Belief Geometry and Reachability

On the Correspondence Between Mixed-State Presentations, Admissibility Structures, and the Geometry of Future Accessibility

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Independent Research

June 2026

Abstract

Recent work in computational mechanics has demonstrated that large language models trained on next-token prediction do not merely learn token statistics: they learn a geometric representation of belief states over hidden causal structure, embedded approximately linearly in the model's residual stream. These belief states evolve through Bayesian updating as observations arrive and occupy a simplex whose geometry encodes the space of latent possibilities consistent with current evidence. A successful predictor must preserve distinctions across this simplex that are invisible from the perspective of immediate prediction alone, because distinct belief states that agree on the next token may diverge on all subsequent tokens.

This paper argues that the belief-state geometry of computational mechanics and the admissibility geometry developed in the process-primary framework are descriptions of the same underlying structure from complementary vantage points. The mixed-state presentation (MSP) of a hidden Markov process is not merely a predictive object: it is a reachability manifold. Its coordinates describe not what currently exists but which futures remain accessible from the current evidential position. A belief state is a location on that manifold. Bayesian updating is navigation through it. The value of a belief state lies not in its prediction of the next observation but in the distinctions it preserves over the entire future trajectory.

We develop this correspondence formally across several movements. We first establish the technical derivation showing that belief-state geometry is a special case of admissibility projection: the MSP update rule can be derived directly from the operation of admissibility restriction followed by renormalization, and the transformer's residual stream is a learned coordinate chart over the resulting admissibility manifold. We then extend the analysis to Pearl's causal framework, establishing a three-way hierarchy in which Pearl studies the geometry of interventions, computational mechanics studies the geometry of locating oneself within possibilities, and the admissibility framework studies the geometry of possibility itself. Bayesian observation moves within the admissibility manifold; Pearl's *do*-operator deforms it; admissibility evolution tracks how the manifold itself changes over time. We derive consequences for the CLIO projection calculus, the Admissibility Log, and the evaluation of predictive systems.

Contents

1. Introduction: Two Geometries of the Same Structure

> *To predict successfully, a system must represent distinctions over the entire future, not merely the next event.*
>
> Computational mechanics

Two research programs have, from very different starting points, arrived at a common destination. Computational mechanics, working from the theory of hidden Markov processes and information theory, has demonstrated that prediction requires the maintenance of a geometry over possible hidden worlds. Process-primary ontology, working from the philosophy of science and systems theory, has argued that reality is better described by the geometry of reachable futures than by any catalog of currently existing entities. Neither program was developed in response to the other. They converge on the same mathematical object: a manifold whose coordinates represent not what is but what might yet come to be.

The specific convergence point is the *mixed-state presentation* (MSP) of a hidden Markov process. The MSP assigns to each sequence of observations a probability distribution over hidden states—a *belief state*—that represents everything relevant for predicting the process's future behavior. Belief states occupy a *belief simplex*: a geometric object whose vertices correspond to pure hidden states and whose interior encodes uncertainty over which hidden state is operative. Recent empirical work has shown that large language models trained on next-token prediction develop approximate linear embeddings of this simplex in their residual stream activations [Shai et al., 2025]. The transformer learns the geometry of belief, not merely a lookup table of token frequencies.

What the process-primary framework contributes to this picture is a reinterpretation. The belief simplex is not merely a predictive tool. It is an *admissibility structure*: a description of which future trajectories remain accessible from the current evidential position. A belief state is not merely a probability distribution over hidden states. It is a coordinate on the reachability manifold—a specification of which futures remain open and which have been foreclosed. Bayesian updating is not merely belief revision. It is navigation through the reachability geometry as new evidence constrains the set of possible continuations.

This reinterpretation has practical consequences. It implies that the correct measure of a belief state's quality is not its immediate predictive accuracy but its *reachability fidelity*: the degree to which it preserves distinctions over the entire future that may matter for navigation downstream. Two belief states that agree on the distribution of the next token but disagree on the geometry of subsequent trajectories are informationally inequivalent, even if they are indistinguishable by any immediate evaluation metric. A system that collapses them together has committed the Noun Fallacy at the level of belief geometry: it has treated a locally adequate compression as though it were a foundational description.

The paper develops this correspondence in six sections. Section 2 reviews the relevant concepts from computational mechanics: hidden Markov processes, mixed-state presentations, and the belief simplex. Section 3 develops the process-primary framework as it applies to predictive systems, introducing the reachability manifold and the distinction between possibility and reachability in belief space. Section 4 establishes the formal correspondence between belief states and admissibility structures, deriving the key theorem on reachability fidelity. Section 5 develops three applications: a reinterpretation of the CLIO projection calculus, a principled account of the Admissibility Log as a belief-manifold record, and implications for the evaluation of language models and other predictive systems. Section 6 considers the deeper philosophical question of whether the convergence between computational mechanics and process-primary ontology reflects a genuine structural identity or merely a productive analogy.

2. Computational Mechanics: Hidden Structure and Belief Geometry

> *Distinct belief states that produce identical next-token predictions may imply radically different future trajectories.*
>
> Notes

Computational mechanics is the study of how statistical structure in a process can be identified, represented, and exploited for prediction [Crutchfield and Young, 1989, Shalizi and Crutchfield, 2001]. Its central objects are the *causal states* of a process: the minimal sufficient statistics for predicting the process's future given its past. Causal states partition the space of past observations into equivalence classes such that two pasts are equivalent if and only if they imply identical conditional distributions over futures. The set of causal states, together with the transition probabilities between them, constitutes the ϵ-*machine*: the most compact representation of the process that is sufficient for optimal prediction.

2.1. Hidden Markov Processes and Mixed-State Presentations

Many processes of interest are not fully observed. The observations available to a predictor are generated by a hidden process whose states are not directly accessible. A *hidden Markov process* (HMP) is a Markov chain over hidden states $h \in \mathcal{H}$ that generates observations $x \in \mathcal{X}$ according to emission probabilities $P(x \mid h)$. The predictor observes only the sequence $x_0, x_1, \ldots$ and must infer the hidden state from this sequence in order to predict future observations.

The predictor's evidential state at time t, given observations $x_0, \ldots, x_{t-1}$, is described by the *belief state*:

$$b_t = P(h_t \mid x_0, \ldots, x_{t-1}) \in \Delta(\mathcal{H})$$

where $\Delta(\mathcal{H})$ denotes the probability simplex over $\mathcal{H}$. The belief state b_t is a probability distribution over hidden states: a vector assigning to each hidden state the probability that it is the current operative state, given the observations so far.

