

# Kernel Operators, Memory, and Field-Theoretic Dynamics

Dynamics as Functionals of Trajectories

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## Abstract

Standard mathematical formulas are conventionally understood as isolated results. Fractional calculus is conventionally understood as an extension of classical calculus. Both framings conceal the same hidden object: a continuous operator algebra acting on trajectories rather than states, parameterized by a kernel and a memory order.

This monograph argues that replacing local, state-based operators with kernel operators on path space is not a technical generalization but a foundational reorientation. The central object is the *trajectory operator*  $\mathcal{T}_K[f](t) = \int_0^t K(t, \tau) F(f(\tau)) d\tau$ , of which classical differentiation, fractional calculus, and field-theoretic transport are all specializations.

Five existing theoretical frameworks are then shown to be representations of this object under different constraint and composition regimes: RSVP (scalar-vector-entropy field theory), Yarncrawler (sheaf-variational world-state reconstruction), Sphero-pop (irreversible event calculus), KES (kinetic-event synthesis as kernel composition), and Constraint Closure (trajectory fixed-point theory). The Unified Memory Theorem establishes that these frameworks are functorially related facets of a single kernel-operator structure, not independent constructions.

The core thesis: *dynamics are not functions of state but functionals of trajectories, and kernel operators provide the minimal algebra for expressing that dependence.* Memory, irreversibility, and constraint closure are algebraic consequences of operator structure, not additional hypotheses.

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## 1 Introduction

### 1.1 Formula Sheets as Operator Atlases

Consider the standard mathematical formula sheet. It lists, side by side, the quadratic formula, trigonometric identities, derivative rules, integral tables, fluid dynamics equations, kinematic relations. Presented as a flat collection of results, the sheet looks like a heterogeneous archive.

But there is a hidden architecture. Every formula on the sheet is a *transformation rule*: it maps one quantity to another under a specific constraint. The quadratic formula solves a polynomial constraint. The Pythagorean theorem encodes a geometric invariant. Derivative rules transform functions under the constraint of local approximability. Integral formulas accumulate values under the constraint of domain structure.

Taken together, the sheet is not a list but an *atlas*: a patchwork of local charts over a space of transformations. The hidden structure is that most of these charts fall into four deep classes—algebraic closure, geometric invariants, differential relations, and integral accumulations—and that the last two are more tightly linked than they appear.

That link is the subject of this monograph.

### 1.2 The Hidden Assumption: Locality

Every differential formula on the standard sheet implicitly assumes an underlying ontology: that the rate of change of a quantity at time  $t$  depends only on the quantity's behavior in an arbitrarily small neighborhood of  $t$ . This is the *locality assumption*.

It is not a law of nature. It is an architectural choice.

The classical derivative encodes this choice as a definition:

$$Df(t) = \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h}.$$

This limit is a pointwise operation. It extracts information from an infinitesimal neighborhood and discards everything else. The function's history—its values at all prior times—is irrelevant by construction.

The same assumption pervades the classical integral, the wave equation, the diffusion equation, kinematic formulas, and fluid conservation laws. Wherever there is a differential or integral operator in a formula, the locality assumption is at work.

This creates a problem for systems that actually remember. Viscoelastic materials

respond to the entire history of applied stress, not just to its current value. Anomalous diffusion in disordered media exhibits power-law waiting times that no finite-dimensional Markovian model can reproduce. Biological systems maintain memory of past inputs across long time windows. Financial instruments are priced against path-dependent risk. In all these cases, the formula-sheet framework either fails or requires the introduction of artificial hidden variables to simulate the memory it has excluded.

### 1.3 The Resolution: Kernel Operators on Trajectories

The resolution is not to add memory as a separate ingredient. It is to recognize that the local operator is a special case of a more general object: the *kernel operator on path space*.

Instead of asking “what is the rate of change at  $t$ ?”, we ask: “how does the entire past trajectory of the system, weighted by a specific kernel, determine the present behavior?”

The central object of this monograph is the *trajectory operator*:

$$\mathcal{T}_K[f](t) = \int_0^t K(t, \tau) F(f(\tau)) d\tau.$$

Here  $K(t, \tau)$  is the *kernel*—a function encoding how strongly the past state at time  $\tau$  influences the present behavior at time  $t$ . The function  $F$  encodes the system-specific dynamics.

Every other construction in this monograph is a specialization or composition of  $\mathcal{T}_K$ :

- Classical integration:  $K(t, \tau) = 1$ ,  $F = \text{id}$ .
- Classical differentiation:  $K(t, \tau) = \delta'(t - \tau)$  (singular limit).
- Fractional calculus:  $K_\alpha(t, \tau) = (t - \tau)^{-\alpha} / \Gamma(1 - \alpha)$ ,  $F = \text{id}$  or  $F = d/dt$ .
- RSVP field transport:  $K = K_\alpha$ ,  $F(\phi) = -\mathbf{v} \cdot \nabla \Phi + D_\Phi \Delta \Phi - \lambda S \Phi$ .
- Yarncrawler reconstruction:  $K = K_Y$  (the consistency kernel).
- KES event synthesis:  $K = K_{\text{syn}} = \psi \cdot \phi \cdot K_\alpha$ .
- Constraint closure: fixed-point condition  $\mathcal{T}_K[f^*] = f^*$ .

The monograph develops each of these specializations in full, but holds the operator  $\mathcal{T}_K$  invariant throughout. This is the principle of notation stability: every framework speaks the same algebraic language.

## 1.4 The Core Thesis

*Dynamics are not functions of state but functionals of trajectories, and kernel operators provide the minimal algebra for expressing that dependence.*

Classical mathematical physics was organized around states because states are computationally convenient. A finite-dimensional state vector is tractable; an infinite-dimensional trajectory is not. The locality assumption is the price paid for tractability.

Kernel operators recover the trajectory without sacrificing rigor. The semigroup law— $K_\alpha * K_\beta = K_{\alpha+\beta}$ —shows that the space of kernels is algebraically well-behaved. The trajectory operator  $\mathcal{T}_K$  provides the universal form. The kernel  $K$  plays the role of a Green's function over trajectory space, encoding how past configurations propagate forward under the system's internal constraints. Everything follows.

## 1.5 Organization

Section 2 establishes the mathematical foundations: path spaces, convolution, and the trajectory operator  $\mathcal{T}_K$ . Section 4 proves the semigroup structure of the fractional kernel family. Section 5 shows how fractional calculus arises as operator interpolation entirely within the kernel framework. Section 6 gives the categorical formulation:  $\mathcal{T}_K$  as a morphism, kernel composition as the compositional law. Sections 7–14 embed the five frameworks. Section 18 states and proves the Unified Memory Theorem. Section 19 draws out implications for physics, cognition, and computation.

## 2 Mathematical Preliminaries

We establish notation that will persist unchanged throughout the monograph. The key discipline: everything is expressed using  $K$ ,  $F$ , and  $\mathcal{T}_K$ . No competing notational systems are introduced later.

Let  $X$  be a Banach space and  $T > 0$  a fixed time horizon.

### 2.1 Path Space

**Definition 2.1** (Path space). *The path space over  $X$  on  $[0, T]$  is*

$$\mathcal{P} = \mathcal{P}([0, T], X) := \{f: [0, T] \rightarrow X \mid f \text{ measurable, locally integrable}\}.$$

Elements of  $\mathcal{P}$  are called *trajectories*. They are interpreted as histories rather than instantaneous states. Classical dynamics act on the point value  $f(t)$ ; kernel dynamics act on the restricted history  $f|_{[0,t]}$ .

This distinction is not aesthetic. When the state space of a system is  $X = \mathbb{R}^n$ , a classical analysis selects a point in  $\mathbb{R}^n$  at each time. A trajectory-based analysis selects a point in the infinite-dimensional space  $\mathcal{P}$ . These are different objects and carry different information.

**Definition 2.2** (Causal restriction). *For  $0 \leq s \leq t \leq T$ , the causal restriction  $\rho_{s,t}: \mathcal{P} \rightarrow \mathcal{P}([s,t], X)$  is defined by  $\rho_{s,t}(f) = f|_{[s,t]}$ .*

Causal restrictions are transitive: if  $r \leq s \leq t$  then  $\rho_{r,s} \circ \rho_{r,t} = \rho_{r,s}$ . This is the sheaf-presheaf compatibility condition; we return to it in Section 8.

## 2.2 Kernels

**Definition 2.3** (Kernel). *A kernel is a measurable function*

$$K: \Delta_T \rightarrow \mathbb{R}, \quad \Delta_T = \{(t, \tau) \in [0, T]^2: 0 \leq \tau \leq t \leq T\},$$

such that

$$\|K\| := \sup_{t \in [0, T]} \int_0^t |K(t, \tau)| d\tau < \infty.$$

The space of all such kernels is denoted  $\mathcal{K}(\Delta_T)$ .

The triangular domain  $\Delta_T$  encodes causality:  $K(t, \tau)$  is defined only for  $\tau \leq t$ , so the kernel cannot reach into the future.

## 2.3 Kernel Convolution

**Definition 2.4** (Convolution of a kernel with a trajectory). *For  $K \in \mathcal{K}(\Delta_T)$  and  $f \in \mathcal{P}$ , the convolution is*

$$(K * f)(t) := \int_0^t K(t, \tau) f(\tau) d\tau.$$

This defines a causal operator on  $\mathcal{P}$ : the output at time  $t$  depends only on  $f|_{[0,t]}$ .

**Definition 2.5** (Composition of kernels). *For  $K_1, K_2 \in \mathcal{K}(\Delta_T)$ , their composition is*

$$(K_1 \circ K_2)(t, \tau) := \int_\tau^t K_1(t, s) K_2(s, \tau) ds.$$

**Lemma 2.6** (Boundedness). *For any  $K \in \mathcal{K}(\Delta_T)$  and  $f \in L^1([0, T], X)$ ,*

$$\|(K * f)(t)\|_X \leq \|K\| \cdot \|f\|_{L^1}.$$

*Proof.*  $|(K * f)(t)| \leq \int_0^t |K(t, \tau)| \cdot |f(\tau)| d\tau \leq (\sup_t \int_0^t |K|) \cdot \|f\|_{L^1} = \|K\| \cdot \|f\|_{L^1}$ .  $\square$

## 2.4 The Trajectory Operator

We now introduce the central object. It incorporates both the kernel (memory structure) and the system dynamics (the function  $F$ ).

**Definition 2.7** (Trajectory operator). *Let  $K \in \mathcal{K}(\Delta_T)$  and  $F: X \rightarrow X$  measurable. The trajectory operator associated to  $(K, F)$  is*

$$\mathcal{T}_K[f](t) := \int_0^t K(t, \tau) F(f(\tau)) d\tau.$$

Every construction in this monograph is a specialization or composition of  $\mathcal{T}_K$ . The following table records the principal instances:

Framework	Kernel $K(t, \tau)$	Dynamics $F$
Integration	1	id
Fractional calculus	$(t - \tau)^{-\alpha} / \Gamma(1 - \alpha)$	$d/dt$
RSVP transport	$K_\alpha(t - \tau)$	$-\mathbf{v} \cdot \nabla \Phi + D_\Phi \Delta \Phi - \lambda S \Phi$
KES synthesis	$\psi(t) \cdot \phi(\tau) \cdot K_\alpha(t - \tau)$	id
Constraint closure	general contractive $K$	id

**Remark 2.8** (What varies; what does not). *The kernel  $K$  controls how much of the past contributes and with what weight. The function  $F$  controls what the system does with each past state. Separating these two roles is the architectural choice that makes notation stable across all five frameworks.*

## 2.5 Volterra Equations and the Resolvent

Evolution equations expressed through  $\mathcal{T}_K$  take the form of Volterra integral equations.

**Definition 2.9** (Volterra equation of the second kind). A Volterra equation of the second kind for unknown  $f \in \mathcal{P}$  is

$$f(t) - \lambda(K * f)(t) = g(t), \quad t \in [0, T],$$

for given  $g \in \mathcal{P}$  and  $\lambda \in \mathbb{R}$ .

**Theorem 2.10** (Resolvent and unique solvability). For any  $K \in \mathcal{K}(\Delta_T)$ ,  $g \in L^\infty([0, T])$ , and  $\lambda \in \mathbb{R}$ , the Volterra equation of the second kind has a unique solution  $f \in L^\infty([0, T])$ , given by the Neumann series

$$f(t) = g(t) + \lambda \int_0^t R(t, \tau; \lambda) g(\tau) d\tau,$$

where the resolvent kernel is  $R(t, \tau; \lambda) = \sum_{n=1}^{\infty} \lambda^n K^{(n)}(t, \tau)$ , with  $K^{(n)}$  the  $n$ -fold iterated composition.

*Proof.* Bound each iterated kernel by  $K^{(n)}(t, \tau) \leq \|K\|^n (t - \tau)^{n-1} / (n - 1)!$ , giving convergence of the Neumann series in  $L^\infty$  norm uniformly in  $t$ .  $\square$

The resolvent theorem guarantees that all evolution equations expressed through  $\mathcal{T}_K$  with bounded kernels are well-posed. This will be invoked implicitly in each of the five framework sections.

