Trigonometry and Transformation

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Introduction: On the Teaching of Trigonometry

Trigonometry occupies a distinctive position within the mathematical curriculum. It is often encountered as a collection of methods for measuring angles, solving triangles, and analyzing periodic phenomena. Yet the subject also serves as an important point of contact between several major branches of mathematics. Geometric reasoning, algebraic symbolism, transformation theory, complex numbers, and harmonic analysis all intersect within its domain. As a result, the manner in which trigonometry is presented has consequences extending well beyond the immediate content of the course. A student's first encounter with trigonometric ideas frequently shapes their understanding of what mathematics itself is and how mathematical knowledge is organized.

The approach adopted in this text proceeds from the view that mathematical understanding is strengthened when individual results are situated within a broader network of relationships. Rather than presenting formulas and techniques as isolated objects of study, the exposition seeks to trace their development from a comparatively small collection of underlying principles. The goal is not merely computational proficiency, although such proficiency remains essential. It is also to cultivate the ability to reconstruct important results from more fundamental ideas when memory alone proves insufficient.

From this perspective, trigonometry may be understood as an extended investigation into the interaction between transformation and invariance. Similarity theory studies those relationships preserved under changes of scale. The unit circle isolates directional information by fixing magnitude. Trigonometric identities emerge from the algebraic structure of rotational composition. Complex numbers provide a unified language for scaling and rotation, while Fourier analysis reveals how periodic structure may be decomposed into elementary rotational modes. The sequence of topics traditionally encountered in trigonometry therefore possesses a deeper coherence than is sometimes apparent from a purely procedural presentation.

Throughout the text, occasional brief excursions consider systems outside traditional mathematics: numeral systems, calendars, linguistic structure, and other symbolic constructions. These comparative notes are not offered for cultural interest alone. Each illustrates, in a different setting, a structural idea also at work in the mathematics under discussion — compositional structure, periodicity, constrained state spaces, or transformations acting on invariant structure. They are intentionally brief and subordinate to the mathematical narrative, intended to remind the reader that the search for structure is not confined to geometry or algebra.

The chapters that follow move gradually from the geometry of triangles to the analysis of periodic phenomena. Along the way, students will encounter ratios, circles, vectors, matrices, complex numbers, and harmonic decompositions. These topics are often taught as separate units. The present text treats them as successive stages in a continuous development. The hope is that readers will come to see trigonometry not merely as a collection of techniques, but as one of the clearest introductions to the study of mathematical structure itself.

Preface

This book supports several styles of instruction. Followed sequentially, the chapters form a conventional undergraduate trigonometry course, self-contained even if the comparative sidebars are skipped for time. Read alongside the derivations, they support a flipped classroom in which class time is reserved for problem solving and proof construction. Read slowly, with the motivating questions at the head of each section taken seriously before the formal definitions that follow them, they support inquiry-based instruction.

Each section follows a consistent rhythm: a motivating problem, a derivation, a worked example, a short practice-now check, and a set of exercises. Major constructions are additionally given as explicit, numbered procedures, since students often need a procedure to lean on before they can appreciate the theory that generated it. Exercises are meant to be worked in order; later exercises frequently depend on results established in earlier ones. Answers to selected exercises appear in the appendix.

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Chapter 0

Structure from Constraint

Before any angle is measured or any triangle solved, it is worth asking a more primitive question: what does it mean for a mathematical object to have structure at all? This short opening chapter proposes an answer using discrete state spaces, in which the objects of study are finite collections of symbols, codes, or configurations rather than continuous geometric figures. The purpose is not to anticipate the content of later chapters but to introduce, in the simplest possible setting, the two ideas that organize the entire book: constraint and invariance.

0.1 Discrete State Spaces

Consider a finite set of symbols, called an *alphabet*, and the collection of all finite sequences that can be formed from it. If the alphabet has k symbols, then there are k^n sequences of length n, and this count grows without any internal organization worth remarking on: every sequence is as good as any other. Structure enters only once some sequences are declared admissible and others are not. It is the constraint, not the alphabet, that produces something worth studying.

