a world becomes livable. The functor F makes this visible geometrically: each commitment sharpens the coherence potential, directs the flow, and moves the probability mass toward a vertex.

32 Conclusion

The paper compresses to three sentences.

Spherepop is the initial object in the category of entropy-decreasing symmetric monoidal rewriting categories, freely generated by Pop, Refuse, Bind, and Collapse, with worldhood as the sheaf condition on local event proposals and meld as policy-induced sheafification.

RSVP is a smooth entropy-witnessed field category whose morphisms carry explicit slack data ensuring categorical closure without boundary-term ambiguity.

$F : \mathbf{SP} \to \mathbf{RSVP}$ *is a symmetric monoidal functor translating option-elimination into boundary sharpening, dependency into directed coupling, and quotient collapse into renormalization, with the structural asymmetry that* **SP** *accumulates constraint while* **RSVP** *redistributes coherence—two scale-dual realizations of irreversible entropy flow.*

Constraint makes worlds. Entropy measures their cost. History defines identity. The functor renders persistence geometrically visible.

A Confluence of Generator Relations

The relations of Definition 7.5 are confluent in the sense that any two reduction paths starting from the same term reach the same normal form. This follows from the fact that: (a) idempotence of Pop is a local rewriting rule with no critical pairs; (b) commutativity of independent Bind events ($\{i, j\} \cap \{k, l\} = \emptyset$) is Church-Rosser by inspection (both orderings produce the same admissibility family since the binding constraints are on disjoint pairs of elements); (c) the monoidal laws are standard and their confluence is part of the theory of symmetric monoidal categories.

B Free Generation: Formal Construction

Let $\mathcal{G}$ be the directed multigraph:

- Vertices: all option-spaces $(\Omega, \mathcal{A})$.
- Edges: for each nonempty $U \subsetneq \Omega$, edges Pop_U and Ref_U; for each pair $i \neq j \in \Omega$ with $\{i \preceq j\} \notin \mathcal{A}$, an edge Bind_{ij}; for each causally admissible $\sim$, an edge $\mathrm{Col}_\sim$.

SP is the quotient of the free category $\mathcal{F}(\mathcal{G})$ by the congruence of Definition 7.5, restricted to entropy-monotone paths.

C Symmetry Entropy for Bind

Define:

$$\mathsf{H}_{\text{sym}}(\Omega, \mathcal{A}) := \log|\Omega| - \log|\text{Aut}(\Omega, \mathcal{A})|,$$

where $\text{Aut}(\Omega, \mathcal{A})$ is the group of admissibility-preserving automorphisms of $(\Omega, \mathcal{A})$. Under Bind_{ij}:

- $|\Omega|$ is unchanged, so $\log|\Omega|$ is unchanged;
- $|\text{Aut}(\Omega, \mathcal{A}')| < |\text{Aut}(\Omega, \mathcal{A})|$ since the automorphism that swaps $i \leftrightarrow j$ (and is in $\text{Aut}(\Omega, \mathcal{A})$ when i, j are symmetric) is no longer admissibility-preserving after binding $i \preceq j$.

Hence $\mathsf{H}_{\text{sym}}(\Omega, \mathcal{A}') > \mathsf{H}_{\text{sym}}(\Omega, \mathcal{A})$: symmetry entropy strictly increases under Bind.

This provides the entropy functional making Axiom 2 non-trivially active for Bind. The monotonicity direction is reversed from the other generators (symmetry entropy increases), which is consistent with Bind being entropy-preserving in the Shannon sense while creating new asymmetry.

D Almost-Sure Normalization

Let Choice events assign probability p_i to branch i. Define expected optionality $\mathbb{E}[\mathsf{Opt}_{t+1}] = \sum_i p_i \mathsf{Opt}(X_i^{(t)})$. If there exists $\delta > 0$ such that $\mathbb{E}[\mathsf{Opt}_{t+1}] \leq \mathsf{Opt}(X_t) - \delta$ whenever $\mathsf{Opt}(X_t) > 0$, then $(\mathsf{Opt}(X_t))_t$ is a non-negative supermartingale, hence $\mathsf{Opt}(X_t) \to 0$ almost surely by Doob's convergence theorem. The condition $\delta > 0$ holds when every branch contains at least one Pop event with nonzero probability—a weak condition satisfied by all well-typed Spherepop programs.