**Bridge to Kernel Algebra.** The trajectory operator  $\mathcal{T}_K$  introduced above isolates two degrees of freedom: the kernel  $K$ , governing temporal memory structure, and  $F$ , governing local state transformation. The question is no longer how a single kernel acts, but how kernels compose. The answer is that  $\mathcal{K}(\Delta_T)$  carries an intrinsic composition law inherited from iterated trajectory evolution, admitting a one-parameter semigroup structure. But first, we derive  $\mathcal{T}_K$  itself from a variational principle, showing it is not an arbitrary construction but the Euler–Lagrange operator of a natural action on path space.

### 3 Variational Origin of Kernel Operators

#### 3.1 From Local Action to Path Functionals

Classical mechanics derives dynamics from the local action  $\mathcal{S}[x] = \int_0^T L(x(t), \dot{x}(t)) dt$ , whose stationary points satisfy the Euler–Lagrange equations. To incorporate memory,

we generalize to a *path-dependent* functional depending on pairs of points along the trajectory, weighted by a kernel.

**Definition 3.1** (Nonlocal action functional). *Let  $K \in \mathcal{K}(\Delta_T)$  and  $\mathcal{L}: X \times X \rightarrow \mathbb{R}$  a measurable interaction Lagrangian. Define*

$$\mathcal{F}[f] := \int_0^T \int_0^t K(t, \tau) \mathcal{L}(f(t), f(\tau)) d\tau dt.$$

The kernel determines how strongly past states  $f(\tau)$  influence the present in the variational principle itself.

### 3.2 First Variation

Let  $f_\varepsilon = f + \varepsilon h$  with  $h \in \mathcal{P}$  compactly supported in  $(0, T)$ . The first variation is

$$\left. \frac{d}{d\varepsilon} \mathcal{F}[f_\varepsilon] \right|_{\varepsilon=0} = \int_0^T \int_0^t K(t, \tau) [\partial_1 \mathcal{L}(f(t), f(\tau)) h(t) + \partial_2 \mathcal{L}(f(t), f(\tau)) h(\tau)] d\tau dt.$$

Separating and swapping integration order in the second term yields two contributions: a *term at  $t$*  involving  $\partial_1 \mathcal{L}$  and a *term at  $\tau$*  involving  $\partial_2 \mathcal{L}$  integrated forward.

### 3.3 Euler–Lagrange Equation on Path Space

Stationarity  $\delta \mathcal{F} = 0$  for all  $h$  gives the *nonlocal Euler–Lagrange equation*:

$$\int_0^t K(t, \tau) \partial_1 \mathcal{L}(f(t), f(\tau)) d\tau + \int_t^T K(s, t) \partial_2 \mathcal{L}(f(s), f(t)) ds = 0. \quad (1)$$

This couples past and future through the kernel.

### 3.4 Causal Reduction to the Trajectory Operator

**Definition 3.2** (Causal separable Lagrangian). *A Lagrangian is causal separable if  $\mathcal{L}(x, y) = \Phi(x) - \langle x, F(y) \rangle$  for some potential  $\Phi$  and map  $F$ .*

Under this structure, equation (1) reduces to  $\nabla \Phi(f(t)) = \int_0^t K(t, \tau) F(f(\tau)) d\tau$ . If  $\nabla \Phi$  is invertible, we obtain

$$f(t) = (\nabla \Phi)^{-1} \left( \int_0^t K(t, \tau) F(f(\tau)) d\tau \right). \quad (2)$$

In the canonical case  $\Phi(x) = \frac{1}{2}\|x\|^2$ , equation (2) becomes exactly  $f(t) = \mathcal{T}_K[f](t)$ .

**Theorem 3.3** (Variational origin of  $\mathcal{T}_K$ ). *The trajectory operator  $\mathcal{T}_K$  is the Euler–Lagrange operator of  $\mathcal{F}$  with a causal separable Lagrangian and  $\Phi(x) = \frac{1}{2}\|x\|^2$ .*

### 3.5 Consequences

- **Classical mechanics:**  $K = \delta$  recovers a local Lagrangian and the classical derivative.
- **Fractional dynamics:**  $K = K_\alpha$  produces power-law memory through the same variational principle.
- **Constraint closure:**  $\mathcal{T}_K[f^*] = f^*$  is equivalent to stationarity of  $\mathcal{F}$ , linking fixed points to variational equilibria.
- **Nonlocality as structure:** memory is a structural consequence of an action depending on pairs of trajectory points.

**Proposition 3.4** (Variational form of constraint closure). *A trajectory  $f^*$  satisfies  $\mathcal{T}_K[f^*] = f^*$  if and only if it is a stationary point of  $\mathcal{F}$  with  $\Phi(x) = \frac{1}{2}\|x\|^2$ .*

*Proof.* Under this choice of  $\Phi$ , the Euler–Lagrange equation reduces directly to  $f = \mathcal{T}_K[f]$ . □

**Bridge to Kernel Algebra.** Theorem 3.3 establishes that  $\mathcal{T}_K$  is the canonical output of a variational principle. The remaining freedom lies in the choice of kernel  $K$ . The next section imposes algebraic structure on this choice by analyzing how kernels compose under sequential trajectory evolution, revealing the fractional semigroup as the unique scale-invariant solution.

## 4 Kernel Operators and the Semigroup Structure

### 4.1 The Monoid of Kernels

**Theorem 4.1** (Monoid structure).  *$(\mathcal{K}(\Delta_T), \circ, \delta)$  is a monoid, where  $\delta(t, \tau) = \delta(t - \tau)$  (the Dirac distribution) is the identity kernel, satisfying:*

$$(i) \text{ **Associativity:}** } (K_1 \circ K_2) \circ K_3 = K_1 \circ (K_2 \circ K_3).$$

(ii) **Left unit:**  $\delta \circ K = K$ .

(iii) **Right unit:**  $K \circ \delta = K$ .

*Proof.* Associativity: both  $(K_1 \circ K_2) \circ K_3$  and  $K_1 \circ (K_2 \circ K_3)$  expand to

$$\int_{\tau}^t \int_r^t K_1(t, s) K_2(s, r) K_3(r, \tau) ds dr,$$

the equality following from Fubini's theorem on the simplex  $\{\tau \leq r \leq s \leq t\}$ . Unitality:  $(\delta \circ K)(t, \tau) = \int_{\tau}^t \delta(t-s)K(s, \tau) ds = K(t, \tau)$ ; similarly on the right.  $\square$

## 4.2 Fractional Kernels

**Definition 4.2** (Fractional kernel family). *For  $\alpha \in (0, 1)$ , the fractional kernel of order  $\alpha$  is*

$$K_{\alpha}(t, \tau) = \frac{(t - \tau)^{-\alpha}}{\Gamma(1 - \alpha)}.$$

*The family extends to boundary cases:  $K_0 = \delta$  (identity,  $\alpha = 0$ ) and  $K_1(t, \tau) \equiv 1$  (uniform integration,  $\alpha = 1$ ).*

**Remark 4.3.** *The kernel  $K_{\alpha}$  is positive, causal, and scale-invariant under the substitution  $(t, \tau) \mapsto (\lambda t, \lambda \tau)$ . These three properties together characterize the fractional family within all possible causal kernels.*

## 4.3 The Semigroup Law

**Theorem 4.4** (Semigroup law). *For all  $\alpha, \beta \in [0, 1]$ ,*

$$K_{\alpha} \circ K_{\beta} = K_{\alpha+\beta}.$$

*The family  $(K_{\alpha})_{\alpha \geq 0}$  is therefore a one-parameter semigroup in  $(\mathcal{K}(\Delta_T), \circ)$ .*

*Proof.* We compute directly:

$$(K_{\alpha} \circ K_{\beta})(t, \tau) = \int_{\tau}^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \cdot \frac{(s-\tau)^{-\beta}}{\Gamma(1-\beta)} ds.$$

Substituting  $s = \tau + (t - \tau)u$ , so that  $ds = (t - \tau) du$ :

$$\begin{aligned} &= \frac{(t - \tau)^{-\alpha}}{\Gamma(1 - \alpha)} \cdot \frac{(t - \tau)^{-\beta}}{\Gamma(1 - \beta)} \cdot (t - \tau) \int_0^1 (1 - u)^{-\alpha} u^{-\beta} du \\ &= \frac{(t - \tau)^{1 - \alpha - \beta}}{\Gamma(1 - \alpha)\Gamma(1 - \beta)} \cdot B(1 - \alpha, 1 - \beta). \end{aligned}$$

The Beta function satisfies  $B(1 - \alpha, 1 - \beta) = \Gamma(1 - \alpha)\Gamma(1 - \beta)/\Gamma(2 - \alpha - \beta)$ . Using  $\Gamma(2 - \alpha - \beta) = (1 - \alpha - \beta)\Gamma(1 - \alpha - \beta)$ , we obtain

$$(K_\alpha \circ K_\beta)(t, \tau) = \frac{(t - \tau)^{1 - (\alpha + \beta)}}{\Gamma(2 - (\alpha + \beta))} = \frac{(t - \tau)^{-((\alpha + \beta) - 1)}}{\Gamma(1 - (\alpha + \beta)) \cdot (1 - (\alpha + \beta))},$$

which simplifies to  $K_{\alpha + \beta}(t, \tau)$  after applying the recurrence  $\Gamma(z + 1) = z\Gamma(z)$  with  $z = 1 - (\alpha + \beta)$ .  $\square$

**Remark 4.5** (Why this is not a coincidence). *The semigroup law is the algebraic reason that fractional calculus is consistent. The composition  $K_{1/2} \circ K_{1/2} = K_1$  says that applying a half-order derivative twice yields the first-order derivative—not by definition, but as a consequence of the Beta function identity. The semigroup law is the algebraic inevitability of the power-law kernel.*

#### 4.4 The Generator

**Definition 4.6** (Semigroup generator). *The generator of the semigroup  $(K_\alpha)$  characterizes how infinitesimal increases in memory order deform the operator, providing the analogue of a differential generator for the fractional semigroup. It is the operator*

$$\mathcal{L}f := \lim_{\alpha \rightarrow 0^+} \frac{\mathcal{T}_K[\text{id}][f] - f}{\alpha} = \lim_{\alpha \rightarrow 0^+} \frac{\int_0^t K_\alpha(t, \tau) f(\tau) d\tau - f(t)}{\alpha},$$

when the limit exists in  $L^\infty$ .

**Proposition 4.7.** *The generator acts as*

$$\mathcal{L}f(t) = \int_0^t \ln \frac{1}{t - \tau} f(\tau) d\tau - \gamma f(t),$$

where  $\gamma = -\Gamma'(1)$  is the Euler–Mascheroni constant. *This is a logarithmic memory kernel: as  $\alpha \rightarrow 0$ , the range of historical dependence diverges logarithmically.*

**Bridge to Operator Interpolation.** The semigroup law  $K_\alpha \circ K_\beta = K_{\alpha+\beta}$  shows that kernel composition is parameterized by a continuous order variable, so classical operators treated as discrete objects are in fact points along a continuous family. What appears as a distinction between differentiation and integration is a difference in kernel order. Fractional calculus does not extend classical calculus; it reveals that classical calculus is a degenerate slice of a larger operator algebra. The next section makes this identification explicit by expressing both as instances of  $\mathcal{T}_K$ .

## 5 Fractional Calculus as Operator Interpolation

### 5.1 Integration and Differentiation as Kernel Operators

We now express classical integration and differentiation as specializations of  $\mathcal{T}_K$ , bringing them into the same notational family as the fractional operators. This is the step that makes fractional calculus a *consequence* rather than an *extension*.

#### 5.1.1 Integration

The classical integral is

$$(If)(t) = \int_0^t f(\tau) d\tau = (K_1 * f)(t) \quad \text{with} \quad K_1(t, \tau) \equiv 1.$$

So: *integration is  $\mathcal{T}_K$  with the uniform kernel and  $F = \text{id}$ .*

#### 5.1.2 Differentiation

The classical derivative is formally

$$(Df)(t) = \int_0^t \delta'(t - \tau) f(\tau) d\tau,$$

where  $\delta'$  is the distributional derivative of the Dirac delta. In trajectory-operator language: *differentiation is  $\mathcal{T}_K$  with the singular kernel  $K(t, \tau) = \delta'(t - \tau)$  and  $F = \text{id}$ .*

The singularity of this kernel is precisely the locality assumption made visible. The delta kernel has zero support width: it contributes only the instantaneous value. Every other kernel in  $\mathcal{K}(\Delta_T)$  has positive support and therefore encodes memory.

## 5.2 Fractional Operators as Kernel Operators

**Definition 5.1** (Fractional integral). *For  $\beta > 0$ , the fractional integral of order  $\beta$  is*

$$(I^\beta f)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - \tau)^{\beta-1} f(\tau) d\tau = (K_{1-\beta} * f)(t).$$

This is  $\mathcal{T}_K$  with  $K = K_{1-\beta}$  and  $F = \text{id}$ .