Definition 0.1. A *discrete state space* is a pair (Σ, n) consisting of a finite alphabet Σ and a length n, together with the set Σ^n of all length-n sequences over Σ. An *admissibility constraint* on (Σ, n) is a rule that partitions Σ^n into two classes: the *admissible* sequences, which satisfy the rule, and the *inadmissible* sequences, which do not. The set of admissible sequences is written $\mathcal{A} \subseteq \Sigma^n$.

A constraint of this kind rarely acts on a sequence all at once. Far more commonly it acts *locally*: whether a given symbol may follow the previous one depends only on that previous symbol, not on the entire history of the sequence. Such constraints are conveniently recorded in a *transition graph*: a directed graph whose vertices are the symbols of Σ, with an edge from symbol a to symbol b exactly when b is permitted to follow a. This single graph then governs every sequence in $\mathcal{A}$, regardless of length.

0.2 Constraints Generate Structure

Theorem 0.2. *Let (Σ, n) be a discrete state space whose admissibility constraint is local, recorded by a transition graph G on vertex set Σ. Then the admissible sequences of length n are in bijection with the directed walks of length $n-1$ in G. Consequently, if M is the adjacency matrix of G, the number of admissible sequences of length n beginning at symbol a and ending at symbol b is the (a, b) entry of M^{n-1}.*

Proof. A sequence $s_1 s_2 \cdots s_n \in \Sigma^n$ is admissible exactly when each consecutive pair (s_i, s_{i+1}) corresponds to an edge of G, for $i = 1, \dots, n-1$. This is precisely the condition for $s_1, s_2, \dots, s_n$ to trace a directed walk of length $n-1$ through G, visiting vertices s_1 through s_n in order. The correspondence between admissible sequences and such walks is one-to-one by construction. The entry-counting claim then follows from the standard fact that the (a, b) entry of M^{n-1} counts directed walks of length $n-1$ from a to b, which is proved by induction on n using the definition of matrix multiplication. $\square$

What the theorem makes precise is something one can already sense informally: once a constraint is imposed, the admissible set stops being an undifferentiated pile of sequences and becomes something with countable, predictable structure, generated by iterating a single local rule. This is the first appearance, in the simplest setting available, of a pattern that will recur throughout the book. A constraint restricts a large space of possibilities to a smaller admissible subset, and that subset, far from being arbitrary, inherits a structure directly traceable to the constraint that produced it.

Example 0.3. Let $\Sigma = \{0, 1, \dots, 9\}$ and consider the discrete state space of length-n digit strings representing n-digit whole numbers, with the constraint that the first symbol s_1 may not be 0. This is a local constraint concentrated entirely on the first position: every symbol may follow every other symbol in positions 2 through n, but only the symbols 1 through 9 are admissible in position 1. The number of admissible sequences is therefore $9 \cdot 10^{n-1}$, obtained by choosing among nine possibilities for the first digit and ten possibilities for each remaining digit. This count is the familiar number of n-digit whole numbers, and it follows immediately from treating positional notation as a constrained discrete state space rather than as a definition to be taken for granted.

> *Practice now.* Let $\Sigma = \{0, 1\}$ and consider binary strings of length n with the constraint that no two consecutive symbols may both equal 1. Draw the transition graph on $\{0, 1\}$ implied by this constraint, and use it to count the admissible strings of length 4 by hand. Compare the result to the count predicted by the adjacency-matrix method of the theorem.

Comparative Note: Cistercian Numerals and Independent Composition.

A striking illustration of composition from constrained parts comes from the Cistercian numeral system, developed by medieval monks to write any whole number from one to nine thousand nine hundred ninety-nine as a single glyph. A vertical stem carries up to four attached strokes, one in each of four fixed positions corresponding to units, tens, hundreds, and thousands, and each position independently carries one of ten possible strokes, drawn from a shared base pattern rotated and reflected into the position it occupies. The system is therefore generated by choosing one symbol from a set of ten for each of four independent positions, giving $10^4 = 10{,}000$ representable values once the empty choice at every position, representing zero, is included. Nothing about the choice made in the units position constrains the choice available in the tens position; the four positions act as independent coordinates of a single glyph, in exactly the sense that a vector in this book is built from independent coordinates along different axes. What makes such a system powerful is not the alphabet of ten strokes by itself, but the fact that composing strokes across four independent positions multiplies the number of representable values far beyond what any single position could express alone, a pattern that will reappear once vectors and matrices are built from independently varying components of their own.