E Scaling Limit Heuristic

Let $\epsilon > 0$. Discretize time as $t_k = k\epsilon$. Suppose events occur at rate $1/\epsilon$ and each Pop eliminates an ϵ-fraction of mass. The discrete action $\mathcal{S}[\gamma_\epsilon] = \sum_k L_k^{(\epsilon)} \approx \epsilon \sum_k (-\dot{\mathsf{Opt}}(\gamma(t_k)))$. As $\epsilon \to 0$:

$$\mathcal{S}[\gamma_\epsilon] \to \int_0^T (-\dot{\mathsf{Opt}}(\gamma(t)))\, dt = \mathsf{Opt}(\Omega_0) - \mathsf{Opt}(\Omega_T).$$

The Euler–Lagrange equation for this action functional under the constraint of entropy-transport is $\partial_t S = \nabla \cdot (\kappa \nabla \Phi)$, the RSVP PDE. A rigorous derivation requires specifying the topology on the space of histories and the sense of convergence; the natural setting is Γ-convergence of functionals on $L^2(\Delta(\Omega))$.

F Faithfulness: Sufficient Conditions

Proposition F.1. *F is faithful on the subcategory generated by* Pop *and* Col *alone, provided the Riemannian metric on $\Delta(\Omega)$ is non-degenerate and the boundary-sharpening profile η_U is injective as a function of U (i.e., $U_1 \neq U_2 \Rightarrow \eta_{U_1} \neq \eta_{U_2}$ as smooth functions on $\Delta(\Omega \setminus U_1) \cap \Delta(\Omega \setminus U_2)$).*

Proof. Two distinct Pop events $\mathrm{Pop}_{U_1} \neq \mathrm{Pop}_{U_2}$ produce distinct face inclusions $\iota_{U_1} \neq \iota_{U_2}$ (they have different domains). Two distinct Col events have different equivalence relations $\sim_1 \neq \sim_2$, hence different coarse-graining maps. The injectivity condition on η ensures that even when the smooth maps φ coincide (which happens only for trivial cases), the slack data distinguish the morphisms. □

G Notation Summary

Symbol	Meaning
SP	Free symmetric monoidal entropy-decreasing rewriting category
RSVP	Smooth entropy-witnessed field category
EDSMC	Category of entropy-decreasing symmetric monoidal categories
$(\Omega, \mathcal{A})$	Option-space with admissibility family
H	Entropy functional $\mathrm{Ob}(\mathbf{SP}) \to \mathbb{R}_{\geq 0}$
Opt	Optionality functional $\mathrm{Ob}(\mathbf{SP}) \to \mathbb{R}_{\geq 0}$
$\mathrm{Pop}, \mathrm{Ref}, \mathrm{Bind}, \mathrm{Col}$	Four generating morphisms
Meld_π	Policy-induced sheafification operator
$\otimes$	Symmetric monoidal tensor (Cartesian product)
I	Unit object (trivial option-space)
$\preceq$	Causal preorder on Ω
$\downarrow x$	Causal past of x
$\sim_q$	Causally admissible equivalence (collapse policy)
$F : \mathbf{SP} \to \mathbf{RSVP}$	Geometric realization functor
(φ, η)	RSVP morphism with entropy-slack witness
$\Delta(\Omega)$	Probability simplex over Ω
Φ, v, S	Coherence potential, velocity field, entropy density
κ	Diffusion coefficient in entropy-transport PDE
$\mathcal{S}[\gamma]$	Action of history γ
π_t	Commitment (conjugate to optionality)
H_t	Hamiltonian (remaining freedom)
$\mathcal{T}$	Presheaf of local event proposals
$a_\pi(\mathcal{T})$	Policy sheafification of $\mathcal{T}$
η_π	Universal π-invariant map
K	Kolmogorov complexity
$\mathcal{B}$	Accounting functor tracking Ref tags

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