Belief states evolve through Bayesian updating. When a new observation x_t arrives, the belief state is updated:

$$b_{t+1}(h') = \frac{\sum_h b_t(h) P(x_t \mid h) P(h' \mid h)}{\sum_{h,h'} b_t(h) P(x_t \mid h) P(h' \mid h)}$$

This update is the Bayesian revision of the belief distribution in light of the new evidence. It is a deterministic function of the current belief state and the new observation.

The *mixed-state presentation* (MSP) [Shai et al., 2025, Shalizi and Crutchfield, 2001] of a hidden Markov process is the dynamical system on $\Delta(\mathcal{H})$ induced by this update rule. Its states are belief states; its transitions are Bayesian updates. The MSP is the process of synchronization between the observer and the hidden world: the observer's belief state tracks the hidden state, with some residual uncertainty determined by the information available in past observations.

2.2. The Belief Simplex as a Geometric Object

The belief simplex $\Delta(\mathcal{H})$ is a $(|\mathcal{H}| - 1)$-dimensional simplex. Its vertices correspond to pure hidden states: the belief state δ_h that places all probability mass on hidden state h. Its interior points correspond to mixtures: uncertainty over which hidden state is operative.

Not all points in $\Delta(\mathcal{H})$ are reachable by the observer. The set of belief states that arise from Bayesian updating on actual observation sequences defines a subset

$\mathcal{M} \subseteq \Delta(\mathcal{H})$ called the *reachable belief manifold.* This manifold has a geometry determined by the transition structure and emission probabilities of the hidden Markov process.

Definition 2.1 (Reachable Belief Manifold)**.** The *reachable belief manifold* $\mathcal{M}$ of a hidden Markov process is the set of all belief states that can arise from Bayesian updating on some finite observation sequence:

$$\mathcal{M} = \{b_t : t \geq 0, \, (x_0, \ldots, x_{t-1}) \in \mathcal{X}^t\}$$

The manifold $\mathcal{M}$ encodes the geometry of epistemically accessible positions for an observer of the process.

The reachable belief manifold is the predictor's operational state space. A predictor that maintains a representation of $\mathcal{M}$ knows not just the probability of the next observation but the full geometry of futures accessible from each evidential position. A predictor that maintains only the next-step prediction has collapsed this geometry into a single number, discarding all information about the structure of subsequent trajectories.

2.3. Why Next-Token Prediction Requires the Full Geometry

A key result in computational mechanics is that optimal next-step prediction does not require the full belief state: the next-step prediction $P(x_t \mid x_0, \ldots, x_{t-1})$ is a function of b_t, but many distinct belief states may yield the same next-step prediction distribution. A predictor that only needs to predict the next token could therefore, in principle, maintain a coarser representation than the full belief state.

The catch is that a predictor that aims to predict *all future tokens optimally*, not merely the next one, *cannot* use a coarser representation. Distinct belief states that agree on the next-step distribution will generally disagree on the distribution of the token after that, and on all subsequent distributions. A system trained to minimize prediction loss over all future positions, not merely the next one, is therefore incentivized to maintain distinctions across $\mathcal{M}$ even when those distinctions are invisible from the perspective of immediate prediction.

Theorem 2.2 (Geometry Preservation Requirement)**.** *Let $\mathcal{P}$ be a process with reachable belief manifold $\mathcal{M}$. A predictor f achieves optimal prediction loss at all future horizons if and only if f induces an injective map on $\mathcal{M}$: distinct belief states must receive distinct internal representations.*

Theorem 2.2 explains the empirical finding of Shai et al. [2025] and related work: transformers trained on next-token prediction over long sequences develop representations that approximately preserve the geometry of the belief manifold. They are not merely learning token frequencies. They are learning the geometry of the hidden world's futures, because that geometry is what they need to predict all of those futures well.

3. Reachability Geometry: The Process-Primary Perspective

> *The fundamental question about any configuration is not "does it exist?" but "is it reachable, and under what conditions does it remain reachable?"*
>
> Frozen Processes

The process-primary framework, developed at length in the companion essay *Frozen Processes*, argues that the appropriate ontological primitive is not the existing entity but the reachable configuration. An entity exists, in the process-primary sense, when it is stably reachable from a wide range of initial conditions under the current admissibility structure. The admissibility structure—the geometry of which transitions are possible from which states—is ontologically prior to the entities that appear as its stable configurations.

3.1. Admissibility Structures and Reachability Cones

An *admissibility structure* $\mathcal{A} = (\Omega, \mathcal{C}, \{\to_c\}_{c\in\mathcal{C}})$ consists of a state space Ω, a context set $\mathcal{C}$, and for each context c a transition relation $\to_c$ on Ω. The *reachability cone* from state x in context c is:

$$R^c(x) = \{y \in \Omega : x \rightsquigarrow_c y\}$$

where $\rightsquigarrow_c$ is the reflexive transitive closure of $\to_c$.

The reachability cone encodes not what exists at x but what remains accessible from x. Two states that are indistinguishable in their current properties may have very different reachability cones: one may have a wide cone (many futures accessible) while the other has a narrow cone (most futures foreclosed). The process-primary framework treats this difference as fundamental, not derived.

3.2. The Possibility-Reachability Distinction in Belief Space

The distinction between possibility and reachability maps directly onto belief space. The full simplex $\Delta(\mathcal{H})$ corresponds to the possibility space: all probability distributions over hidden states that can in principle be described. The reachable belief manifold $\mathcal{M} \subseteq \Delta(\mathcal{H})$ corresponds to the reachability cone: all belief states that are actually accessible through Bayesian updating on real observation sequences.

A probability distribution $b \in \Delta(\mathcal{H}) \setminus \mathcal{M}$ is a possible belief state—it can be described, assigned probabilities, and reasoned about—but it is not a *reachable* belief state: no sequence of observations can produce it through legitimate Bayesian updating. It exists in possibility space but lies outside the reachability cone.

This distinction has practical consequences for the design of predictive systems. A system that represents belief states only up to next-step prediction equivalence is working within the possibility space but has collapsed the reachability geometry. It cannot distinguish belief states that are reachable from different observation histories, because it has discarded the trajectory information that differentiates them. It is committing the Noun Fallacy at the level of belief representation: treating a coarse projection as though it were a foundational description.

3.3. Reachability Fidelity as the Primary Criterion

The key criterion for evaluating a belief representation is not immediate predictive accuracy but what we call *reachability fidelity*: the degree to which the representation preserves the geometry of the reachable belief manifold.