0.3 From Discrete Constraint to Continuous Invariance

The state spaces considered so far have been finite, and their constraints have been combinatorial rules about which symbols may follow which. Nothing in the underlying logic, however, depends on finiteness. A continuous analogue appears as soon as one asks which points (x, y) in the plane satisfy $x^2 + y^2 = 1$. Here the ambient space $\mathbb{R}^2$ plays the role of Σ^n, the equation plays the role of the admissibility constraint, and the resulting unit circle plays the role of $\mathbb{A}$: a lower-dimensional admissible set carved out of a larger, unconstrained space by a single governing condition. Chapter 4 will take up this continuous case in detail, but it is the same idea encountered here in miniature. A constraint isolates an admissible substructure, and that substructure, rather than the ambient space that contained it, becomes the proper object of study. Much of what follows in this book can be read as a sustained exploration of what happens when that single observation is taken seriously.

0.4 Exercises

Exercise 0.1. Let $\Sigma = \{a, b, c\}$ with transition graph given by the edges $a \to b$, $b \to c$, $c \to a$, and $c \to b$. List all admissible sequences of length 3 beginning at a, and verify your count against the adjacency-matrix method of the theorem.

Exercise 0.2. Explain, in a short paragraph, why the count $9 \cdot 10^{n-1}$ from the worked example would change if the constraint instead forbade the digit 0 from appearing in *any* position rather than only the first. Compute the new count and identify the transition graph that generates it.

Part I

Similarity, Ratios, and Coordinates

Chapter 1

Similarity, Ratios, and Scale

1.1 Why Size Matters

Imagine two maps of the same city, one small enough to fit on a pocket card and the other large enough to cover a classroom wall. The streets, rivers, and landmarks occupy different physical distances on the two sheets of paper, yet both maps represent the same place: the larger map has not created new streets, and the smaller map has not removed any, since one is simply a scaled copy of the other. This observation introduces one of the oldest and most important ideas in mathematics, namely that some properties change when size changes while others remain the same. A line segment that measures two centimeters on one map may measure ten centimeters on another, and areas and volumes change along with it, yet certain relationships survive: the ratio between the lengths of two streets is unchanged, the angle at which two roads meet is unchanged, and the overall shape of the city persists even as its scale varies from map to map. Mathematics often begins by asking a deceptively simple question — what remains unchanged when everything else changes? — and the study of similarity is one of the oldest and most productive answers to that question.

1.2 Scale Transformations

Definition 1.1. Let $P = (x, y)$ be a point of the Cartesian plane and let k be a positive real number. The *scale transformation*, or *dilation*, by factor k sends P to the point $P' = (kx, ky)$.

Every point of a figure moves proportionally under this transformation: if $k > 1$ the figure expands away from the origin, and if $0 < k < 1$ the figure contracts toward it. The map $(x, y) \mapsto (kx, ky)$ changes the size of a figure while preserving its shape, and the question this chapter takes up is which properties survive such a transformation and which do not.

1.3 Similar Figures

Definition 1.2. Two figures are *similar* if one can be obtained from the other by some combination of scaling, translation, reflection, and rotation. Informally, similar figures share a common shape without necessarily sharing a common size.

Two squares of different sizes are similar, as are two circles of different radii and two equilateral triangles of different side lengths, since in each case one figure is obtained from the other by scaling alone. A square and a rectangle whose sides are unequal, by contrast, are not similar, since no combination of scaling, translation, reflection, and rotation carries one onto the other. Similarity is therefore a statement about structure rather than about measurement: it asks whether two figures agree in shape, leaving their absolute size unspecified.

1.4 Similar Triangles

Triangles provide the most important examples of similarity encountered in this book. Consider two triangles $\triangle ABC$ and $\triangle DEF$. If their corresponding angles agree, so that $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$, the triangles are said to be similar, written $\triangle ABC \sim \triangle DEF$. When two triangles are similar, every pair of corresponding sides occurs in the same ratio,

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF},$$

and this common ratio is called the *scale factor* relating the two triangles.