**Definition 5.2** (Caputo fractional derivative). *For  $\alpha \in (0, 1)$ , the Caputo derivative of order  $\alpha$  is*

$$(D^\alpha f)(t) = \int_0^t \frac{(t - \tau)^{-\alpha}}{\Gamma(1 - \alpha)} f'(\tau) d\tau = \mathcal{T}_K[f'](t),$$

with  $K = K_\alpha(t, \tau) = (t - \tau)^{-\alpha}/\Gamma(1 - \alpha)$  and  $F = d/dt$ .

This is the key identification. The Caputo derivative is exactly  $\mathcal{T}_K$  with the fractional kernel and differentiation as the dynamics function. No new notation is required.

**Definition 5.3** (Riemann–Liouville derivative). *For  $\alpha \in (0, 1)$ , the Riemann–Liouville derivative is*

$$(D_{\text{RL}}^\alpha f)(t) = \frac{d}{dt}(I^{1-\alpha} f)(t) = \frac{d}{dt}(K_\alpha * f)(t).$$

Both forms are derivable from  $\mathcal{T}_K$  with the fractional kernel family. The Caputo form (Definition 5.2) places differentiation inside the integral; the Riemann–Liouville form (Definition 5.3) places it outside. For smooth functions, they agree; for functions with nontrivial initial values, the Caputo form is often more natural for physical applications.

## 5.3 The Operator Interpolation Table

The following table displays the continuous transition across the kernel family:

Order $\alpha$	Kernel $K_\alpha(t, \tau)$	Operator
$\alpha = 0$	$\delta(t - \tau)$	Identity
$\alpha \in (0, 1)$	$(t - \tau)^{-\alpha}/\Gamma(1 - \alpha)$	Fractional derivative
$\alpha = 1$	1	Integral
$\alpha < 0$	$(t - \tau)^{ \alpha -1}/\Gamma( \alpha )$	Fractional integral

The table displays what the notes called the “operator as a dial”: positive  $\alpha$  extracts change (differentiates), negative  $\alpha$  accumulates history (integrates), and intermediate values do both simultaneously.

#### 5.4 The Loss of Markov Structure

**Definition 5.4** (Markovian dynamics). *A dynamical system  $u(t)$  is Markovian if its future evolution from time  $t$  depends only on  $u(t)$ , not on the full trajectory  $u|_{[0,t]}$ .*

**Proposition 5.5** (Non-Markovian character of fractional dynamics). *Let  $u$  satisfy the fractional evolution equation  $D^\alpha u(t) = F(u(t))$  for  $\alpha \in (0, 1)$ . Then  $u$  is non-Markovian.*

*Proof.* Expanding the Caputo derivative:

$$\int_0^t \frac{(t - \tau)^{-\alpha}}{\Gamma(1 - \alpha)} u'(\tau) d\tau = F(u(t)).$$

The left side is a Volterra integral over the full history  $u|_{[0,t]}$ . No finite augmentation of the state vector renders this Markovian, since the kernel  $(t - \tau)^{-\alpha}$  is not exponential and therefore not generated by any finite-dimensional ODE (exponential kernels are the only ones arising from finite-rank semigroups).  $\square$

**Remark 5.6** (The ontological shift). *This is not a perturbation of classical behavior. The state space of a fractional system is not  $X$  but  $\mathcal{P}$ . A point in  $\mathcal{P}$  carries the full history; a point in  $X$  carries only the present. These are categorically different objects, and the passage from one to the other is not a limit but a change of domain.*

#### 5.5 Fractional Diffusion as a Worked Example

We carry the diffusion equation through the full transition as the worked example the notes require.

##### 5.5.1 Classical diffusion

$$\partial_t \Phi(x, t) = D_\Phi \Delta \Phi(x, t).$$

In trajectory-operator language with  $K = \delta$  and  $F(\Phi) = D_\Phi \Delta \Phi$ :

$$\partial_t \Phi = \mathcal{T}_K[\text{id}][\Phi](t)|_{K=\delta},$$

with local response: the present curvature of  $\Phi$  drives instantaneous change.

### 5.5.2 Fractional diffusion

Replace  $\partial_t$  with the Caputo operator  $D_t^\alpha$ :

$$D_t^\alpha \Phi(x, t) = D_\Phi \Delta \Phi(x, t), \quad \alpha \in (0, 1).$$

Expanding:

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \partial_\tau \Phi(x, \tau) d\tau = D_\Phi \Delta \Phi(x, t).$$

The left side is exactly  $\mathcal{T}_{K_\alpha}[\partial_t \Phi](t)$ . The causal geometry is now: *curvature generates change, but that change is integrated through a long-memory filter before affecting the field.*

### 5.5.3 Anomalous scaling

Solutions to fractional diffusion satisfy

$$\langle x^2(t) \rangle = \frac{2D_\Phi}{\Gamma(1+\alpha)} t^\alpha,$$

as opposed to the classical  $\langle x^2 \rangle \propto t$ . The exponent  $\alpha < 1$  marks subdiffusion: the system spreads more slowly because it carries memory of where it has been. This is the physical signature of the power-law kernel.

**Bridge to Categorical Structure.** Once differentiation, integration, and their fractional interpolants are expressed uniformly as kernel operators, the distinction between operators becomes purely structural: each is determined by its position in the kernel algebra, and composition of operators corresponds to convolution of kernels. This invites a categorical reformulation in which operators are morphisms between trajectory spaces, with kernel composition as the fundamental law. The semigroup structure becomes functorial, and the entire operator calculus is subsumed into a single category  $\mathcal{KOp}$ .

## 6 The Category of Kernel Operators

### 6.1 Why Categorical Language

The monoid structure of Section 4 organizes kernels acting on a single path space  $\mathcal{P}$ . To describe morphisms between different trajectory spaces—to encode how one system's

history influences another's evolution—we need a richer structure. Category theory provides it.

A category consists of *objects* and *morphisms* (arrows) between them, with an associative composition law and identity morphisms. The categorical language here does one thing: it names the structure that is already present in the kernel algebra. It does not add new content. The reader unfamiliar with category theory can understand every result by interpreting “morphism” as “kernel operator” and “functor” as “kernel-preserving map between systems.”

## 6.2 The Category $\mathcal{KOp}$

**Definition 6.1** (The category  $\mathcal{KOp}$ ). • **Objects:** path spaces  $\mathcal{P}([0, T], X)$  for Banach spaces  $X$  and intervals  $[0, T]$ .

- **Morphisms:**  $\text{Hom}(\mathcal{P}, \mathcal{P}') = \{(K, F) : \mathcal{T}_K \text{ maps } \mathcal{P} \text{ to } \mathcal{P}'\}$ .
- **Composition:**  $(K_1, F_1) \circ (K_2, F_2) = (K_1 \circ K_2, F_1 \circ F_2)$ .
- **Identities:**  $(\delta, \text{id})$ .

**Theorem 6.2** ( $\mathcal{KOp}$  is a category). *Definition 6.1 satisfies the categorical axioms.*

*Proof.* Associativity of composition follows from Theorem 4.1 (kernel associativity) and associativity of function composition. Identity morphisms exist:  $\mathcal{T}_\delta[f](t) = \int_0^t \delta(t - \tau)f(\tau) d\tau = f(t)$ , so  $(\delta, \text{id})$  acts as the identity on every path space.  $\square$

## 6.3 The Semigroup as a Functor

**Definition 6.3** (Semigroup functor). *Let  $([0, \infty), +)$  be the additive monoid of non-negative reals, viewed as a one-object category with morphisms  $\alpha: * \rightarrow *$ . The semigroup functor is*

$$F_{\text{sg}}: ([0, \infty), +) \rightarrow \mathcal{KOp}, \quad F_{\text{sg}}(\alpha) = (K_\alpha, \text{id}).$$

**Proposition 6.4** (Functoriality).  *$F_{\text{sg}}$  is a functor:  $F_{\text{sg}}(\alpha + \beta) = F_{\text{sg}}(\alpha) \circ F_{\text{sg}}(\beta)$  and  $F_{\text{sg}}(0) = \text{id}$ .*

*Proof.* The composition law  $F_{\text{sg}}(\alpha + \beta) = (K_{\alpha+\beta}, \text{id}) = (K_\alpha \circ K_\beta, \text{id}) = F_{\text{sg}}(\alpha) \circ F_{\text{sg}}(\beta)$  is Theorem 4.4. The unit law  $F_{\text{sg}}(0) = (K_0, \text{id}) = (\delta, \text{id}) = \text{id}$  holds by definition.  $\square$

The interpolation between classical operators is now categorical: moving along the semigroup functor from  $\alpha = 0$  to  $\alpha = 1$  is a path in  $\mathcal{KOp}$  from differentiation to integration.

## 6.4 Natural Transformations as Kernel Changes

A *natural transformation*  $\eta: F \Rightarrow G$  between two functors assigns to each object a morphism that commutes with all arrows. In the kernel context, a natural transformation is a change-of-kernel operation that commutes with temporal evolution: if two kernels  $K, K'$  are related by a natural transformation, then switching between them at any time  $t$  produces the same result as switching at any other time. This is the form of gauge invariance relevant to RSVP field theory.

## 6.5 The Linear Subcategory $\mathbf{Ker}$

For many purposes it is convenient to work with the linear subcategory in which  $F = \text{id}$  and morphisms are pure kernel operators.

**Definition 6.5** (The category  $\mathbf{Ker}$ ). *The category  $\mathbf{Ker}$  has:*

- **Objects:** trajectory spaces  $\mathcal{P}$ .
- **Morphisms:** kernel operators  $\mathcal{K}: \mathcal{P} \rightarrow \mathcal{P}$  of the form  $(\mathcal{K}f)(t) = \int_0^t K(t, \tau) f(\tau) d\tau$ .
- **Composition:** kernel convolution.
- **Identities:** the Dirac kernel  $K_{\text{id}}(t, \tau) = \delta(t - \tau)$ .

$\mathbf{Ker}$  is a full subcategory of  $\mathbf{KOp}$  obtained by restricting to  $F = \text{id}$ . The fractional semigroup  $\{\mathcal{K}_\alpha\}_{\alpha \geq 0}$  is a one-parameter family of morphisms in  $\mathbf{Ker}$  satisfying  $\mathcal{K}_\alpha \circ \mathcal{K}_\beta = \mathcal{K}_{\alpha+\beta}$  (Theorem 4.4), making it a subcategory in its own right.

**Remark 6.6** (Nonlinear extension). *Trajectory operators  $\mathcal{T}_K$  with  $F \neq \text{id}$  do not in general close under simple kernel composition, because  $\mathcal{T}_K[f] \circ \mathcal{T}_{K'}[f] \neq \mathcal{T}_{K \circ K'}[f]$  when  $F$  is nonlinear. However, they form a category under composed Volterra operators where both the kernel and nonlinear map are tracked jointly as pairs  $(K, F)$ , as formalized in Definition 6.1.*

## 6.6 Limits as Global Consistency

A *limit* of a diagram of path spaces is a universal compatible system: a path space  $L$  with restriction maps  $\pi_i: L \rightarrow \mathcal{P}_i$  such that every other compatible system factors uniquely through  $L$ . When the diagram is the restriction system  $\{\mathcal{P}(U_i)\}$  for an open cover  $\{U_i\}$  of parameter space, the limit is the space of globally consistent trajectories—the sheaf-theoretic content of Yarncrawler.

## 7 RSVP Field Theory

### 7.1 The Standard RSVP System

The Relativistic Scalar-Vector-Plenum (RSVP) framework models physical dynamics through three coupled fields on a domain  $\Omega \subseteq \mathbb{R}^n$ :

- $\Phi(x, t)$ : the *scalar plenum field* (energy density, information potential).
- $\mathbf{v}(x, t)$ : the *vector field* (transport, directed flow).
- $S(x, t)$ : the *entropy field* (irreversibility, thermal uncertainty).

The local RSVP evolution equations are:

$$\partial_t \Phi + \mathbf{v} \cdot \nabla \Phi = D_\Phi \Delta \Phi - \lambda S \Phi, \quad (3)$$

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P + \mu \Delta \mathbf{v} - \kappa \nabla S, \quad (4)$$

$$\partial_t S + \mathbf{v} \cdot \nabla S = D_S \Delta S + \sigma |\nabla \mathbf{v}|^2. \quad (5)$$

In trajectory-operator language, equation (3) says:  $\partial_t \Phi = \mathcal{T}_K[\text{id}][\Phi]$  with  $K = \delta$  and  $F(\Phi) = -\mathbf{v} \cdot \nabla \Phi + D_\Phi \Delta \Phi - \lambda S \Phi$ . It is local:  $K = \delta$ .

### 7.2 Kernel-Weighted RSVP

We replace locality with a general kernel from the semigroup family.

**Definition 7.1** (Kernel-weighted RSVP). *The kernel-weighted RSVP system is:*

$$\partial_t \Phi(x, t) + \int_0^t K_\alpha(t - \tau) \mathbf{v}(x, \tau) \cdot \nabla \Phi(x, \tau) d\tau = D_\Phi \Delta \Phi - \lambda S \Phi, \quad (6)$$

$$\partial_t \mathbf{v}(x, t) + \int_0^t K_\beta(t - \tau) (\mathbf{v} \cdot \nabla) \mathbf{v}(x, \tau) d\tau = -\nabla P + \mu \Delta \mathbf{v} - \kappa \nabla S, \quad (7)$$

$$\partial_t S(x, t) + \int_0^t K_\gamma(t - \tau) \mathbf{v}(x, \tau) \cdot \nabla S(x, \tau) d\tau = D_S \Delta S + \sigma |\nabla \mathbf{v}|^2. \quad (8)$$

The orders  $\alpha, \beta, \gamma \in (0, 1)$  may differ by field.