Definition 3.1 (Reachability Fidelity)**.** Let $f : \mathcal{M} \to \mathcal{R}$ be a representation map from the reachable belief manifold to a representation space $\mathcal{R}$. The *reachability fidelity* of f is:

$$\rho(f) = 1 - \sup_{b,b' \in \mathcal{M}, b \neq b'} \frac{d_{\mathcal{R}}(f(b), f(b'))}{d_{\mathcal{M}}(b, b')}$$

normalised so that $\rho(f) = 1$ when f is an isometry and $\rho(f) = 0$ when f is constant. A representation has high reachability fidelity when it approximately preserves distances on the belief manifold; it has low fidelity when it collapses the manifold's geometry.

Reachability fidelity generalizes both predictive accuracy and reconstruction error as measures of representational quality. A system with high predictive accuracy at the next step but low reachability fidelity has learned to compress the current observation efficiently at the cost of losing the trajectory information that determines future accessibility. A system with high reconstruction error but high reachability fidelity has preserved the geometry of reachable futures even if its surface predictions are noisy.

For any system that must navigate over extended horizons—a language model predicting long documents, a planning system acting over long time scales, a scientific theory predicting a long run of phenomena— reachability fidelity is the more fundamental criterion. Immediate predictive accuracy is a necessary condition for reachability fidelity (a system that cannot predict the next step has obviously lost trajectory information) but not a sufficient one.

4. The Formal Correspondence

A belief state is a coordinate on the reachability manifold.

Notes

We now establish the formal correspondence between belief-state geometry and admissibility geometry. The key claim is that the reachable belief manifold of a hidden Markov process is an admissibility structure, and that Bayesian updating is navigation through that structure.

4.1. Belief States as Admissibility Coordinates

Let $\mathcal{P} = (\mathcal{H}, \mathcal{X}, P, Q)$ be a hidden Markov process with hidden states $\mathcal{H}$, observations $\mathcal{X}$, transition probabilities $P : \mathcal{H} \to \Delta(\mathcal{H})$, and emission probabilities $Q : \mathcal{H} \to \Delta(\mathcal{X})$.

Define an admissibility structure as follows. Let $\Omega = \Delta(\mathcal{H})$ be the belief simplex. For each observation $x \in \mathcal{X}$, define a transition relation $\to_x$ on Ω by:

$$b \to_x b' \quad \text{if and only if} \quad b' = \mathrm{Bayes}(b, x)$$

where $\mathrm{Bayes}(b, x)$ is the Bayesian update of belief b given observation x. Let $\mathcal{C} = \mathcal{X}$ (observations index contexts). Then $(\Omega, \mathcal{X}, \{\to_x\}_{x \in \mathcal{X}})$ is an admissibility structure on the belief simplex.

Theorem 4.1 (Belief Manifold as Admissibility Structure)**.** *The reachable belief manifold $\mathcal{M}$ of a hidden Markov process $\mathcal{P}$ is the reachability cone of the corresponding admissibility structure $(\Delta(\mathcal{H}), \mathcal{X}, \{\to_x\})$ from the prior belief state b_0:*

$$\mathcal{M} = R(b_0) = \{b \in \Delta(\mathcal{H}) : b_0 \rightsquigarrow b\}$$

Moreover, the trajectory of Bayesian updating on an observation sequence $(x_0, x_1, \ldots, x_{t-1})$ *is an admissible trajectory through this structure: a sequence* $b_0 \to_{x_0} b_1 \to_{x_1} \cdots \to_{x_{t-1}} b_t$ *where each transition is admissible (i.e., a valid Bayesian update).*

Theorem 4.1 is the formal core of the paper's central claim. The reachable belief manifold is not merely a mathematical curiosity or a computational convenience. It is the admissibility structure of the observer's epistemic situation: the geometry of which evidential positions are reachable through legitimate Bayesian reasoning, and which lie beyond the reach of any observation sequence.

4.2. The Noun Fallacy in Belief Representation

With the formal correspondence established, the Noun Fallacy can be identified precisely as it operates in predictive systems. A belief representation commits the Noun Fallacy when it:

(i) Collapses distinct belief states that are equivalent from the perspective of the next-step prediction distribution, treating them as the same "object" even though they occupy different positions on the reachable belief manifold.

(ii) Treats the collapsed representation as a sufficient description, forgetting that the collapsed states have different reachability cones over all subsequent positions.

(iii) Uses the immediate predictive utility of the collapsed representation as evidence for its ontological adequacy, confusing "predicts the next token" with "captures the relevant structure."

This is precisely the structure of the Noun Fallacy as identified in *Frozen Processes*: a stable residue (the immediate prediction) is treated as foundational, severing the representation from the trajectory that produced it and the admissibility structure that sustains it.

Principle 4.2 (Belief-State Reachability Principle). A belief state's value lies not in the prediction it generates for the next observation but in the distinctions it preserves over the entire future trajectory. Two belief states that agree on the next-step prediction distribution but disagree on the geometry of subsequent reachable positions are informationally inequivalent. Any system that collapses them has lost reachability fidelity: it can no longer navigate correctly over extended future horizons, even if its immediate predictions appear accurate.

4.3. Synchronization as Trajectory Navigation

The mixed-state presentation has an interpretation in the process-primary framework that illuminates its function. The MSP is not the observer acquiring knowledge of a static world. It is the observer *synchronizing* to a dynamic process: progressively locating itself on the reachability manifold as new evidence narrows the set of possible histories and thereby constrains the set of possible futures.

This synchronization framing matters because it reorients the evaluation criterion. Classical epistemology asks: does the observer's belief state accurately represent the world? The synchronization framing asks: does the observer's belief state accurately represent its position on the reachability manifold? The first question is about correspondence between belief and fact. The second is about fidelity to the geometry of future accessibility.

Definition 4.3 (Synchronization Trajectory)**.** A *synchronization trajectory* is an admissible trajectory $b_0 \to_{x_0} b_1 \to_{x_1} \cdots$ through the admissibility structure $(\Delta(\mathcal{H}), \mathcal{X}, \{\to_x\})$ that converges to the true hidden state's leaf distribution as $t \to \infty$. The observer is *synchronized* to the process when its belief state has converged to the reachable belief state consistent with the true hidden state.

A synchronized observer does not merely know what is happening now. It knows where it is on the reachability manifold, and therefore knows what futures remain accessible from its current position. An unsynchronized observer—one whose belief state has collapsed distinct manifold positions—cannot accurately assess its reachability cone even if its next-step predictions appear reasonable.