1.5 Derivation: Why Ratios Survive Scaling

Theorem 1.3. *Let a and b be the lengths of two segments, and suppose both are scaled by the same factor $k > 0$, becoming ka and kb. Then the ratio of the new lengths equals the ratio of the original lengths: $\frac{ka}{kb} = \frac{a}{b}$.*

Proof. Since $k \neq 0$, it may be cancelled from numerator and denominator:

$$\frac{ka}{kb} = \frac{k}{k} \cdot \frac{a}{b} = \frac{a}{b}.$$

□

The scale factor cancels because it appears identically in numerator and denominator, and this is the fundamental reason ratios play such a central role in geometry. A ratio ignores absolute size and records only structure: whenever every length in a figure changes by the same factor, the ratios between those lengths remain exactly as they were before the figure was scaled.

1.6 The Geometry of Invariance

Example 1.4. Consider a right triangle with sides 3, 4, and 5, and suppose every side is doubled, producing a new triangle with sides 6, 8, and 10. The lengths, the area, and the perimeter of the new triangle all differ from those of the original. Yet

$$\frac{3}{5} = \frac{6}{10} \qquad \text{and} \qquad \frac{4}{5} = \frac{8}{10},$$

so the ratios between corresponding sides remain exactly the same: only the scale has changed, not the shape.

This observation lies at the foundation of everything that follows in this book. The trigonometric functions to be introduced in later chapters arise precisely because certain ratios of a right triangle depend only on the triangle's angles and not on its size, and before sine, cosine, or tangent can be defined in any meaningful way, it is necessary to first understand why such size-independent quantities can exist at all. Similarity provides the answer: because ratios are invariant under scaling, a ratio computed from any triangle similar to a given one will agree with the ratio computed from the original, which is exactly the property that will let the trigonometric ratios of Chapter 3 depend on angle alone.

1.7 Similarity Criteria

Three standard criteria determine whether two triangles are similar without requiring every angle and every side to be checked individually.

Theorem 1.5 (Angle–Angle)**.** *If two angles of one triangle equal two corresponding angles of another, the two triangles are similar.*

Proof. Since the angles of a triangle sum to a fixed total, agreement of two corresponding angles forces agreement of the third as well, so all three corresponding angles agree and the triangles are similar by definition. □

Theorem 1.6 (Side–Angle–Side)**.** *If two pairs of corresponding sides of two triangles are proportional and the included angles are equal, the triangles are similar.*

Theorem 1.7 (Side–Side–Side)**.** *If all three pairs of corresponding sides of two triangles occur in the same ratio, $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$, the triangles are similar.*

Procedure 1.1 (How to Determine Whether Two Triangles Are Similar).
Given two triangles, first check whether two pairs of corresponding angles are known and equal; if so, the Angle–Angle criterion applies and no further check is needed. If instead two pairs of corresponding sides and their included angle are known, compute the ratio of the two known sides in each triangle; if the ratios agree and the included angles are equal, the Side–Angle–Side criterion applies. If all three pairs of corresponding sides are known, compute all three ratios; if the three ratios agree, the Side–Side–Side criterion applies. If none of these conditions can be verified from the given information, similarity cannot yet be concluded, and more information about the triangles is required.

Example 1.8. Determine whether a triangle with sides 3, 4, 5 is similar to a triangle with sides 6, 8, 10. Computing the three ratios of corresponding sides gives

$$\frac{6}{3} = 2, \qquad \frac{8}{4} = 2, \qquad \frac{10}{5} = 2,$$

and since all three ratios agree, the Side–Side–Side criterion applies: the triangles are similar, with scale factor $k = 2$.

1.8 Applications: Similarity as a Tool for Measurement

Similarity allows unknown lengths to be determined without direct measurement. Suppose a tree casts a shadow 12 meters long, and at the same moment a vertical meter stick casts a shadow 0.8 meters long. Because sunlight arrives at essentially parallel angles, the triangle formed by the tree and its shadow is similar to the triangle formed by the meter stick and its shadow, so the ratio of height to shadow length must agree between them:

$$\frac{h}{12} = \frac{1}{0.8}.$$

Solving for h gives $h = 15$, so the tree is fifteen meters tall. No direct measurement of the tree itself was required; similarity alone was enough to convert an inaccessible length into an accessible one.