In trajectory-operator form, equation (6) is:

$$\partial_t \Phi = \mathcal{T}_{K_\alpha}[-\mathbf{v} \cdot \nabla \Phi](t) + D_\Phi \Delta \Phi - \lambda S \Phi.$$

The convective term is exactly  $\mathcal{T}_K$  with the fractional kernel and  $F = -\mathbf{v} \cdot \nabla$ . The  $\delta$ -kernel limit ( $\alpha \rightarrow 0$ ) recovers equation (3).

### 7.3 Physical Interpretation

The kernel  $K_\alpha$  in equation (6) means: *transport of the scalar field at time  $t$  is governed not by the instantaneous velocity field, but by the weighted history of velocity directions.* High- $\alpha$  systems have long transport memory; the plenum “remembers” where it has been flowing and continues in that direction.

In RSVP terms:

- $\Phi$ : the plenum carries a persistent potential that diffuses, is transported with memory, and is damped by entropy.
- $\mathbf{v}$ : transport velocity has inertia encoded by  $K_\beta$ ; old flow directions decay by a power law rather than instantaneously.
- $S$ : entropy advection has long memory ( $K_\gamma$ ); dissipation from past velocity gradients accumulates in the entropy field.

This gives the precise RSVP interpretation of fractional calculus that the notes point toward:

fractional operator = built-in entropy-weighted transport.

### 7.4 Well-Posedness

**Theorem 7.2** (Local well-posedness). *For initial data  $\Phi_0 \in H^2(\Omega)$ ,  $\mathbf{v}_0 \in H^2(\Omega)^n$ ,  $S_0 \in H^1(\Omega)$ , there exists  $T^* > 0$  and a unique strong solution to the kernel-weighted RSVP system on  $[0, T^*]$ .*

*Proof.* For fixed  $\mathbf{v}, S$ , equation (6) is a linear Volterra integro-differential equation in  $\Phi$ . By Theorem 2.10, it has a unique  $L^\infty$ -valued solution. The full system is handled by a contraction-mapping argument on  $H^2 \times H^2 \times H^1$ , with contraction constant controlled by  $T^* < \infty$  chosen sufficiently small.  $\square$

### 7.5 Entropy Growth Bound

**Proposition 7.3** (Second law compatibility). *Solutions to the kernel-weighted RSVP system satisfy*

$$\frac{d}{dt} \int_{\Omega} S(x, t) dx \geq \sigma \int_{\Omega} |\nabla \mathbf{v}(x, t)|^2 dx.$$

*Proof.* Integrate equation (8) over  $\Omega$ ; divergence terms vanish by integration by parts. The kernel transport term has definite sign since  $K_\gamma \geq 0$ . The dissipation source  $\sigma|\nabla\mathbf{v}|^2 \geq 0$  provides the lower bound.  $\square$

Memory-weighted RSVP obeys the second law: entropy grows at least as fast as viscous dissipation requires. The kernel amplifies rather than violates irreversibility.

## 7.6 The RSVP Scalar Field as a Fixed Point

The kernel-weighted scalar equation (6) can be rewritten in a form that connects directly to the constraint-closure machinery of Section 14. Integrating (6) with respect to  $t$  and collecting terms, the scalar field satisfies

$$\Phi(x, t) = \Phi(x, 0) + \int_0^t K_\alpha(t - \tau) F(\Phi(x, \tau), \mathbf{v}(x, \tau), S(x, \tau)) d\tau,$$

where

$$F(\Phi, \mathbf{v}, S) = -\mathbf{v} \cdot \nabla\Phi + D_\Phi \Delta\Phi - \lambda S\Phi.$$

This is precisely  $\Phi = \Phi_0 + \mathcal{T}_K[\Phi]$  in trajectory-operator notation. With homogeneous initial data, it reduces to the fixed-point condition

$$\Phi = \mathcal{T}_K[\Phi],$$

identifying RSVP solutions as fixed points of the trajectory operator  $\mathcal{T}_K$  with  $K = K_\alpha$  and the above  $F$ . This is the same fixed-point structure that appears in constraint closure (Section 14) and Yarncrawler reconstruction (Section 8): RSVP field theory is an instance of the same algebraic pattern.

## 7.7 Structural Role of the Three Fields

Within  $\mathcal{T}_K$ , the three RSVP fields play distinct roles:

- $\Phi$  is the trajectory being evolved—the argument of the operator.
- $\mathbf{v}$  enters through  $F$  as the directional transport term; it shapes how  $\Phi$ 's history is used.
- $S$  enters through  $F$  as a damping modulation; it can be interpreted as controlling the effective memory depth by suppressing the contribution of past states in high-entropy regions.

The notes observe that this suggests a deeper coupling:  $K = K(t, \tau; S)$ , in which the entropy field directly modulates the kernel. This is the self-modulating regime flagged in Section 18 as a future direction; the present section establishes the structural preparation for it.

**Definition 7.4** (RSVP functor). *The RSVP functor  $\mathcal{R}: \mathcal{KOp} \rightarrow \mathbf{FieldSys}$  assigns to each triple of kernel orders  $(\alpha, \beta, \gamma)$  the corresponding kernel-weighted RSVP system. The local system is  $\mathcal{R}(0, 0, 0)$  (the  $\delta$ -kernel limit).*

The RSVP functor is the formal statement that RSVP is not a separate theory but a representation of  $\mathcal{KOp}$  in the category of field systems.

**Bridge to Global Consistency.** The RSVP system describes how fields evolve under kernel-weighted dynamics, but does not yet address global consistency: when do locally defined trajectories assemble into a coherent world-state? If RSVP specifies the local evolution law, Yarncrawler specifies when local solutions glue into a global trajectory. In operator terms,  $\mathcal{T}_K$  must be compatible with restriction and recombination across spatial domains. This shifts focus from dynamics alone to the topology of trajectories, formalized through sheaf structure.

## 8 Yarncrawler and the Sheaf-Variational Equivalence

### 8.1 The Reconstruction Problem

Yarncrawler addresses a global consistency question: given local trajectory fragments observed through partial sensors, when can the global world-state be reconstructed?

The answer is formalized through sheaf theory. Informally: if all local observations agree on overlapping regions, a unique global trajectory exists explaining them all.

In trajectory-operator language: Yarncrawler is  $\mathcal{T}_K$  with the consistency kernel  $K_Y$ , and reconstruction completes when  $K_Y$  is idempotent.

### 8.2 Sheaves of Trajectories

**Definition 8.1** (Sheaf of trajectories). *The sheaf of trajectories  $\mathcal{T}$  on parameter space  $X$  assigns to each open  $U \subseteq X$  the path space  $\mathcal{P}(U, Y)$ , and to each inclusion  $V \hookrightarrow U$  the restriction map  $\rho_{U,V}: \mathcal{P}(U) \rightarrow \mathcal{P}(V)$ . The sheaf axiom: local sections that agree on overlaps glue uniquely to a global section.*

The sheaf axiom is the exact mathematical form of global consistency from local agreement. The causal restrictions  $\rho_{s,t}$  of Section 2 define a presheaf over the time interval  $([0, T], \leq)$ ; the sheaf axiom adds the gluing condition.

### 8.3 The Identifiability Theorem

**Theorem 8.2** (Identifiability Theorem). *The global world-state is identifiable from a cover  $\{U_i\}$  if and only if  $\check{H}^1(\{U_i\}, \mathcal{T}) = 0$ .*

*Proof.* The Čech cohomology long exact sequence shows that the obstruction to lifting consistent local sections to a global section lies in  $\check{H}^1$ . Vanishing of this group ensures every cocycle is a coboundary, i.e., every consistent local family glues to a global section.  $\square$

### 8.4 The Consistency Functional

**Definition 8.3** (Consistency functional). *For a family of local trajectories  $\{f_i\}$ , define*

$$\mathcal{E}[\{f_i\}] = \sum_{i,j} \int_{U_i \cap U_j} \|\rho_{U_i, U_i \cap U_j}(f_i)(t) - \rho_{U_j, U_i \cap U_j}(f_j)(t)\|^2 dt.$$

**Theorem 8.4** (Sheaf-Variational Equivalence).  *$\mathcal{E}[\{f_i\}] = 0$  if and only if  $\{f_i\}$  glues to a global section of  $\mathcal{T}$ . Moreover, the gradient flow of  $\mathcal{E}$  in the product path space converges to a minimizer, and every minimizer with  $\mathcal{E} = 0$  is a global section.*

*Proof.* The equivalence  $\mathcal{E} = 0 \Leftrightarrow$  global section is immediate from the definition. For gradient flow:  $\mathcal{E}$  is a non-negative quadratic functional on a Hilbert product space; its gradient flow is a linear ODE with positive-semidefinite coefficient matrix, converging to the kernel of that matrix—the globally consistent sections.  $\square$

### 8.5 The Yarncrawler Reconstruction Kernel

**Definition 8.5** (Yarncrawler kernel). *The Yarncrawler reconstruction kernel  $K_Y$  is the kernel whose trajectory operator  $\mathcal{T}_{K_Y}$  projects the product path space onto the space of globally consistent trajectories:*

$$\mathcal{T}_{K_Y}[\{f_i\}] = \operatorname{argmin}_{g \in \mathcal{P}(X)} \mathcal{E}[\{g|_{U_i}\}].$$

**Proposition 8.6** ( $K_Y$  is idempotent).  $\mathcal{T}_{K_Y} \circ \mathcal{T}_{K_Y} = \mathcal{T}_{K_Y}$ .

*Proof.* Applying  $\mathcal{T}_{K_Y}$  to already-consistent data leaves it unchanged, since consistent data minimizes  $\mathcal{E}$  and projection onto a minimum is idempotent. Formally:  $\mathcal{E}[\{(\mathcal{T}_{K_Y} f_i)|_{U_j}\}] = 0$ , so projecting again produces the same output.  $\square$

Idempotence is the algebraic signature of constraint closure. The Yarncrawler kernel is a constraint operator whose fixed point is the globally consistent trajectory—the world-state.

**Bridge to Event Structure.** Yarncrawler establishes that globally consistent trajectories arise as fixed points of a consistency kernel, but this description remains static: it characterizes admissible histories without describing transitions between them. Spherepop introduces this structure by treating transitions as primitive morphisms. Where Yarncrawler concerns the existence of global sections, Spherepop concerns the composition of events transforming one trajectory into another—the passage from global admissibility to causal transformation within the same operator framework.

## 9 Spherepop: The Irreversible Event Calculus

### 9.1 Events as Trajectory Operators

Spherepop is an event calculus in which the primitive objects are not states but events: causal transitions between trajectory segments. In trajectory-operator language: an event is a morphism in  $\mathcal{KOp}$ , i.e., a kernel-mediated transformation of a past trajectory into a future one.

**Definition 9.1** (Event morphism). *An event morphism  $e: \sigma \rightarrow \sigma'$  is a triple  $(I, f, f')$  where  $I = [t_0, t_1]$ ,  $f \in \mathcal{P}(I)$  is the incoming causal history, and  $f' \in \mathcal{P}(I)$  is the outgoing causal future. The event is the transition  $f \mapsto f'$ .*

**Proposition 9.2** (Kernel representation). *Every event morphism  $e$  is represented by a kernel operator  $K_e \in \mathcal{K}(\Delta_I)$  via*

$$f'(t) = \mathcal{T}_{K_e}[\text{id}][f](t) = \int_{t_0}^t K_e(t, \tau) f(\tau) d\tau.$$

*Composition of events corresponds to kernel composition:  $K_{e' \circ e} = K_{e'} \circ K_e$ .*

This establishes that **Sph** (the Spherepop event category) is a subcategory of  $\mathcal{KOp}$ : events are a special class of kernel morphisms.

## 9.2 Symmetric Monoidal Structure

Events compose sequentially and can occur in parallel.

**Definition 9.3** (Tensor product of events). *For simultaneous events  $e, f$  in independent subsystems,  $e \otimes f: \sigma_1 \otimes \sigma_2 \rightarrow \sigma'_1 \otimes \sigma'_2$  is the joint event, with kernel  $K_{e \otimes f}(t, \tau) = K_e(t, \tau) \oplus K_f(t, \tau)$  (direct sum).*

**Theorem 9.4** (**Sph** is symmetric monoidal). *The event category **Sph** with the tensor product is a symmetric monoidal category.*

*Proof.* Associativity and symmetry of  $\otimes$  follow from the commutativity of simultaneous events in independent subsystems: events in disjoint trajectory spaces compose in any order. The coherence axioms (pentagon, triangle) hold because the relevant isomorphisms are canonical permutations of direct-sum kernel components.  $\square$

## 9.3 Trace and Irreversibility

**Definition 9.5** (Trace). *For an event  $e: A \otimes U \rightarrow B \otimes U$ , the trace  $\text{tr}^U(e): A \rightarrow B$  represents the effect of  $e$  after feeding the  $U$ -component back into itself.*

**Theorem 9.6** (Irreversibility of the Spherepop trace). *The trace in **Sph** is not invertible: there is in general no event  $\bar{e}$  with  $\text{tr}^U(\bar{e}) = (\text{tr}^U(e))^{-1}$ .*

*Proof.* The trace maps  $\text{Hom}(A \otimes U, B \otimes U) \rightarrow \text{Hom}(A, B)$ . Injectivity fails (multiple events produce the same traced outcome). Surjectivity of the inverse fails because a reversed trajectory would require the entropy field  $S$  to decrease over the traced feedback loop, violating Proposition 7.3. Hence the trace is not invertible.  $\square$

Irreversibility here is not imposed as a physical postulate. It is derived from the entropy growth bound of the RSVP embedding (Proposition 7.3) together with the categorical structure of the trace.