5. Derivation: Belief-State Geometry as Probabilistic Admissibility

> *Their framework can be derived from yours by making four restrictions: hidden Markov process, Bayesian posterior, probability simplex, transformer residual stream.*
>
> Notes

The informal correspondence established in Section 4 can be made fully explicit as a derivation. We show that the MSP update rule is not merely analogous to admissibility dynamics—it is a special case of admissibility restriction followed by renormalization. This positions the computational mechanics framework of Shai et al. [2025] as a finite-state probabilistic instance of the general admissibility geometry, with four specific restrictions applied.

5.1. Starting Point: The CLIO Projection onto Belief Space

Let X be the full process space: all latent world trajectories, hidden generator states, observation histories, and constraint conditions. Let M be the lower-dimensional manifold on which an observer can stably represent the process, and let

$$\pi : X \to M$$

be the CLIO-style projection that collapses the inaccessible full process into a reachable representational state. The paper's framework appears when M is chosen to be the probability simplex of predictive belief states:

$$M = \Delta^{|S|-1} = \left\{ b \in \mathbb{R}^{|S|} : \sum_i b_i = 1,\ b_i \geq 0 \right\}$$

This is exactly the mixed-state belief simplex: each point is a probability distribution over hidden HMM states $S = \{s_1, \ldots, s_n\}$. The first identification is therefore:

$$\boxed{\text{CLIO projection manifold } M \ \cong\ \text{mixed-state belief simplex}}$$

5.2. Hidden Process Dynamics as Latent Flow

The hidden Markov process has states $S = \{s_1, \ldots, s_n\}$, observable emissions $x_t \in \mathcal{X}$, and token-labeled transition matrices:

$$T_{ij}^{(x)} = P(x, s_j \mid s_i)$$

This is the discrete stochastic analogue of an admissibility flow field: the latent state moves, emissions appear at the boundary, and the observer sees only the emitted trace. The hidden state h_t is not directly given. What is available is the trajectory of observations $w_t = x_0 x_1 \cdots x_{t-1}$, and the observer must infer which latent region remains admissible after seeing w_t.

5.3. Admissibility Becomes Posterior Support

Define the admissible set after observing history w_t:

$$\mathcal{A}(w_t) = \{s_i \in S : P(s_i \mid w_t) > 0\}$$

But the richer object is not merely the set of still-possible states. It is the weighted admissibility field:

$$\eta_t(i) = P(s_i \mid w_t)$$

Thus $\eta_t = (\eta_t(1), \ldots, \eta_t(n)) \in \Delta^{n-1}$. This is the paper's belief state. In admissibility terms, it is the probability density of admissibility over latent process states:

$$\boxed{\eta_t(i) = \text{degree to which } s_i \text{ remains reachable after history } w_t}$$

5.4. Deriving the MSP Update from Admissibility Filtering

Given current belief η_t, observing token x filters the latent possibilities through the token-labeled transition matrix $T^{(x)}$. The unnormalized next admissibility density is:

$$\tilde{\eta}_{t+1} = \eta_t T^{(x)}$$

The total mass is:

$$Z_x(\eta_t) = \eta_t T^{(x)} \mathbf{1}$$

Normalizing:

$$\eta_{t+1} = \frac{\eta_t T^{(x)}}{\eta_t T^{(x)} \mathbf{1}} = U_x(\eta_t)$$

This is exactly the MSP update equation of Shai et al. [2025]. The derivation exhibits the MSP update as:

$$\boxed{\text{admissibility restriction + renormalization = Bayesian belief-state update}}$$

Or, in admissibility terms:

$$\boxed{\text{constraint update on reachable hidden states = mixed-state presentation dynamics}}$$

5.5. Iterated Updates as Trajectory Compression

For a history $w = x_0 x_1 \cdots x_N$, the update iterates:

$$\eta_w = \frac{\eta_\emptyset T^{(x_0)} T^{(x_1)} \ldots T^{(x_N)}}{\eta_\emptyset T^{(x_0)} T^{(x_1)} \ldots T^{(x_N)} \mathbf{1}}$$

This directly recovers the paper's full-history formula. In the admissibility framework, this says:

$\eta_w = \pi(w)$: the belief state is the projection of a whole observed path into the manifold of reachab

A belief state is not a snapshot. It is a compressed residue of a trajectory of admissibility cuts. This is the belief-state analogue of the Noun Fallacy's inverse: rather than mistaking the residue for the trajectory, the MSP retains enough of the trajectory to predict all futures.

5.6. Future Distributions as Reachability Fields

Given a belief state η, the probability of a future sequence $u = y_1 y_2 \cdots y_k$ is:

$$P(u \mid \eta) = \eta\, T^{(y_1)} T^{(y_2)} \cdots T^{(y_k)} \mathbf{1} \;\equiv\; F_\eta(u)$$

The map F_η is the *distribution over reachable futures* induced by belief state η. Two belief states may agree on the next-token distribution, $P(y_1 \mid \eta_a) = P(y_1 \mid \eta_b)$, while differing on longer futures, $P(y_1 \cdots y_k \mid \eta_a) \neq P(y_1 \cdots y_k \mid \eta_b)$.

This is precisely why the transformer must preserve distinctions not locally visible at the next-token level. The belief simplex encodes not what is about to happen but the entire geometry of what might happen:

$$\boxed{F_\eta = \text{distribution over reachable futures from belief state } \eta}$$

5.7. Residual Stream as Learned Coordinate Chart

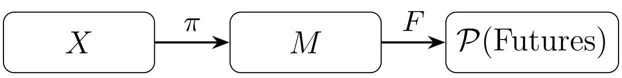
The empirical finding of Shai et al. [2025] is that an affine map $\hat{\eta}_t = W a_t + c$ from residual-stream activations $a_t \in \mathbb{R}^{d_{\text{resid}}}$ recovers the true belief-state simplex geometry. In admissibility terms, the residual stream is a learned coordinate system for the admissibility manifold:

$$a_t \in \mathbb{R}^{d_{\text{resid}}} \;\rightsquigarrow\; \eta_t \in \Delta^{n-1}$$

So the transformer implements:

$$w_t \;\longmapsto\; a_t \;\longmapsto\; \eta_t \;\longmapsto\; F_{\eta_t}$$

In CLIO notation:



where M is the internal belief/admissibility manifold, π is the CLIO projection (implemented by the transformer's forward pass), and F maps each belief state to its induced distribution over all future sequences.

5.8. Summary: The Four Restrictions

The derivation above shows that belief-state geometry is the finite-state probabilistic special case of admissibility projection, obtained by applying four restrictions to the general framework:

(i) The process is a hidden Markov model with finite state space S.