> *Practice now.* A flagpole casts a shadow 9 meters long at the same moment that a 1.5-meter fence post casts a shadow 2 meters long. Set up the similar-triangle ratio implied by these measurements and find the height of the flagpole.

1.9 Looking Ahead

This chapter has introduced the idea that will support everything that follows: a scale transformation changes lengths but preserves the ratios between them, and those preserved quantities are the first examples of what this book will repeatedly call invariants. The next chapter introduces coordinates and vectors, and with them a second kind of transformation — rotation — that changes orientation while preserving other quantities in turn. Where similarity studies what survives a change of size, the chapters ahead study what survives a change of orientation, and the path from triangles to the trigonometric functions begins with the same simple cancellation derived above: $ka/kb = a/b$. The scale factor vanishes, and what remains is structure.

1.10 Exercises

Exercise 1.1. A triangle has sides 5, 12, 13. A second triangle has sides 10, 24, 26. Determine whether the triangles are similar, and if so, state the scale factor.

Exercise 1.2. Two triangles share an angle of $40°$, and the sides adjacent to that angle are in the ratio $3 : 5$ in both triangles. Which similarity criterion applies, and why is that criterion sufficient without knowing the remaining angles or sides?

Exercise 1.3. A person of height 1.7 meters stands beside a building and casts a shadow 2.1 meters long. At the same moment, the building casts a shadow 34 meters long. Find the height of the building, and explain in a sentence why the assumption of parallel sunlight is essential to the argument.

Chapter 2

Coordinates and Vectors

2.1 Why Position Is Not Enough

The previous chapter studied similarity and scale, and found that certain relationships survive changes in size: a triangle may grow or shrink while its shape, recorded in the ratios between its sides, remains unchanged. Geometry must answer a different question as well. A hiker begins at one location and ends at another; a ship leaves a harbor and travels across open water; a package is moved across a warehouse floor. In each case what matters is not merely where something is, but how it moved. A single point describes a location, but it does not describe a displacement, and to describe movement requires a mathematical object capable of representing magnitude and direction at once. The development of such an object begins, as it must, with a way of describing position in the first place.

2.2 Coordinates as a System of Description

Definition 2.1. The *Cartesian coordinate system* consists of two perpendicular number lines meeting at a point called the *origin*: a horizontal *x-axis* and a vertical *y-axis*. Every point in the plane is assigned an ordered pair (x, y), where x measures horizontal displacement from the origin and y measures vertical displacement.

Historical Note: Why "Cartesian".

The coordinate system of this chapter is named for the French philosopher and mathematician René Descartes, whose Latinized name, *Cartesius*, gives the adjective *Cartesian* [5]. Descartes did not present the system in the clean (x, y)-axis form given here; his 1637 appendix *La Géométrie* used a single reference line and measured a second distance obliquely from it, arriving at essentially the same idea, that a curve could be described by an algebraic relationship between two measured quantities, without drawing the perpendicular axes now named after him. The now-standard picture of two perpendicular axes meeting at an origin was assembled gradually by later mathematicians working from Descartes' insight, so that the name Cartesian honors the founding idea, the translation of geometric questions into algebraic ones, more than it records the exact picture students draw today.

The pair $(3, 2)$, for instance, identifies the point three units to the right of the origin and two units above it. Coordinates should not be mistaken for part of the plane itself; they are a convention for labeling positions within it, and different coordinate systems may describe the same geometric reality in different ways. The usefulness of Cartesian coordinates lies in the fact that geometric questions can be translated directly into algebraic ones, a translation this chapter now puts to use.

2.3 Vectors

Consider the points $A = (1, 1)$ and $B = (4, 3)$. Moving from A to B requires a horizontal change of $4 - 1 = 3$ units and a vertical change of $3 - 1 = 2$ units, and this displacement is recorded by the ordered pair $\langle 3, 2\rangle$.

Definition 2.2. A *vector* is an object possessing magnitude and direction but no fixed location. Two vectors are *equal* whenever they share the same magnitude and direction, regardless of where each is drawn.