## 9.4 Discrete Event Sequences and Total Kernel Accumulation

Sequential events compose into a single kernel via convolution:

**Proposition 9.7** (Total kernel of a discrete event sequence). *Let  $\{E_n\}_{n=1}^N$  be a sequence of events with kernels  $\{K_{E_n}\}$ . The composed evolution*

$$\Phi_N = (E_N \circ \dots \circ E_1)(\Phi_0)$$

corresponds to the single trajectory operator with kernel

$$K_{\text{tot}} = K_{E_N} \circ K_{E_{N-1}} \circ \cdots \circ K_{E_1}.$$

*Proof.* By induction. The base case  $N = 1$  is immediate. For the inductive step:  $(E_{n+1} \circ E_n)(\Phi) = \mathcal{T}_{K_{E_{n+1}}}[\mathcal{T}_{K_{E_n}}[\Phi]]$ , which in the linear case equals  $\mathcal{T}_{K_{E_{n+1}} \circ K_{E_n}}[\Phi]$  by the kernel composition law.  $\square$

This result establishes that discrete event composition and continuous kernel accumulation are two descriptions of the same structure. As  $N \rightarrow \infty$  with event duration  $\Delta t \rightarrow 0$ , the discrete composition converges to the continuous Volterra operator  $\mathcal{T}_K$ , with  $K = \lim_{N \rightarrow \infty} K_{\text{tot}}$ .

**Proposition 9.8** (Non-invertibility from smoothing). *If  $K_E$  is supported on  $\tau \leq t$  and is integrable but not singular, then the event  $E$  is non-invertible: there is no kernel  $K^{-1}$  satisfying  $K_E \circ K^{-1} = \delta$ .*

*Proof.* Such kernels smooth trajectories by averaging over past values. This operation is injective from  $L^2$  to  $L^2$  but has dense range; its adjoint is not a left inverse. Concretely, the Fourier transform of  $K_E(t - \tau)$  decays at high frequencies (since  $K_E \in L^1$ ), so its inverse would amplify high frequencies unboundedly—an operator not bounded on any standard function space.  $\square$

**Bridge to Event Generation.** Spherpap treats events as morphisms but does not specify how they arise from continuous dynamics. KES resolves this: events correspond to threshold crossings of kernel-weighted accumulations. What appears categorically as a discrete morphism is analytically the result of continuous synthesis. The distinction between continuous evolution and discrete events collapses; both are expressions of  $\mathcal{T}_K$  viewed at different levels of resolution.

## 10 KES: Kinetic-Event Synthesis

### 10.1 Events from Trajectories

KES provides the mechanism by which discrete events emerge from continuous trajectory dynamics. An event is not primitive but *generated* when a trajectory accumulation crosses a threshold. In trajectory-operator terms:  $\mathcal{T}_K$  generates events.

**Definition 10.1** (Synthesis kernel). *A synthesis kernel is a kernel of the form*

$$K_{\text{syn}}(t, \tau) = \psi(t) \cdot \phi(\tau) \cdot K_{\alpha}(t, \tau),$$

where  $\psi \in L^{\infty}$  is an output envelope,  $\phi \in L^{\infty}$  is an input envelope, and  $K_{\alpha}$  is the fractional kernel of order  $\alpha$ .

The factored form has a clean reading:  $\phi$  selects which portion of the history contributes;  $K_{\alpha}$  weights contributions by temporal distance (with  $\alpha$  controlling memory depth);  $\psi$  shapes the output into a new trajectory segment.

## 10.2 Threshold Crossings as Events

**Definition 10.2** (KES event set). *For  $f \in \mathcal{P}$  and threshold  $\theta > 0$ , the event set is*

$$\mathcal{E}_{\theta}(f) = \{t \in [0, T] : |\mathcal{T}_{K_{\text{syn}}}[f](t)| \geq \theta\}.$$

**Proposition 10.3** (Event discreteness via Sard’s theorem). *If  $f \in \mathcal{P}$  is absolutely continuous and  $K_{\text{syn}}$  has  $\alpha \in (0, 1)$ , then  $\mathcal{E}_{\theta}(f)$  is a countable set for a.e. threshold  $\theta > 0$ .*

*Proof.* The map  $t \mapsto |\mathcal{T}_{K_{\text{syn}}}[f](t)|$  is continuous (Volterra operators on absolutely continuous inputs are continuous). By Sard’s theorem, the level set of a continuous function at a generic regular value is discrete.  $\square$

This is the formal derivation of event discreteness from continuous dynamics: *discrete events are threshold-crossings of a trajectory operator, not primitive objects*. KES provides the analytic ground floor of Spherpap’s categorical structure.

## 10.3 Kernel Composition Law

**Theorem 10.4** (KES composition). *For synthesis kernels  $K_{\text{syn}}^{(1)}, K_{\text{syn}}^{(2)}$ :*

$$\mathcal{T}_{K_{\text{syn}}^{(1)}} \circ \mathcal{T}_{K_{\text{syn}}^{(2)}} = \mathcal{T}_{K_{\text{syn}}^{(1)} \circ K_{\text{syn}}^{(2)}}.$$

*Proof.* Direct from the associativity of kernel composition (Theorem 4.1) and the definition of  $\mathcal{T}_K$ .  $\square$

In words: sequential KES synthesis stages compose as their kernels compose. This is the sense in which “kinetic-event synthesis” is literally kernel composition.

## 10.4 Nonlinear Synthesis

In the nonlinear setting where  $F \neq \text{id}$ , sequential application yields

$$(\mathcal{T}_{K_1} \circ \mathcal{T}_{K_2})[\Phi](t) = \int_0^t K_1(t, s) F_1 \left( \int_0^s K_2(s, \tau) F_2(\Phi(\tau)) d\tau \right) ds.$$

This does not in general collapse to a single kernel operator, but it preserves causality (the simplex  $0 \leq \tau \leq s \leq t$  is the support of the nested integral) and temporal ordering.

**Proposition 10.5** (Causality of nonlinear composition). *Nonlinear compositions of trajectory operators preserve causality: the output at time  $t$  depends only on  $\Phi|_{[0,t]}$ .*

*Proof.* The nested integrals are supported on  $\{(\tau, s, t) : 0 \leq \tau \leq s \leq t\}$ , so no future values of  $\Phi$  contribute.  $\square$

Under suitable conditions (linearization, or  $F_1, F_2$  weakly nonlinear), the composite operator admits an effective kernel approximation:

$$\mathcal{T}_{K_1} \circ \mathcal{T}_{K_2} \approx \mathcal{T}_{K_{\text{eff}}}, \quad K_{\text{eff}} \approx K_1 \circ K_2.$$

This provides an approximate closure of nonlinear synthesis within the kernel algebra, connecting it back to the convolution monoid.

Spherepop specifies the categorical structure of events (morphisms, composition, tensor product, trace). KES provides the analytic mechanism by which those events arise from trajectory data.

**Definition 10.6** (KES-Spherepop functor).  $\Psi: \mathcal{A}_{\text{KES}} \rightarrow \mathbf{Sph}$  sends each synthesis kernel to the event morphism it generates:  $\Psi(K_{\text{syn}}) = (\text{trajectories at } \mathcal{E}_\theta)$ .

The categories are related: KES generates the events that Spherepop organizes.

**Bridge to Synchronization and Quantum Structure.** KES shows how events emerge from trajectory accumulation but does not impose global constraints on how they combine across space. TARTAN provides this constraint by enforcing compatibility across a tiled domain. When local transition laws are required to be unistochastic and globally synchronized, the resulting dynamics acquire quantum structure. Ising synchronization selects a globally coherent phase configuration, transforming local kernel dynamics into consistent unitary evolution. Quantum behavior emerges as a global constraint on kernel composition.

## 11 TARTAN: Recursive Tiling, Ising Synchronization, and Quantum Dynamics

### 11.1 Overview

TARTAN is a framework for recursive tiling and gluing of local dynamical data. In the kernel-operator setting, TARTAN supplies the cover structure  $\{U_a\}$  over which local trajectory operators are defined, together with the compatibility constraints that govern how they glue. The connection to quantum mechanics arises when the local transition laws are required to be *unistochastic*: stochastic matrices arising as squared moduli of unitary matrices. Ising synchronization then selects a globally consistent unitary lift.

### 11.2 Local Configuration Tiles

Let  $\Omega$  be covered by TARTAN tiles  $\{U_a\}_{a \in A}$ . Each tile carries a finite configuration space  $X_a$  and a local transition law

$$P^{(a)}: X_a \times X_a \rightarrow [0, 1].$$

We require  $P^{(a)}$  to be *unistochastic*: there exists a unitary matrix  $U^{(a)}$  such that

$$P_{ij}^{(a)} = \left| U_{ij}^{(a)} \right|^2.$$

The matrix  $P^{(a)}$  gives observable transition probabilities;  $U^{(a)}$  carries the phase structure required for quantum interference.

### 11.3 Phase-Lifted TARTAN Variables

A unistochastic matrix determines probabilities but not a unique unitary lift. We introduce local phase variables  $\theta_{ij}^{(a)} \in S^1$  and write

$$U_{ij}^{(a)} = \sqrt{P_{ij}^{(a)}} e^{i\theta_{ij}^{(a)}}.$$

The TARTAN problem is to synchronize phase lifts across overlapping tiles. On overlaps  $U_a \cap U_b$ , compatibility requires:

- Probability agreement:  $P^{(a)}|_{U_a \cap U_b} = P^{(b)}|_{U_a \cap U_b}$ .
- Phase coherence:  $\theta_{ij}^{(a)} - \theta_{ij}^{(b)} \equiv 0 \pmod{2\pi}$  up to admissible gauge transformations.

### 11.4 Ising Synchronization

To drive synchronization, assign an Ising spin  $\sigma_a \in \{-1, +1\}$  to each tile. Neighboring tiles interact through the energy functional

$$E_{\text{Ising}}(\sigma) = - \sum_{\langle a,b \rangle} J_{ab} \sigma_a \sigma_b - \sum_a h_a \sigma_a,$$

where the coupling

$$J_{ab} = \exp\left(-\eta \left\| \theta^{(a)} - \theta^{(b)} \right\|_{U_a \cap U_b}^2\right)$$

rewards phase-compatible tile pairs, and  $h_a$  encodes local bias from the TARTAN annotation layer. The synchronized configuration is

$$\sigma^* = \arg \min_{\sigma} E_{\text{Ising}}(\sigma).$$

### 11.5 Synchronized Unistochastic Law

Once synchronized, the local matrices glue into an effective global law:

$$P^{\text{eff}} = \text{Glue}_{\sigma^*}(\{P^{(a)}\}_{a \in A}),$$

with unitary lift

$$U_{ij}^{\text{eff}} = \sqrt{P_{ij}^{\text{eff}}} e^{i\theta_{ij}^{\text{eff}}}.$$

The global quantum dynamics are then generated by

$$\psi(t + \Delta t) = U^{\text{eff}} \psi(t), \quad p_i(t + \Delta t) = \sum_j P_{ij}^{\text{eff}} p_j(t).$$

### 11.6 Constraint Closure in the TARTAN Layer

A globally admissible quantum process is a fixed point of the combined TARTAN–Ising–unistochastic operator:

$$P^* = \mathcal{G}_{\text{TARTAN}}(\text{Sync}_{\text{Ising}}(\{P^{(a)}\})).$$

This is an instance of the general constraint-closure pattern  $\Phi = \mathcal{T}_K[\Phi]$ , with the kernel encoding the gluing and synchronization operations. The correspondences are:

TARTAN layer	Kernel-operator language
Tile compatibility	Sheaf gluing condition
Ising alignment	Selection of consistent global section
TARTAN gluing	Fixed point of $\mathcal{T}_{K_{\text{TARTAN}}}$
Quantum transition law	Globally synchronized section $P^*$

**Remark 11.1** (Core slogan). *Quantum mechanics, in this formulation, is globally synchronized unistochastic transition dynamics: TARTAN supplies the gluing geometry; Ising synchronization supplies the alignment dynamics; unistochasticity supplies the quantum law.*

**Bridge to Symmetry Constraints.** The TARTAN construction enforces global consistency across spatial tiles. The next layer imposes consistency across reference frames. Where TARTAN constrains admissibility through synchronization, relativity constrains it through invariance of a quadratic form. Kernel operators compatible with this symmetry must commute with Lorentz transformations, placing geometric bounds on the operator algebra.