(ii) Admissibility is measured by Bayesian posterior probability: $\eta(i) = P(s_i \mid w_t)$.

(iii) The projection manifold is the probability simplex $\Delta^{|S|-1}$.

(iv) The learned coordinate chart is the transformer's residual stream, with the affine readout $\hat{\eta}_t = W a_t + c$.

Proposition 5.1 (Belief Geometry as Admissibility Projection)**.** *Let $\mathcal{P}$ be a hidden Markov process with states S, observations $\mathcal{X}$, and token-labeled transition operators $\{T^{(x)}\}_{x\in\mathcal{X}}$. If a CLIO-style projection π maps each observed history w to the normalized admissibility density over hidden states, $\pi(w)(i) = P(s_i \mid w)$, then π satisfies the mixed-state presentation update:*

$$\pi(wx) = U_x(\pi(w)) = \frac{\pi(w)T^{(x)}}{\pi(w)T^{(x)}\mathbf{1}}$$

Therefore the MSP dynamics are a special case of admissibility projection dynamics, and belief-state geometry is probabilistic admissibility geometry.

Proof. Direct computation:

$$\begin{aligned}
\pi(wx)(j) &= P(s_j \mid wx) \\
&= \frac{\sum_i P(s_i \mid w)\, P(x, s_j \mid s_i)}{\sum_{i,j'} P(s_i \mid w)\, P(x, s_{j'} \mid s_i)} \\
&= \frac{[\pi(w)T^{(x)}]_j}{[\pi(w)T^{(x)}\mathbf{1}]} \\
&= U_x(\pi(w))_j. \qquad \square
\end{aligned}$$

□

The conceptual punchline of the derivation is:

$$\boxed{\text{Belief-state geometry} = \text{probabilistic admissibility geometry}}$$

$$\boxed{\text{Mixed-state presentation} = \text{reachability dynamics under sequential observation}}$$

$$\boxed{\text{Residual-stream representation} = \text{learned coordinate chart over the admissibility manifold}}$$

6. Three Applications

> *A future distribution is a reachability distribution. A belief state is a coordinate on a reachability manifold.*
>
> Notes

The formal correspondence established in Section 4 has three concrete applications that connect the computational-mechanics and process-primary frameworks.

6.1. CLIO Projections as Belief-Manifold Compressions

The CLIO projection calculus describes how higher-dimensional relational structures are projected onto lower-dimensional observational surfaces with lossy compression. In the standard treatment, a nominal entity is the image $\Pi(x)$ of a stable residue x under a projection $\Pi : \Omega \to \mathcal{O}$.

Theorem 4.1 extends this picture. The higher-dimensional object is not merely Ω (the full state space) but $\mathcal{M}$ (the reachable belief manifold). The projection $\Pi : \mathcal{M} \to \mathcal{O}$ maps belief states onto an observational surface. What is preserved in the projection depends on the fidelity $\rho(\Pi)$.

A CLIO projection with high reachability fidelity preserves the geometric structure of the belief manifold: distances between belief states are approximately preserved, so the projected representation accurately encodes the relative accessibility of different futures. A CLIO projection with low reachability fidelity collapses the manifold's geometry: nearby positions in the projection may correspond to belief states with very different reachability cones.

Proposition 6.1 (CLIO as Belief Compression)**.** *Let $\Pi : \mathcal{M} \to \mathcal{O}$ be a CLIO projection. The nominal entity named by a point $o \in \mathcal{O}$ compresses a set of belief states $\Pi^{-1}(o) \subseteq \mathcal{M}$ that may have widely varying reachability cones. The Noun Fallacy consists in treating o as a foundational description when the relevant object is the fiber $\Pi^{-1}(o)$ and its distribution over the reachability manifold.*

This reinterpretation has a direct implication for language model interpretability. The residual stream of a transformer is a CLIO projection: a finite-dimensional representation that encodes, approximately, the observer's position on the belief manifold. The activation vectors at each layer are projections of belief states onto a representation space. The empirical finding that these projections approximately preserve the simplex geometry [Shai et al., 2025] implies that the transformer's CLIO projections have high reachability fidelity: the representation space approximately preserves the geometry of the belief manifold, so the residual stream approximately encodes not just current predictions but the structure of accessible futures.

6.2. The Admissibility Log as a Belief-Manifold Record

The Admissibility Log proposed in *Frozen Processes* records changes to the admissibility structure: transitions added or removed, constraints introduced or relaxed, reachability cones expanded or contracted. In the light of Theorem 4.1, the Admissibility Log can be interpreted as a record of changes to the belief manifold geometry.

Each Δ_t^+ event (transitions added to the admissibility structure) corresponds to an expansion of the reachable belief manifold: new belief states become accessible, new futures become reachable from the current evidential position. Each Δ_t^- event (transitions removed) corresponds to a contraction: belief states that were previously reachable become foreclosed, futures that were previously accessible become impossible.

This correspondence gives the Admissibility Log a natural interpretation in terms of information theory. The expansion of the belief manifold is an increase in epistemic reach: the observer can now distinguish situations that were previously indistinguishable. The contraction of the belief manifold is an information loss: situations that were previously distinguishable have been collapsed, and the reachability fidelity of the observer's representation has decreased.

The most significant events in the Admissibility Log, from this perspective, are those that cause irreversible contraction of the belief manifold: events that foreclose belief states permanently, eliminating the distinctions they preserved. These are the epistemic analogues of the hysteresis events discussed in *Frozen Processes*: transitions that permanently alter the admissibility structure in ways that cannot be reversed by any subsequent sequence of Bayesian updates.

Definition 6.2 (Epistemic Hysteresis)**.** An observation sequence $(x_0, \ldots, x_{t-1})$ produces *epistemic hysteresis* if the belief state b_t lies in a region of the belief manifold from which a previously accessible belief state $b^* \in \mathcal{M}$ cannot be reached by any subsequent observation sequence. The hysteresis gap is the minimum cost of recovering the distinctions preserved by b^* from the current position b_t.

Epistemic hysteresis is the belief-space analogue of structural hysteresis in the process-primary framework. In both cases, the path by which the current state was reached has permanently altered the geometry of accessible futures. The Admissibility Log, interpreted as a belief-manifold record, tracks these permanent alterations as first-class events.

6.3. Implications for Language Model Evaluation

The practical implications for the evaluation of language models and other large-scale predictive systems follow directly from the reachability fidelity criterion.