The displacement $\langle 3, 2\rangle$ describes the same movement whether it begins at A, at B, or at any other point of the plane, which is precisely why a vector is not tied to a particular location the way a point is. For this reason vectors are often drawn as arrows, with the length of the arrow representing magnitude and its orientation representing direction.

2.4 Vector Addition

Suppose one displacement is $\mathbf{u} = \langle 2, 1\rangle$ and another is $\mathbf{v} = \langle 3, 2\rangle$. Performing the first displacement and then the second produces the combined displacement

$$\mathbf{u} + \mathbf{v} = \langle 2 + 3,\ 1 + 2\rangle = \langle 5, 3\rangle.$$

Geometrically, this sum may be visualized by placing the tail of the second vector at the head of the first, so that the resulting displacement runs from the tail of $\mathbf{u}$ to the head of $\mathbf{v}$; this construction is known as the *tip-to-tail method*, and an equivalent picture is provided by the parallelogram rule.

2.5 Scalar Multiplication

Chapter 1 studied scale transformations of the form $(x, y) \mapsto (kx, ky)$, and vectors behave in exactly the same way under scaling. Given a vector $\mathbf{v} = \langle x, y\rangle$ and a scalar k, the product $k\mathbf{v} = \langle kx, ky\rangle$ points in the same direction as $\mathbf{v}$ when $k > 0$, with $3\langle 2, 1\rangle = \langle 6, 3\rangle$ illustrating a vector three times as long as the original. If $0 < k < 1$ the vector becomes shorter without changing direction, and if $k < 0$ the vector reverses direction entirely. Scalar multiplication is therefore nothing more than the dilation of Chapter 1, applied to a displacement rather than to a whole figure.

2.6 Derivation: The Length of a Vector

A vector possesses both direction and magnitude, and this section derives a formula for the latter. Consider $\mathbf{v} = \langle x, y\rangle$ with its tail placed at the origin. The horizontal and vertical components of $\mathbf{v}$, together with $\mathbf{v}$ itself, form a right triangle whose legs have lengths $|x|$ and $|y|$ and whose hypotenuse is $\mathbf{v}$.

Theorem 2.3. *For a vector $\mathbf{v} = \langle x, y\rangle$, its magnitude is*

$$|\mathbf{v}| = \sqrt{x^2 + y^2}.$$

Proof. The vector $\mathbf{v}$ forms the hypotenuse of a right triangle whose legs have lengths $|x|$ and $|y|$. By the Pythagorean theorem, $|\mathbf{v}|^2 = x^2 + y^2$, and taking the positive square root gives $|\mathbf{v}| = \sqrt{x^2 + y^2}$. □

This formula is one of the most important results of the chapter: it will later become the bridge connecting vectors to the trigonometric ratios of Chapter 3, since every vector, by virtue of possessing a horizontal and a vertical component, already carries a right triangle within it.

2.7 Components and Decomposition

Every vector in the plane may be viewed as the sum of a horizontal displacement and a vertical displacement. For instance,

$$\langle 5, 3\rangle = \langle 5, 0\rangle + \langle 0, 3\rangle,$$

and these two simpler vectors are called the *components* of the original. The process of expressing a vector as a sum of its components is called *decomposition*, and it allows complicated motion to be analyzed one direction at a time; many problems in physics, engineering, and navigation become tractable only after such a decomposition has been made.

Procedure 2.1 (How to Decompose a Vector into Components).
Given a vector $\mathbf{v} = \langle x, y\rangle$, identify the horizontal coordinate x and the vertical coordinate y separately. Form the horizontal component $\langle x, 0\rangle$ and the vertical component $\langle 0, y\rangle$, and confirm that their sum recovers the original vector, $\mathbf{v} = \langle x, 0\rangle + \langle 0, y\rangle$.

Example 2.4. An aircraft travels 120 kilometers east and 50 kilometers north. Its displacement is the vector $\mathbf{v} = \langle 120, 50\rangle$, and by the magnitude formula,

$$|\mathbf{v}| = \sqrt{120^2 + 50^2} = \sqrt{14400 + 2500} = \sqrt{16900} = 130.$$

The aircraft is displaced 130 kilometers from its starting point.