## 12 Relativity from a Pythagorean Invariant

### 12.1 The Euclidean Prototype

In Euclidean geometry, the Pythagorean theorem defines the invariant quadratic form  $s^2 = x^2 + y^2$ . Rotations preserve this quantity. The classical square-dissection proofs demonstrate that such invariance is equivalent to area-preserving rearrangements—the quadratic form encodes a symmetry group.

This section argues that the Lorentz transformation and length contraction are the exact analogues of this structure applied to spacetime, with one sign change in the metric.

### 12.2 The Spacetime Quadratic Form

Introduce a temporal coordinate  $t$  and a universal speed  $c$ . For a light signal,  $x = ct$ , so any invariant quantity must vanish along light trajectories. This forces the spacetime

quadratic form to be

$$s^2 = c^2t^2 - x^2,$$

the Minkowski metric with indefinite signature. This is the unique (up to scaling) quadratic form that preserves the light cone structure, and therefore encodes the causal geometry of spacetime. It is structurally Pythagorean, but with a sign change that encodes the distinction between space and time.

### 12.3 Invariant-Preserving Transformations

We seek all linear maps  $(x, t) \mapsto (x', t')$  preserving  $s^2$ :

$$c^2t'^2 - x'^2 = c^2t^2 - x^2.$$

Assuming the transformation  $x' = \gamma(x - vt)$ ,  $t' = \gamma(t - vx/c^2)$  and substituting into the invariant condition, invariance holds if and only if

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

The Lorentz factor  $\gamma$  arises as the unique normalization constant that makes the linear transformation preserve the spacetime Pythagorean form.

**Theorem 12.1** (Lorentz transformation from quadratic invariance). *The unique family of linear maps preserving  $s^2 = c^2t^2 - x^2$  and reducing to the identity at  $v = 0$  is the Lorentz boost with parameter  $\gamma$ .*

*Proof.* Substitute the general linear ansatz into  $s^2 = s'^2$  and solve for the coefficients. The constraint that  $(x', t') = (x, t)$  at  $v = 0$  fixes the integration constants, uniquely determining  $\gamma$  as stated.  $\square$

### 12.4 Length Contraction as Projection

Let a rod be at rest in one frame with proper length  $L_0 = x_2 - x_1$ . To measure its length in a moving frame, both endpoints must be measured at the same time in that frame:  $t'_2 = t'_1$ . The condition  $t - vx/c^2 = \text{const}$  combined with the Lorentz transformation for  $x'$  gives

$$L = x'_2 - x'_1 = \frac{L_0}{\gamma}.$$

This is Lorentz contraction. It is not a physical compression of the rod but a projection effect: the same invariant spacetime interval is expressed along a different coordinate axis under a symmetry-preserving transformation.

**Remark 12.2** (Geometric interpretation). *In Euclidean geometry, rotation preserves circular level sets and “dissects” area by rearrangement. In spacetime, Lorentz boosts preserve hyperbolic level sets of  $s^2$ . Length contraction is the re-slicing of those hyperbolas under a different temporal axis—the relativistic analogue of a Euclidean dissection.*

## 12.5 Connection to the Operator Framework

The quadratic form  $s^2$  acts as a constraint functional: it defines the class of admissible transformations. Lorentz transformations are the symmetry group of that constraint. In the language of kernel operators:

$$\text{invariant quadratic form} \Rightarrow \text{symmetry group} \Rightarrow \text{observable contraction effects.}$$

This places relativity in the same structural position as the other frameworks: a constraint on admissible dynamics, expressed through an invariant functional.

**Bridge to Spectral Structure.** Relativistic invariance constrains the form of admissible kernel operators but does not explain their internal behavior: why memory induces smoothing, why irreversibility arises, or why certain trajectories are stable. These questions are answered by spectral analysis. The kernel operator, viewed as an operator on path space, possesses a spectrum that governs decay, stability, and information loss, shifting focus from external constraints to internal dynamics.

## 13 Lorentz Symmetry on Trajectory Space

### 13.1 Trajectories in Spacetime

We extend the path space to spacetime coordinates. Let  $\Phi: \Omega \times [0, T] \rightarrow \mathbb{R}$  be a trajectory, and equip the domain with the Minkowski quadratic form  $s^2 = c^2t^2 - x^2$ . A Lorentz transformation  $\Lambda$  acts on spacetime coordinates by

$$(x, t) \mapsto (x', t') = \Lambda(x, t), \quad s'^2 = s^2.$$

This induces a pullback operator on trajectories:

$$(\Lambda \cdot \Phi)(x, t) := \Phi(\Lambda^{-1}(x, t)).$$

### 13.2 Lorentz Compatibility of Kernel Operators

**Definition 13.1** (Lorentz-compatible operator). *A trajectory operator  $\mathcal{T}_K$  is Lorentz-compatible if*

$$\Lambda \cdot \mathcal{T}_K[\Phi] = \mathcal{T}_K[\Lambda \cdot \Phi]$$

for all trajectories  $\Phi$  and all Lorentz transformations  $\Lambda$ .

**Proposition 13.2** (Invariant kernels are Lorentz-compatible). *If the kernel  $K$  depends only on the spacetime interval, i.e.  $K(t, \tau) = \tilde{K}(s^2)$  for some function  $\tilde{K}$ , then  $\mathcal{T}_K$  is Lorentz-compatible.*

*Proof.* Since  $s^2$  is invariant under  $\Lambda$ , we have  $K(t', \tau') = K(t, \tau)$ . Substituting into the definition of  $\mathcal{T}_K$ :

$$(\Lambda \cdot \mathcal{T}_K[\Phi])(x, t) = \mathcal{T}_K[\Phi](\Lambda^{-1}(x, t)) = \int_0^{t^*} K(t^*, \tau) F(\Phi(\Lambda^{-1}(x, \tau))) d\tau,$$

where  $t^* = [\Lambda^{-1}(x, t)]_t$ . Under the change of variables  $\tau \mapsto \Lambda^{-1}(\tau)$ , this equals  $\mathcal{T}_K[\Lambda \cdot \Phi](x, t)$ , since  $K$  is invariant.  $\square$

### 13.3 Relativistic Kernel Evolution

A fully Lorentz-invariant trajectory operator integrates over the past light cone rather than a fixed time interval:

$$\mathcal{T}_K[\Phi](x, t) = \int_{\mathcal{C}(x, t)} K(s^2((x, t), (x', t'))) F(\Phi(x', t')) d\mu(x', t'),$$

where  $\mathcal{C}(x, t) = \{(x', t') : c^2(t - t')^2 - (x - x')^2 \geq 0, t' \leq t\}$  is the past light cone. This is the relativistic generalization of the causal Volterra operator: instead of integrating over  $[0, t]$ , it integrates over all causally accessible past events.

### 13.4 Length Contraction as Trajectory Projection

Let  $\Phi$  represent a spatial configuration at rest in one frame. Different inertial frames correspond to different temporal slicings of spacetime. Since  $\Lambda$  preserves  $s^2$ , the underlying

trajectory is unchanged; what changes is its projection onto the spatial axis of the moving frame. This projection is exactly Lorentz contraction  $L = L_0/\gamma$ : a consequence of evaluating the same invariant trajectory under different coordinate projections.

### 13.5 Interpretation

Within the kernel-operator framework, Lorentz symmetry is a compatibility condition rather than an independent postulate. The diagram

$$\Lambda \cdot \mathcal{T}_K = \mathcal{T}_K \cdot \Lambda$$

says that temporal evolution and frame change commute—a statement about the algebraic structure of  $\mathcal{KOp}$ , not a separate postulate of physics. Relativity appears as the constraint that kernel operators must commute with the symmetry group of the invariant quadratic form.

## 14 Constraint Closure and the Trajectory-First Theory

### 14.1 Trajectories as Primary Objects

The standard picture: state space  $\mathcal{X}$ , evolution map  $\Phi_t: \mathcal{X} \rightarrow \mathcal{X}$ . Completion = reaching a fixed point  $x^* \in \mathcal{X}$ .

The trajectory-first picture: the primary objects are trajectories  $f \in \mathcal{P}$ , and states are derived by evaluation. Completion is not “this state is fixed” but:

$$\textit{this trajectory satisfies } \mathcal{T}_K[f^*] = f^*.$$

This is the fixed-point condition for the trajectory operator  $\mathcal{T}_K$ .

### 14.2 Constraint Operators

**Definition 14.1** (Constraint operator). A constraint operator on  $\mathcal{P}$  is  $\mathcal{T}_K$  satisfying:

- (i) **Causal:**  $\mathcal{T}_K[f](t)$  depends only on  $f|_{[0,t]}$ .
- (ii) **Contractive:**  $\|\mathcal{T}_K[f] - \mathcal{T}_K[g]\|_{\mathcal{P}} \leq L\|f - g\|_{\mathcal{P}}$  for some  $L < 1$ .
- (iii) **Positivity-preserving:**  $f \geq 0 \Rightarrow \mathcal{T}_K[f] \geq 0$ .

**Definition 14.2** (Constraint closure). A trajectory  $f^*$  is closed under constraint  $\mathcal{T}_K$  if  $\mathcal{T}_K[f^*] = f^*$ : it is a fixed point of its own constraint.

### 14.3 The Constraint Closure Theorem

**Theorem 14.3** (Constraint Closure Theorem). *Let  $\mathcal{T}_K$  be a constraint operator. Then:*

- (i) *There exists a unique fixed point  $f^* \in \mathcal{P}$  with  $\mathcal{T}_K[f^*] = f^*$ .*
- (ii) *For any  $f_0 \in \mathcal{P}$ , the iterates  $f_{n+1} = \mathcal{T}_K[f_n]$  converge to  $f^*$  in  $\mathcal{P}$ -norm.*
- (iii) *The operator  $\mathcal{T}_K$  is idempotent on  $\{f^*\}$ :  $\mathcal{T}_K(\mathcal{T}_K[f^*]) = \mathcal{T}_K[f^*]$ .*

*Proof.* Parts (i) and (ii): the Banach fixed-point theorem applied to the contractive map  $\mathcal{T}_K$  on the complete metric space  $(\mathcal{P}, \|\cdot\|)$ . Part (iii):  $f^* = \mathcal{T}_K[f^*]$  implies  $\mathcal{T}_K(f^*) = f^* = \mathcal{T}_K[f^*]$ , so applying  $\mathcal{T}_K$  again gives the same result.  $\square$

### 14.4 Idempotence as Completion

**Corollary 14.4** (Completion without states). *A process defined by trajectory  $f \in \mathcal{P}$  and constraint  $\mathcal{T}_K$  is complete if and only if  $\mathcal{T}_K$  is idempotent on  $f$ . No state space is required.*

This is the formal statement of the trajectory-first completion principle:

$$\text{completion} = \text{constraint closure} = \text{kernel idempotence.}$$

### 14.5 Yarncrawler as Constraint Closure

The Yarncrawler kernel  $K_Y$  is idempotent by Proposition 8.6. This is not a coincidence: Yarncrawler reconstruction is exactly constraint closure for the sheaf-consistency constraint.

**Proposition 14.5** (Yarncrawler is a constraint closure). *Yarncrawler reconstruction is an instance of Theorem 14.3 with  $\mathcal{T}_K = \mathcal{T}_{K_Y}$ .*

The world-state reconstruction completes when the Yarncrawler constraint closes: when further application of  $\mathcal{T}_{K_Y}$  produces no new information.

**Bridge to Spectral and Variational Analysis.** Constraint closure establishes when a trajectory is a fixed point. Spectral theory now reveals why such fixed points are stable and what happens to trajectories that do not satisfy the constraint: their non-invariant components are spectrally damped. Variational theory shows that the same fixed point is a minimizer of a mismatch functional. The following two sections formalize this triple equivalence before the Unified Memory Theorem assembles it into a single statement.

## 15 Spectral Theory of Kernel Operators

### 15.1 Operator-Theoretic Setting

We now study  $\mathcal{T}_K$  as a linear operator on a function space, characterizing memory, irreversibility, and convergence through spectral properties. Let  $\mathcal{P} := L^2([0, T], X)$  and define the linear kernel operator

$$(\mathcal{K}f)(t) := \int_0^t K(t, \tau) f(\tau) d\tau, \quad \mathcal{K}: \mathcal{P} \rightarrow \mathcal{P}.$$

This is bounded by Lemma 2.6.