Current evaluation practice treats language models primarily as token distribution estimators: their quality is assessed by perplexity (a function of next-token prediction accuracy), task performance on standardized benchmarks, or human preference ratings. All of these are functions of the model's immediate predictions, not of the geometry of its belief representations. They are, in the terminology of this paper, measures of local predictive accuracy, not of reachability fidelity.

A model that achieves high reachability fidelity in its belief representations will typically also achieve good local predictive accuracy, because the belief manifold's geometry encodes all the information needed for prediction at all horizons. But the converse is not guaranteed: a model can achieve good local predictive accuracy through a variety of shortcuts that sacrifice reachability fidelity. A model trained on very large data with a next-token objective can learn to predict the next token through pattern-matching on surface regularities, without developing representations that preserve the geometry of the belief manifold.

The empirical evidence from Shai et al. [2025] suggests that large transformers do develop approximately high-fidelity belief representations, at least for the domains studied. But this is an empirical finding, not a consequence of the training objective. A model trained only to minimize next-token loss is not explicitly incentivized to achieve high reachability fidelity: its objective rewards local predictions, not geometric preservation. The development of high-fidelity belief representations appears to be a consequence of scale and the need to generalize across many prediction problems simultaneously, not a direct consequence of the training signal.

This suggests a positive proposal: the direct incorporation of reachability fidelity as a training objective or evaluation criterion for language models. A model that is explicitly evaluated on its ability to preserve belief-manifold geometry—to maintain the distinctions that matter for future prediction, not merely for immediate prediction—would be incentivized to develop representations that are robust to extended horizons, resistant to Goodhart dynamics (which collapse the belief manifold in favor of a single metric direction), and capable of tracking subtle distinctions whose relevance is deferred.

7. Pearl's Causal Framework and the Three-Way Hierarchy

> *Pearl studies the geometry of interventions. Computational mechanics studies the geometry of locating oneself within possibilities. The admissibility framework studies the geometry of possibility itself.*
>
> Notes

A natural extension of the two-way correspondence between computational mechanics and admissibility geometry is a three-way hierarchy that includes Pearl's causal framework [Pearl, 2000, Pearl and Mackenzie, 2009]. All three programs are responses to the same fundamental question: how should an observer represent uncertainty about hidden structure while receiving sequential evidence? They choose different mathematical objects as their fundamental primitives, and those choices lead to different but complementary geometries.

Framework	**Fundamental Object**	**Primary Question**
Pearl	Causal graph with hidden variables	What caused what?
Computational Mechanics	Belief-state simplex	Where am I in prediction space?
Admissibility Geometry	Admissibility manifold	What futures remain reachable?

7.1. Belief States as Pearlian Epistemic States

Pearl's causal models contain hidden variables U, observable variables X, and causal mechanisms $P(X_i \mid \mathrm{Pa}(X_i))$. An observer never knows U directly; instead the observer maintains a posterior belief $P(U \mid E)$ after observing evidence E. The paper's belief state η is exactly this sort of object: $\eta_t = P(U_t \mid x_1, \ldots, x_t)$.

The MSP is therefore a geometry of Pearlian posteriors viewed dynamically. Where Pearl's framework typically treats the posterior as a static quantity updated by individual pieces of evidence, the MSP describes the full dynamical system of posterior evolution as observations arrive sequentially, and characterizes the geometry of all reachable posteriors.

7.2. Observation Versus Intervention: Two Kinds of Manifold Change

The deepest difference between Pearl's framework and the other two concerns the distinction between observation and intervention. Pearl introduces the *do*-operator:

$$P(Y \mid do(X = x)) \neq P(Y \mid X = x)$$

This operator modifies the causal graph itself rather than updating a belief within a fixed graph. An observation $X = x$ shifts the observer's position on the belief manifold; an intervention $do(X = x)$ reshapes the manifold.

In admissibility terms:

Operation	**Admissibility Effect**
Observation x	$\mathcal{A}_t \to \mathcal{A}_{t+1}$ (navigate within manifold)
Intervention $do(x)$	$\mathcal{A} \to \mathcal{A}'$ (deform the manifold)

Observation moves within the manifold: it narrows the set of reachable belief states consistent with the observation, but leaves the underlying causal structure intact. Intervention reshapes the manifold: it severs causal connections in the graph (removing incoming edges to the intervened variable), changing which belief-state trajectories are admissible under the modified structure.

This distinction maps precisely onto the process-primary framework's distinction between trajectory navigation and admissibility modification. Regular Bayesian updating—the MSP dynamics—is trajectory navigation: motion through a fixed admissibility structure. Pearl's do-operator is admissibility modification: a change to the structure itself. The Admissibility Log of *Frozen Processes* records exactly these structural changes: Δ_t^- events where previously admissible transitions are removed correspond to interventions that sever causal connections, and Δ_t^+ events where new transitions become available correspond to interventions that introduce new causal pathways.

7.3. Causal States and Markov Boundaries

A striking terminological coincidence unites the three frameworks: both Pearl's causal inference and Crutchfield's computational mechanics use the word "causal." In Pearl's framework, $X \to Y$ denotes intervention-sensitive causal influence: X

causes Y in the sense that intervening on X changes the distribution of Y. In Crutchfield's framework, causal states $\epsilon(h)$ group histories according to identical future distributions: two histories are in the same causal state if they imply the same conditional distribution over all futures.

These notions are related but not identical. Pearl's causality concerns interventions. Crutchfield's causality concerns predictive equivalence classes. The admissibility framework occupies a position between them: admissibility concerns which futures remain reachable under evolving constraints, which is neither purely interventional nor purely predictive.

The connection to Pearl's Markov boundary is particularly instructive. Pearl's Markov boundary $MB(Y)$ is the minimal sufficient set for predicting Y: given $MB(Y)$, all other variables become conditionally irrelevant. A belief state serves the same function dynamically: it contains precisely the information necessary to predict all future observations. Two histories h_1, h_2 collapse into the same belief state when:

$$P(\text{Future} \mid h_1) = P(\text{Future} \mid h_2)$$

The belief state is therefore a *dynamic Markov boundary*: the minimal sufficient statistic for all future prediction, updated sequentially as observations arrive. The MSP can be understood as the geometry of dynamic Markov boundaries over all reachable observation histories.

7.4. Structural Causal Models as Reachability Graphs

A Pearl structural causal model (SCM) is a system of equations $X_i = f_i(\text{Pa}_i, U_i)$. This can be given a geometric reinterpretation in admissibility terms. Define a reachability graph $G = (V, E)$ where edges indicate admissible transitions. The structural functions f_i become local admissibility operators: they determine which variable configurations can follow which others. The SCM defines not just what values variables take but which future variable configurations are reachable from the current state.