Practice now. A hiker travels 24 kilometers east and 7 kilometers north. Represent the displacement as a vector and determine its magnitude.

2.8 Applications

Vectors provide a mathematical language for describing motion, force, velocity, acceleration, and displacement. Navigation describes courses and bearings with vectors; physics represents the forces acting on an object with them; engineering describes stresses and motions within structures in the same terms; computer graphics determines the positions and movements of objects within virtual environments the same way still. A vector's magnitude can already be computed from its components, but its orientation remains, for now, unspecified: no formal method for measuring direction has yet been introduced, even though the need for one should already be apparent. Supplying that method is the task of the next chapter, and the tool it introduces is the angle.

2.9 Looking Ahead

This chapter introduced vectors as mathematical descriptions of displacement, and showed how to add them, scale them, decompose them into components, and determine their magnitudes. Most importantly, it showed that every vector naturally generates a right triangle, an observation that is not accidental: right triangles are the bridge between length and direction. The next chapter begins the study of angles and the ratios associated with right triangles, ratios that will eventually describe a vector's direction as precisely as its magnitude can already be described today. The transition from geometry to trigonometry begins with a single question: if a vector's magnitude can be measured, how can its direction be measured?

2.10 Exercises

Exercise 2.1. Find the magnitude of each vector: (a) $\langle 3, 4\rangle$, (b) $\langle 5, 12\rangle$, (c) $\langle 8, 15\rangle$.

Exercise 2.2. Decompose each vector into horizontal and vertical components: (a) $\langle 7, 2\rangle$, (b) $\langle -4, 5\rangle$, (c) $\langle 3, -6\rangle$.

Exercise 2.3. A boat travels 15 kilometers east and 20 kilometers north. Represent the displacement as a vector and determine its magnitude.

Exercise 2.4. Explain why two vectors with the same magnitude and direction are considered equal even when they begin at different points.

Exercise 2.5. A vector is multiplied by a scalar k. Explain how its magnitude and direction change as k varies, and describe what happens in the particular case $k < 0$.

Chapter 3

Right Triangle Trigonometry

Chapter 1 established that certain ratios survive changes of scale. Chapter 2 established vectors, and showed that every vector generates a right triangle whose legs are its components. This chapter asks the question that those two results were building toward: given a right triangle with a fixed acute angle, what remains unchanged as the triangle is enlarged or reduced? The answer will turn out to be a small, fixed collection of ratios, and naming those ratios is all that defining sine, cosine, and tangent amounts to.

3.1 Motivation: The Problem of Direction

A vector $\mathbf{v} = \langle x, y \rangle$ has a magnitude, $|\mathbf{v}| = \sqrt{x^2 + y^2}$, computed in the previous chapter. Magnitude alone, however, does not determine a vector: a vector of length 10 may point in infinitely many directions, and nothing established so far can distinguish one from another. Measuring direction will require a new idea, and that idea begins with the right triangle every vector already carries inside it.

3.2 Similar Right Triangles

Construct several right triangles sharing the same acute angle θ: for instance, triangles with sides 3-4-5, 6-8-10, and 9-12-15. Each is obtained from the 3-4-5 triangle by scaling, so by Angle–Angle similarity all three triangles are similar, and Chapter 1 guarantees that corresponding side ratios agree across all of them:

$$\frac{3}{5} = \frac{6}{10} = \frac{9}{15}, \qquad \frac{4}{5} = \frac{8}{10} = \frac{12}{15}.$$

Theorem 3.1. *For any collection of right triangles sharing a common acute angle θ, the ratio of any two corresponding sides is the same across every triangle in the collection.*

Proof. Two right triangles sharing an acute angle θ also share their right angle, hence agree in two corresponding angles and are similar by the Angle–Angle criterion of Chapter 1. Similar triangles have proportional corresponding sides, so the ratio of any two corresponding sides is the common scale factor's cancellation, exactly as shown in the derivation of Section 1.5: the scale factor appears in both the numerator and denominator of the ratio and cancels, leaving a value depending only on θ and not on the size of the triangle chosen. $\square$

This is the central theorem of the chapter, and everything that follows is simply a matter of giving names to the ratios it guarantees exist.