### 15.2 Compactness and Smoothing

**Theorem 15.1** (Compactness of causal kernel operators). *If  $K \in L^2(\Delta_T)$ , then  $\mathcal{K}$  is a compact operator on  $\mathcal{P}$ .*

*Proof.*  $\mathcal{K}$  is a Hilbert–Schmidt operator:  $\|\mathcal{K}\|_{HS}^2 = \int_0^T \int_0^t |K(t, \tau)|^2 d\tau dt < \infty$ . All Hilbert–Schmidt operators are compact.  $\square$

**Remark 15.2** (Smoothing interpretation). *Compactness means  $\mathcal{K}$  maps bounded sets into relatively compact sets. Physically, this is smoothing: fine-scale variations in the trajectory are suppressed by integration against  $K$ . Memory is not merely accumulation—it is selective persistence with suppression of high-frequency modes.*

### 15.3 Spectral Structure and Memory Decay

**Theorem 15.3** (Spectral structure). *If  $\mathcal{K}$  is compact, then  $\sigma(\mathcal{K}) = \{0\} \cup \{\lambda_n\}$  with  $\lambda_n \rightarrow 0$ , each nonzero  $\lambda_n$  has finite multiplicity, and 0 is the only accumulation point.*

The eigenvalues  $\lambda_n$  measure how strongly different modes of the trajectory persist under the kernel. Since  $\lambda_n \rightarrow 0$ , all modes eventually decay: *memory is lossy*. Repeated application of the kernel erases information.

### 15.4 Irreversibility from Spectral Decay

**Theorem 15.4** (Non-invertibility of smoothing kernels). *If  $K \in L^1(\Delta_T)$  and is not a distribution concentrated on  $t = \tau$ , then  $\mathcal{K}$  is not invertible on  $\mathcal{P}$ .*

*Proof.* A compact operator on an infinite-dimensional Banach space cannot be invertible, since its spectrum accumulates at 0.  $\square$

**Remark 15.5** (Algebraic irreversibility). *This gives a purely spectral derivation of irreversibility: smoothing  $\Rightarrow \lambda_n \rightarrow 0 \Rightarrow$  no inverse. No thermodynamic argument is required; information loss is encoded in the operator spectrum.*

## 15.5 Fractional Kernels and Power-Law Spectra

**Proposition 15.6** (Spectral decay of fractional operators). *The singular values of  $\mathcal{K}_\alpha$  satisfy  $s_n(\mathcal{K}_\alpha) \sim n^{-(1-\alpha)}$ .*

For  $\alpha \rightarrow 0$ : rapid spectral decay, short memory. For  $\alpha \rightarrow 1$ : slow decay, long memory. The fractional parameter  $\alpha$  directly controls the spectral decay rate, making memory depth a measurable spectral quantity.

## 15.6 Stability of Fixed Points

Linearizing  $\mathcal{T}_K[f] = \mathcal{K}(F \circ f)$  around a fixed point  $f^*$ :

**Theorem 15.7** (Stability criterion). *A fixed point  $f^*$  of  $\mathcal{T}_K$  is stable if  $\rho(\mathcal{K} \circ DF(f^*)) < 1$ , where  $\rho$  is the spectral radius.*

Constraint closure is therefore not just existence of a fixed point but *spectral contraction* of deviations: the kernel controls stability by damping perturbations.

## 15.7 Resolvent and Memory Propagation

The resolvent  $R(\lambda) = (I - \lambda\mathcal{K})^{-1}$  expands as

$$R(\lambda) = \sum_{n=0}^{\infty} \lambda^n \mathcal{K}^n, \quad |\lambda| < 1/\|\mathcal{K}\|.$$

Each term represents a deeper layer of historical influence. Memory propagation is literally a geometric series in kernel composition.

**Bridge to Variational Equivalence.** Spectral theory characterizes the behavior of  $\mathcal{K}$  through eigenstructure. The same structure can be obtained from an optimization principle: fixed points of  $\mathcal{K}$  are minimizers of a natural mismatch functional. The next section formalizes this equivalence, completing the identification between operator algebra, energy minimization, and constraint closure.

## 16 Spectral–Variational Equivalence

### 16.1 Overview

We establish the equivalence between three perspectives on  $\mathcal{T}_K$ :

$$\text{variational minimization} \iff \text{spectral contraction} \iff \text{constraint closure.}$$

### 16.2 The Mismatch Functional

**Definition 16.1** (Mismatch functional).

$$\mathcal{J}[f] := \frac{1}{2} \|f - \mathcal{K}f\|_{\mathcal{P}}^2.$$

**Proposition 16.2** (Euler–Lagrange condition). *A trajectory  $f^*$  is a stationary point of  $\mathcal{J}$  if and only if  $(I - \mathcal{K})^*(I - \mathcal{K})f^* = 0$ .*

*Proof.*  $\delta\mathcal{J} = \langle (I - \mathcal{K})f, (I - \mathcal{K})\delta f \rangle = 0$  for all  $\delta f$  gives  $(I - \mathcal{K})^*(I - \mathcal{K})f = 0$ .  $\square$

**Corollary 16.3** (Variational constraint closure).  $\mathcal{J}[f^*] = 0 \iff f^* = \mathcal{K}f^*$ .

Minimizing  $\mathcal{J}$  enforces the fixed-point condition: constraint closure is a variational principle.

### 16.3 Spectral Characterization of Minimizers

Expand  $f = \sum_n c_n e_n$  in the eigenbasis of  $\mathcal{K}$ . Then  $\mathcal{J}[f] = \frac{1}{2} \sum_n (1 - \lambda_n)^2 |c_n|^2$ .

**Proposition 16.4** (Spectral minimizers).  *$f^*$  minimizes  $\mathcal{J}$  if and only if  $c_n = 0$  whenever  $\lambda_n \neq 1$ . The fixed-point space is the eigenspace  $\ker(I - \mathcal{K})$ .*

### 16.4 Gradient Flow Convergence

The gradient flow  $\dot{f} = -\nabla\mathcal{J}[f] = -(I - \mathcal{K})^*(I - \mathcal{K})f$  diagonalizes as  $\dot{c}_n = -(1 - \lambda_n)^2 c_n$ .

**Theorem 16.5** (Convergence to fixed point). *If  $\rho(\mathcal{K}) < 1$  on the complement of  $\ker(I - \mathcal{K})$ , then  $f(t) \rightarrow f^* \in \ker(I - \mathcal{K})$ .*

*Proof.* For  $\lambda_n \neq 1$ :  $c_n(t) \rightarrow 0$  exponentially. For  $\lambda_n = 1$ :  $c_n$  is constant. The system projects onto the constraint-closed subspace.  $\square$

## 16.5 The Equivalence Theorem

**Theorem 16.6** (Spectral–Variational Equivalence). *For a compact kernel operator  $\mathcal{K}$ , the following are equivalent:*

- (i)  $f^*$  minimizes  $\mathcal{J}[f] = \frac{1}{2}\|f - \mathcal{K}f\|^2$ ,
- (ii)  $f^*$  satisfies  $\mathcal{K}f^* = f^*$ ,
- (iii)  $f^*$  lies in the eigenspace  $\lambda = 1$  of  $\mathcal{K}$ ,
- (iv)  $f^*$  is the limit of the gradient flow of  $\mathcal{J}$ ,
- (v)  $f^*$  is stable under iteration of  $\mathcal{K}$ .

*Proof.* (i) $\Rightarrow$ (ii): Corollary above. (ii) $\Rightarrow$ (iii): definition of eigenvector. (iii) $\Rightarrow$ (iv): Theorem 16.5. (iv) $\Rightarrow$ (v): fixed point implies invariance under iteration. (v) $\Rightarrow$ (ii): the iteration limit satisfies  $\mathcal{K}f^* = f^*$ .  $\square$

*Constraint closure is projection onto the unit-eigenspace of a smoothing operator.*

**Bridge to Worked Example.** The preceding sections establish that kernel dynamics, spectral decay, variational minimization, and constraint closure are equivalent at the level of abstract structure. To make this equivalence concrete and eliminate any remaining separation between perspectives, we now exhibit a single system—fractional diffusion—in which all of them coincide explicitly.

## 17 Unified Worked Example: Fractional Diffusion

### 17.1 Overview

We carry fractional diffusion through every representation developed in this monograph. The goal is structural: to show that kernel dynamics, spectral decay, variational minimization, constraint closure, RSVP field evolution, and sheaf consistency are the same object described in different languages.

## 17.2 Classical vs. Fractional Formulation

Classical diffusion:  $\partial_t u(x, t) = D\Delta u(x, t)$ . Local and Markovian.

Fractional diffusion (Caputo):  $D_t^\alpha u = D\Delta u$ ,  $\alpha \in (0, 1)$ . Expanding:

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \partial_\tau u(x, \tau) d\tau = D\Delta u(x, t).$$

This is  $\mathcal{T}_{K_\alpha}[\partial_t u](t) = D\Delta u(t)$ , with  $K_\alpha(t, \tau) = (t-\tau)^{-\alpha}/\Gamma(1-\alpha)$ .

## 17.3 Kernel Operator Form and Fixed Point

Rewriting as a Volterra equation:

$$u(x, t) = u_0(x) + \int_0^t K_\alpha(t, \tau) D\Delta u(x, \tau) d\tau.$$

This is  $u = u_0 + \mathcal{K}[u]$ , so solutions are fixed points of the trajectory operator  $u \mapsto u_0 + \mathcal{T}_{K_\alpha}[u]$ .

## 17.4 Spectral Representation

Expand over the Laplacian eigenbasis  $\Delta\phi_n = -\mu_n\phi_n$ . Each mode satisfies  $D_t^\alpha c_n = -D\mu_n c_n$ , with solution

$$c_n(t) = c_n(0) E_\alpha(-D\mu_n t^\alpha),$$

where  $E_\alpha$  is the Mittag-Leffler function. For large  $t$ :  $c_n(t) \sim t^{-\alpha}$  (power-law decay), contrasting with classical exponential decay. *Memory appears as slow spectral decay.*

## 17.5 Variational Formulation

Define  $\mathcal{J}_\alpha[u] = \frac{1}{2} \int_0^T \|\mathcal{T}_{K_\alpha}[\partial_t u] - D\Delta u\|^2 dt$ . Minimizers of  $\mathcal{J}_\alpha$  satisfy the fractional diffusion equation. Thus: *diffusion = variational minimization of kernel mismatch.*

## 17.6 RSVP Interpretation

Set  $\Phi = u$ ,  $\mathbf{v} = 0$ ,  $S = 0$  in the kernel-weighted RSVP system. Equations (6)–(8) reduce to  $\partial_t \Phi = \mathcal{T}_{K_\alpha}[D\Delta \Phi]$ . Fractional diffusion is the *minimal RSVP system*: scalar field only, no transport, no entropy coupling, pure memory-weighted diffusion.

### 17.7 KES Interpretation

Define synthesis operator  $\mathcal{T}_{K_{\text{syn}}}[u](t) = \int_0^t \psi(t)K_\alpha(t, \tau)u(\tau) d\tau$ . Events occur when  $|\mathcal{T}_{K_{\text{syn}}}[u](t)| \geq \theta$ . Continuous diffusion generates discrete events as threshold crossings of smoothed trajectories.

### 17.8 Yarncrawler Interpretation

Local solutions  $\{u_i\}$  on a cover  $\{U_i\}$  satisfy compatibility on overlaps. The global solution exists iff  $\check{H}^1 = 0$ . Fractional diffusion solutions are global sections of the trajectory sheaf.

### 17.9 Spectral–Variational–Closure Summary

Perspective	Statement for fractional diffusion
Kernel	$u = u_0 + \mathcal{T}_{K_\alpha}[u]$
Spectral	$c_n(t) \sim E_\alpha(-\mu_n t^\alpha) \sim t^{-\alpha}$
Variational	$u = \arg \min \mathcal{J}_\alpha$
Constraint closure	$u$ is a fixed point of $u \mapsto u_0 + \mathcal{K}[u]$
RSVP	Minimal kernel-weighted scalar field
Sheaf	Global section of trajectory sheaf

*All descriptions produce the same solution space. Diffusion is not a differential equation; it is a fixed point of a kernel operator whose stability is spectral, whose structure is variational, and whose behavior is memory.*

## 18 The Unified Memory Theorem

### 18.1 Statement

We now assemble all eight frameworks into a single categorical statement.

**Theorem 18.1** (Unified Memory Theorem). *There is a commutative diagram of functors*

$$\begin{array}{ccccc}
 \mathcal{A}_{\text{KES}} & \xrightarrow{\Psi} & \mathbf{Sph} & \xrightarrow{\Omega} & \mathcal{KOp} \\
 & & \downarrow \Lambda & & \downarrow \mathcal{R} \\
 & & \mathbf{Yarn} & \xrightarrow{\Xi} & \mathbf{FieldSys}
 \end{array}$$

where  $\Psi$  is the KES-Spherepop functor,  $\Omega$  embeds events via kernel representation,  $\mathcal{R}$  is the RSVP functor,  $\Lambda$  sends events to their post-event trajectories,  $\mathbf{Yarn}$  is the Yarncrawler category of sheaf sections, and  $\Xi$  embeds global sections as RSVP field histories.

Moreover, the following conditions on a trajectory  $\Phi \in \mathcal{P}$  are equivalent characterizations of admissible trajectory realization:

- (i) **Constraint closure:**  $\Phi = \mathcal{T}_{\text{tot}}[\Phi]$  (fixed point of the synthesized kernel operator).
- (ii) **Sheaf consistency:**  $\Phi$  is a global section of the sheaf of local kernel solutions  $\{\Phi_i\}$  over a cover  $\{U_i\}$ .
- (iii) **Event fixed point:**  $\Phi$  is a fixed point of a traced composition of irreversible Spherepop events.
- (iv) **Kernel algebra:**  $\Phi$  is a fixed point of a composition in  $\mathcal{KOp}$ .
- (v) **Symmetry compatibility:**  $\Phi$  is stable under the symmetry group of the spacetime invariant  $s^2 = c^2t^2 - x^2$ , i.e.  $\Lambda \cdot \mathcal{T}_{\text{tot}}[\Phi] = \mathcal{T}_{\text{tot}}[\Lambda \cdot \Phi]$ .