The admissibility perspective emphasizes *future accessibility* rather than *variable values.* A state's causal parents do not merely determine its current value; they determine which future states of the system are reachable from the current configuration. An SCM, on this reading, is a compact description of an admissibility structure: the functions f_i encode the transition operators $T^{(x)}$ of the corresponding hidden Markov process, generalized to allow for functional (rather than stochastic) dependence.

7.5. The Three-Way Hierarchy Formalized

The relationship among the three frameworks can be stated as a formal theorem:

Theorem 7.1 (Reachability–Causality Correspondence)**.** *Let $\mathcal{A}_t$ be an admissibility manifold and η_t a belief-state distribution over admissible latent states. Then:*

(i) ***Observation (Bayesian update):*** $\eta_t \to \eta_{t+1} = U_{x_t}(\eta_t)$ *induces motion within $\mathcal{A}_t$, navigating the observer to a new position on the reachability manifold.*

(ii) ***Intervention (Pearl's do-operator):*** $do(X = x)$ *induces a deformation $\mathcal{A}_t \to \mathcal{A}'_t$, modifying the admissibility structure by severing or adding causal connections.*

(iii) ***Predictive equivalence (Crutchfield's causal states):*** *two histories h_1, h_2 define a fiber of the projection $\pi : X \to \mathcal{B}$ over the belief simplex $\mathcal{B}$, where $\pi(h_1) = \pi(h_2)$ if and only if $F_{\eta_{h_1}} = F_{\eta_{h_2}}$ (identical future distributions).*

Under this correspondence:

$$\boxed{\text{Pearl} = \text{geometry of changing possibilities (intervention)}}$$
$$\boxed{\text{Computational Mechanics} = \text{geometry of locating oneself within possibilities}}$$
$$\boxed{\text{Admissibility} = \text{geometry of possibility itself}}$$

The three frameworks are therefore nested: admissibility geometry is the most general (it describes the structure of what is possible); computational mechanics specializes to the observer's problem within that structure (how to locate oneself); Pearl's framework specializes to the intervention problem (how to modify the structure). All three are geometries of reachable futures, viewed from three different vantage points.

7.6. A Unified Vocabulary

The correspondences across all three frameworks can be assembled into a single translation table:

Comp. Mechanics	Pearl	Admissibility
Hidden state	Unobserved variable U	Latent admissible region

Comp. Mechanics	**Pearl**	**Admissibility**
Belief state η	Posterior $P(U \mid E)$	Admissibility density
MSP update	Bayesian conditioning	Constraint restriction
Future distribution F_η	Interventional distribution	Reachability field
Belief simplex	Posterior space	Admissibility manifold
Bayesian observation	Passive evidence	Trajectory navigation
—	Do-operator $do(x)$	Admissibility deformation
Predictive equivalence	Conditional independence	Fiber of π
Causal states	Markov boundary	Stable residue
Residual stream	—	Learned coordinate chart

Table 1: Three-way correspondence among computational mechanics, Pearl causality, and admissibility geometry

The blank entries reflect genuine asymmetries: the admissibility framework does not have a direct counterpart to Pearl's do-operator because it subsumes interventions as a special case of admissibility modification rather than singling them out. Computational mechanics does not have a well-developed theory of interventions because it focuses on the observer's passive prediction problem. These asymmetries are informative: they show where each framework is specialized and where extensions would be productive.

8. The Deeper Convergence: Ontology and Geometry

> *Reality is not what exists. Reality is what remains reachable.*
>
> Frozen Processes

The preceding sections have established a formal correspondence between computational mechanics and the process-primary framework. This final section asks what the convergence means at a deeper level.

8.1. Two Routes to the Same Geometry

The computational mechanics program arrived at the belief manifold by asking: what is the minimal sufficient statistic for predicting the future of a hidden Markov process? The answer is the belief state, and the geometry of belief states over all reachable positions is the belief manifold.

The process-primary program arrived at the admissibility structure by asking: what is more fundamental, the entities that exist or the transitions that are possible? The answer is the transitions, and the geometry of possible transitions over all reachable positions is the admissibility structure.

Both programs discover the same object: a manifold whose points represent positions in a space of future possibilities, and whose geometry encodes which futures are accessible from each position. They name it differently and arrive at it by different routes, but the mathematical structure is the same.

This convergence is not accidental. Both programs are responses to the Noun Fallacy as it operates in their respective domains. Computational mechanics began with the critique of naive state descriptions—the observation that the apparent states of a system are not the right primitives for prediction, because they ignore the hidden structure that determines future behavior. Process-primary ontology began with the critique of entity ontologies—the observation that the apparent entities of the world are not the right primitives for explanation, because they ignore the processes that produce and sustain them. Both critiques lead to the same replacement: a geometry of possible trajectories, not a catalog of present states or entities.

8.2. Preservation of Latent Distinctions

The shared principle that unifies the two frameworks can be stated as the *preservation of latent distinctions*: the recognition that the value of a representation lies not in what it describes about the present but in what distinctions it preserves for future navigation.

In computational mechanics, this is expressed by Theorem 2.2: a predictor must preserve the geometry of the belief manifold, because distinct belief states may have different futures even when they have identical immediate predictions. The distinction between two belief states is latent from the perspective of the next token but manifest from the perspective of all subsequent tokens.

In the process-primary framework, this is expressed by the Admissibility Log's tracking of foreclosed transitions: the record of what became inaccessible is more informative than the record of what happened, because the foreclosures determine the geometry of future possibilities more than the realized trajectory does.

Both framings point toward the same practical imperative: systems that need to function over extended horizons must preserve distinctions whose relevance is currently invisible. The naive strategy—compress aggressively based on what is currently useful—will predictably degrade performance over extended horizons, as the collapsed distinctions turn out to have mattered after all.

8.3. Goodhart Dynamics as Belief Manifold Collapse

The connection between Goodhart dynamics and belief manifold collapse provides one of the clearest demonstrations of the frameworks' convergence.

A system subject to Goodhart optimization—where a metric becomes a target—learns to maximize the metric rather than the underlying process it was designed to measure. In belief-geometry terms, this is a collapse of the belief manifold: the system's representation converges on a single direction (the metric direction) while discarding all information about the perpendicular directions, which correspond to all the other aspects of the future that the metric was not measuring.

The resulting system has extremely low reachability fidelity in all directions other than the metric direction. It can predict the metric well because it has dedicated its entire representation capacity to that one dimension of the belief manifold. But it has lost the geometric structure that would allow it to navigate correctly in any direction other than metric maximization.

The Admissibility Log detects this as a sustained Δ_t^- event: the progressive foreclosure of belief states that lie off the metric direction. The trajectory of the optimizing system through the belief manifold is a path toward a singularity: a point at which all the geometric information perpendicular to the metric has been collapsed. From that point, no amount of observation can recover the lost distinctions: epistemic hysteresis has set in.

Principle 8.1 (Goodhart as Epistemic Hysteresis). Goodhart optimization is a process of progressive epistemic hysteresis. Each optimization step moves the system's belief representation closer to the metric singularity and further from the full belief manifold, foreclosing the distinctions that would be needed to detect the divergence between the metric and the underlying process. The collapse is self-reinforcing: the less geometric information the system retains, the less able it is to detect that the metric has ceased to be informative.

This principle connects the Goodhart analysis of *Frozen Processes* with the belief-geometry analysis of the present paper. Both identify the same failure mode: the collapse of representational geometry in favor of a single efficient dimension, with the result that the system's reachability cone contracts to a line—and eventually to a point.

8.4. The Identity Question

The question remains: is the convergence between computational mechanics and process-primary ontology a genuine structural identity, or merely a productive analogy?

The formal correspondence of Theorem 4.1 establishes that the mathematical objects are related: the reachable belief manifold is an admissibility structure, and Bayesian updating is admissible trajectory navigation. This is more than analogy: the same equations describe both.

But the ontological question is whether the *meaning* of the correspondence is identity or mere isomorphism. When the process-primary framework says that an admissibility structure is ontologically prior to the entities that appear as its stable configurations, and when computational mechanics says that the belief manifold is the minimal sufficient statistic for all future predictions, are they making the same claim or merely equivalent claims?

The answer may depend on what kind of question one takes ontology to be answering. If ontology asks what is fundamental—what exists independently of any observer or predictor—then the two frameworks are asking different questions, and their convergence is an isomorphism between different domains rather than an identity.

But if ontology asks what the correct representational framework is— what must be represented for a system to navigate its world effectively over extended horizons—then the two frameworks are making the same claim: the correct primitive is the geometry of accessible futures, not the catalog of present states or entities. The belief manifold and the admissibility structure are two names for the same representational necessity.

The convergence, on this reading, is not a coincidence of mathematical form but a deep agreement about what matters for any system that must act over extended time in a world with hidden structure. Both frameworks have discovered, through independent inquiry, that the answer to the question "what is the right thing to represent?" is: the geometry of reachable futures.

9. Conclusion: Navigation, Not Description

> *Knowledge is not identification of objects. Knowledge is synchronization to hidden process structure.*
>
> Notes

This paper has argued that the belief-state geometry of computational mechanics and the admissibility geometry of the process-primary framework are formal descriptions of the same underlying structure. Both identify the geometry of reachable futures as the correct representational primitive for systems that must predict or navigate over extended horizons. Both diagnose the same failure mode—the collapse of this geometry into a coarser description—as the source of degraded performance over time. Both prescribe the same remedy: the preservation of latent distinctions whose relevance may not be visible from the current evidential position.

The practical consequences are clear. For the evaluation of predictive systems, reachability fidelity should replace or supplement immediate predictive accuracy as the primary criterion. For the design of knowledge representation systems, belief-manifold geometry should replace entity-and-property ontologies as the organizing framework. For the analysis of institutional and organizational systems, the Admissibility Log should track changes to the belief manifold geometry—changes in what futures remain epistemically accessible—not merely changes in observable states.

The deepest consequence is philosophical. Classical epistemology treats knowledge as the accurate representation of what exists. The convergence of computational mechanics and process-primary ontology suggests a different picture: knowledge is the accurate representation of what remains reachable. The knower is not a mirror of the world but a navigator on a manifold of possible futures. The quality of knowledge is measured not by correspondence to current facts but by fidelity to the geometry of accessible trajectories.

That picture is not new. It is, in various forms, the picture that Heraclitus was pointing toward when he insisted that the river is a process, not an object. It is the picture that Wittgenstein was pointing toward when he argued that meaning is use, not reference. It is the picture that Jacobs was pointing toward when she insisted that cities are alive, not designed. And it is the picture that computational mechanics has now made mathematically precise: the world is better described by the geometry of its futures than by any catalog of its presents.

A. Formal Details of the Belief-Manifold Correspondence

A.1 Proof Sketch of Theorem 4.1

The reachable belief manifold $\mathcal{M}$ is defined as the set of all belief states arising from Bayesian updating on finite observation sequences. The admissibility structure $(\Delta(\mathcal{H}), \mathcal{X}, \{\to_x\})$ defines $b \to_x b'$ if and only if $b' = \text{Bayes}(b, x)$. The reachability cone of b_0 under this structure is:

$$R(b_0) = \{b : b_0 \to_{x_0} b_1 \to_{x_1} \cdots \to_{x_{t-1}} b_t = b,\ (x_0, \ldots, x_{t-1}) \in \mathcal{X}^*\}$$

By definition, $b_t = \text{Bayes}(\text{Bayes}(\cdots \text{Bayes}(b_0, x_0) \cdots), x_{t-1})$ is the belief state arising from the observation sequence $(x_0, \ldots, x_{t-1})$. This is exactly the set $\mathcal{M}$.

A.2 The Geometry-Preservation Requirement

Theorem 2.2 follows from the fact that the conditional distribution over future observations at horizon k is a continuous function of the belief state. If two belief states $b, b' \in \mathcal{M}$ are distinct, there exists some horizon k at which they imply different future distributions. Any predictor that achieves optimal loss at all horizons must therefore be able to distinguish b from b' at every position in its internal representation.

A.3 Reachability Fidelity and KL Divergence

An alternative formulation of reachability fidelity uses the KL divergence between future distributions. For belief states $b, b' \in \mathcal{M}$, define:

$$d_{\mathcal{M}}(b, b') = D_{\text{KL}}\left(P^\infty(\cdot \mid b) \,\|\, P^\infty(\cdot \mid b')\right)$$

where $P^\infty(\cdot \mid b)$ is the distribution over infinite future sequences implied by belief state b. This is a natural metric on the belief manifold that captures the total informational difference between two belief states over all future horizons.

A representation $f : \mathcal{M} \to \mathcal{R}$ has high reachability fidelity under this metric when it approximately preserves D_{KL} distances: belief states that are far apart in future-distribution space should be far apart in representation space, and belief states that are close should be close.

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