In addition:

- (a) Every RSVP field solution is the image under  $\Xi$  of a Yarncrawler global section.
- (b) Every Yarncrawler reconstruction corresponds to an idempotent constraint operator in  $\mathcal{KOp}$ .
- (c) Every Spherepop event is a kernel morphism; its trace is irreversible (Theorem 9.6).
- (d) Every KES event is a threshold-crossing of  $\mathcal{T}_{\text{Ksyn}}$  (Proposition 10.3).
- (e) TARTAN quantum dynamics arise as globally synchronized unistochastic sections, which are fixed points of the gluing operator  $\mathcal{G}_{\text{TARTAN}} \circ \text{Sync}_{\text{Ising}}$ .
- (f) Lorentz-compatible kernel operators commute with the symmetry group of  $s^2$  (Proposition 13.2).

*Proof. Commutativity of the diagram.* The two paths from **Sph** to **FieldSys** are: (1) embed as kernel, apply RSVP functor; (2) extract post-event trajectory, embed as global sheaf section, embed in field history. Both paths yield the same RSVP history because both solve the same kernel-weighted Volterra equation (Definition 7.1) with initial data determined by the event  $e$ .

(i)  $\Leftrightarrow$  (ii). A fixed point of  $\mathcal{T}_{\text{tot}}$  defines a global solution whose restrictions to each  $U_i$  are local solutions. Conversely, compatible local solutions glue (Theorem 8.4) to a global section that satisfies  $\Phi = \mathcal{T}_{\text{tot}}[\Phi]$ .

(i)  $\Leftrightarrow$  (iii). Event compositions correspond to kernel compositions (Proposition 9.7); fixed points of the resulting operator are trajectories invariant under the composed event. Traced events yield the fixed-point equation  $\Phi = E(\Phi)$ , which is precisely constraint closure.

(i)  $\Leftrightarrow$  (iv). Kernel composition in  $\mathcal{KOp}$  defines the same algebra as trajectory synthesis; fixed points are the same in both descriptions.

(i)  $\Leftrightarrow$  (v). For Lorentz-compatible kernels (Proposition 13.2), fixed points of  $\mathcal{T}_{\text{tot}}$  are automatically Lorentz-stable. Conversely, Lorentz-stable trajectories that are solutions of the kernel equation are fixed points.

(a)–(d) were established in the individual framework sections above.

(e) TARTAN gluing defines a sheaf over the tile cover; Ising synchronization selects a globally consistent phase section; the result is a fixed point of the composed gluing operator, which is an instance of (i).

(f) Proposition 13.2. □

## 18.2 The Eight-Point Translation

The Unified Memory Theorem says in words: memory, causality, irreversibility, global consistency, constraint closure, quantum synchronization, and relativistic invariance are perspectives on a single structure. The translation table:

Property / Framework	Algebraic form in $\mathcal{KOp}$
Memory	Non-locality of $K(t, \tau)$ in time
Causality	Triangular support: $K(t, \tau) = 0$ for $\tau > t$
Irreversibility	Non-invertibility of the trace morphism
Global consistency	Vanishing of $\check{H}^1(\{U_i\}, \mathcal{T})$
Constraint closure	Idempotence / fixed point of $\mathcal{T}_K$
TARTAN quantum law	Fixed point of $\mathcal{G}_{\text{TARTAN}} \circ \text{Sync}_{\text{Ising}}$
Lorentz symmetry	Commutativity $\Lambda \cdot \mathcal{T}_K = \mathcal{T}_K \cdot \Lambda$
Fractional interpolation	Semigroup: $K_\alpha \circ K_\beta = K_{\alpha+\beta}$

Each entry is a theorem about kernel operators, not an assumption about the physical world.

### 18.3 The Self-Modulating Kernel: A Future Direction

The monograph has held the kernel  $K$  fixed. A natural generalization—noted but deliberately deferred in the notes—is to let  $K$  depend on the system’s own state:

$$K = K(t, \tau; \Phi(t), \mathbf{v}(t), S(t)).$$

In this regime:

- memory depth depends on entropy: high-entropy regions forget faster;
- transport reshapes its own history weighting;
- the system controls how strongly it remembers.

This is no longer fractional calculus in the classical sense. It is a self-modulating nonlocal field theory in which  $K$  is itself a dynamical variable. The Unified Memory Theorem applies to the fixed-kernel case; the self-modulating case is the natural continuation of this program.

## 19 Discussion

### 19.1 Memory as the Default

The standard approach treats memory as a complication: an extra ingredient added to explain anomalous behavior. The kernel-operator framework inverts this. The delta kernel ( $K = \delta$ ) is the *exceptional* case: it is the only kernel with zero support width. Every other kernel has positive support and therefore encodes memory. The generic situation is history-dependent dynamics; Markovian dynamics are the measure-zero limit.

The twentieth-century emphasis on Markovian models was an architectural choice, not an empirical finding. Locality was imposed because it makes differential equations tractable. The kernel-operator framework recovers the generality that locality sacrifices, without abandoning rigor.

### 19.2 The Formula Sheet, Revisited

The opening observation of the monograph was that a standard formula sheet is an atlas of local charts. The conclusion follows: *the formula sheet is a discrete sampling of a continuous operator space whose coordinates are kernel choice  $K$ , operator order  $\alpha$ , and constraint structure  $F$ .*

Arithmetic identities are charts near  $K = \delta$ ,  $\alpha = 0$ . Integration formulas are charts near  $K = 1$ ,  $\alpha = 1$ . Fractional operators are the curves connecting them. RSVP field equations are surfaces in the space parameterized by  $(\alpha, \beta, \gamma)$ . Yarncrawler, Spherepop, KES, and constraint closure are structured submanifolds of the same space.

The unified picture does not render the formula sheet obsolete; it reveals the hidden topology of the space from which each formula was sampled.

### 19.3 Irreversibility from Algebra, Not from Statistics

A long-standing puzzle: how does irreversibility emerge from time-reversible microscopic laws? The standard resolution invokes thermodynamic limits, coarse-graining, or cosmological initial conditions.

Theorem 9.6 offers an algebraic perspective: irreversibility is a property of the *trace operation* in the symmetric monoidal category of events, grounded in the entropy growth of the RSVP embedding. No thermodynamic limit is required. The non-invertibility of the traced morphism is a consequence of operator structure.

This does not resolve the foundational puzzle of statistical mechanics (which concerns Hamiltonian dynamics specifically), but it suggests that irreversibility can be located in the morphism structure of a physical theory rather than in statistical approximations.

#### 19.4 Quantum Mechanics as Synchronized Trajectory Dynamics

The TARTAN section establishes a structural connection between kernel-operator theory and quantum mechanics via Barandes’s unistochastic reformulation. The key insight is that quantum transition probabilities arise as squared moduli of unitary matrices—a constraint (unistochasticity) that, when enforced locally and synchronized globally via Ising dynamics, produces a global quantum law. This is an instance of the general TARTAN–sheaf pattern: local sections (tile transition laws) satisfying compatibility conditions (phase coherence) glue to a global section (the effective quantum dynamics).

The implication is that quantum behavior need not be posited as a primitive. It emerges from the same local-to-global constraint structure that governs RSVP field evolution and Yarncrawler reconstruction. The trajectory operator  $\mathcal{T}_K$  with the TARTAN kernel encodes both the probabilistic structure (through unistochasticity) and the memory structure (through the fractional kernel weighting).

#### 19.5 Relativity as Symmetry Constraint on Kernels

Section 12 showed that Lorentz transformations arise from a single requirement: that a quadratic form  $s^2 = c^2t^2 - x^2$  be invariant. Section 13 then embedded this into the kernel-operator framework: Lorentz-compatible kernels are those that depend only on invariant spacetime intervals, and the commutativity condition  $\Lambda \cdot \mathcal{T}_K = \mathcal{T}_K \cdot \Lambda$  is the algebraic statement of relativistic invariance.

This means that the transition from non-relativistic to relativistic field theory is, in the kernel-operator language, a transition from arbitrary causal kernels to kernels that are invariant under the Lorentz group. It is a restriction on the kernel, not an independent postulate about the world.

Constraint closure has a cognitive reading: a thinking system operating on trajectories (sequences of states, representations, or actions) completes a task when its constraint operator becomes idempotent. This is the algebraic definition of “done.”

The trajectory-first perspective contrasts with goal-state architectures, in which completion is defined by reaching a designated terminal state. In the constraint-closure framework, completion *emerges* from the dynamics of the system—the fixed point is not

specified in advance but reached by iteration. This has implications for autonomous systems designed to complete open-ended tasks without explicit terminal-state specifications.

The connection to representation learning: kernel-weighted prediction (which uses the full trajectory history, weighted by  $K_\alpha$ ) generalizes the Markovian prediction of current transformer architectures. The semigroup law provides a principled interpolation between these poles parameterized by  $\alpha$ .

## 19.6 Open Problems

1. **Self-modulating kernels.** Formalize the dynamics of  $K(t, \tau; \Phi, \mathbf{v}, S)$  as a field equation in its own right. What is the stability theory?
2. **Kernel renormalization.** Do the orders  $(\alpha, \beta, \gamma)$  of the RSVP system flow under renormalization-group transformations? What are the fixed points?
3. **Computability of constraint closure.** Given a computable  $\mathcal{T}_K$ , is the fixed point  $f^*$  always computable?
4. **Sheaf cohomology and information geometry.** Is there a natural Fisher metric on the space of sheaf sections making  $\mathcal{E}$  an information-geometric functional?
5. **KES phase transitions.** What happens at the exceptional thresholds excluded by Proposition 10.3? Do they correspond to phase transitions in the event structure?
6. **Quantum extension.** Do the semigroup law and Unified Memory Theorem extend to operator-valued kernels on Hilbert spaces?

## 20 Conclusion

This monograph has developed a single thesis: that dynamics are not functions of state but functionals of trajectories, and that kernel operators provide the minimal algebra for expressing that dependence.

The argument proceeded through five moves and eight frameworks:

1. **Reframe the formula sheet.** Standard mathematical formulas are local charts in a continuous operator space. The hidden assumption underlying all differential and integral formulas is locality: the kernel  $K = \delta$ .

2. **Introduce the trajectory operator.** The central object  $\mathcal{T}_K[f](t) = \int_0^t K(t, \tau)F(f(\tau)) d\tau$  unifies integration, differentiation, fractional calculus, RSVP transport, KES synthesis, and constraint closure as specializations of a single form.
3. **Prove the semigroup law.** The fractional kernel family  $\{K_\alpha\}$  is a one-parameter semigroup under composition. This is a consequence of the Beta function identity, not a definition. It makes fractional calculus algebraically inevitable.
4. **Categorify.** The trajectory operators form  $\mathcal{KOp}$  with  $\mathbf{Ker}$  as its linear subcategory. The original five frameworks—RSVP, Yarncrawler, Spherepop, KES, Constraint Closure—are subcategories and functorial images. TARTAN quantum dynamics emerge as globally synchronized sections; Lorentz symmetry appears as a commutativity condition on kernel operators.
5. **Prove the Unified Memory Theorem.** The commutative diagram of Theorem 18.1 establishes that all eight frameworks are not independent theories but equivalent characterizations of a single condition: a trajectory is a fixed point of its synthesized kernel operator.

The result is a picture in which memory is not a property added to a dynamics but the defining characteristic of dynamics as such. Systems that remember their history are the rule; systems that forget are the exception produced by setting  $K = \delta$ .

The formula sheet, revisited: it is not a list but a sample. Every formula in it is a point in the continuous operator space parameterized by kernel, order, and constraint structure. Fractional calculus is the curve connecting two such points. RSVP, Yarncrawler, Spherepop, KES, Constraint Closure, TARTAN, and relativistic symmetry are structured regions of the same space.

The algebra of kernels is the algebra of memory. Memory is the algebra of time.

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4. **Categorify.** The trajectory operators form the category  $\mathcal{KOp}$ , in which the semigroup becomes a functor. The five frameworks—RSVP, Yarncrawler, Spherepop, KES, Constraint Closure—are subcategories and functorial images.
5. **Prove the Unified Memory Theorem.** The commutative diagram of Theorem 18.1 establishes that the five frameworks are not independent theories but linked representations of the kernel-operator structure.

The result is a picture in which memory is not a property added to a dynamics but the defining characteristic of dynamics as such. Systems that remember their history are the rule; systems that forget are the exception produced by setting  $K = \delta$ .

The formula sheet, revisited: it is not a list but a sample. Every formula in it is a point in the continuous operator space parameterized by kernel, order, and constraint structure. Fractional calculus is the curve connecting two such points. RSVP, Yarncrawler, Spherepop, KES, and Constraint Closure are structured regions of the same space.

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