3.3 Naming the Invariants

For an acute angle θ in a right triangle, call the side across from θ the *opposite* side, the side adjacent to θ that is not the hypotenuse the *adjacent* side, and the side across from the right angle the *hypotenuse*.

Definition 3.2. For an acute angle θ of a right triangle,

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}.$$

Historical Note: Where the Word "Sine" Comes From.

The word *sine* has an unusually crooked history for a term this central to mathematics [5]. Indian astronomers of the first millennium worked with a quantity called *jyā-ardha*, literally "half-chord," shortened in later Sanskrit texts to *jyā* or *jivā*. When these texts were translated into Arabic, the word was transliterated phonetically as *jiba*, a term with no meaning of its own in Arabic. Arabic is ordinarily written without vowels, so later readers encountering the consonants of *jiba* reinterpreted them as the ordinary Arabic word *jaib*, meaning a bay, fold, or pocket in a garment. When the twelfth-century translator Gerard of Cremona rendered this word into Latin, he translated the meaning of *jaib* rather than transliterating the original Sanskrit, giving the Latin word *sinus*, meaning a bay, curve, or fold of cloth, from which English inherited *sine*. The name of one of the most important functions in mathematics therefore descends from a fold in a piece of clothing, by way of a translator's honest misreading of an unfamiliar word. Cosine and cotangent follow more transparently: the prefix "co-" abbreviates *complementi*, since the cosine of θ is the sine of its complementary angle, $90^\circ - \theta$.

It is worth pausing on what has actually happened here. These three equations are not arbitrary formulas handed down by convention; they are names attached to quantities the theorem of the previous section has already proved to be invariant under scaling. Nothing about $\sin\theta$, $\cos\theta$, or $\tan\theta$ depends on which triangle with acute angle θ happens to be drawn, precisely because the ratio defining each one was shown, in general, to be scale-independent before it was ever given a name.

Comparative Note: Why Do We Name Invariants?.

A quantity earns a name in mathematics and science largely because it survives changes that are, for the purpose at hand, irrelevant. Temperature is useful because it survives the shape of its container; density is useful because it survives the size of the sample measured; a trigonometric ratio is useful because it survives the scale of the triangle it is computed from. In each case a name is attached not to a number but to an invariant, a quantity that stays fixed while everything else around it changes, and it is this stability that makes the quantity worth comparing across situations in the first place. Mathematics repeatedly performs this same maneuver — identify what a family of transformations leaves unchanged, then name it — and sine, cosine, and tangent are among the earliest and most durable examples a student will encounter.

3.4 The Remaining Three Functions

Three further ratios are traditionally defined as reciprocals of the ones above:

$$\csc\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}.$$

These reciprocal functions — cosecant, secant, and cotangent — appear throughout the historical literature and remain useful in certain identities and calculus formulas encountered later in this book, but the overwhelming majority of applications in this text and elsewhere rely on sine, cosine, and tangent alone.

3.5 Finding Unknown Sides

Procedure 3.1 (How to Find an Unknown Side Using a Trigonometric Ratio).
Given a right triangle with one known acute angle and one known side, first identify which of the two known quantities the unknown side is related to by determining whether the unknown side is opposite, adjacent, or the hypotenuse relative to the known angle. Next, select whichever of $\sin\theta$, $\cos\theta$, or $\tan\theta$ relates the known side, the unknown side, and the known angle. Write the resulting equation and solve it for the unknown side.

Example 3.3. A ten-meter ladder leans against a wall, making a 65° angle with the ground. How high up the wall does the ladder reach? The height h is opposite the given angle, and the ladder itself is the hypotenuse, so

$$\sin(65^\circ) = \frac{h}{10}, \qquad h = 10\sin(65^\circ) \approx 9.06 \text{ meters.}$$

Practice now. A twelve-meter ladder leans against a wall, making a 72° angle with the ground. Find the height the ladder reaches up the wall.

3.6 Finding Unknown Angles

The reverse problem also arises naturally: given the ratio of two sides, how is the corresponding angle recovered? If $\tan\theta = 7/4$, for example, some process must undo the tangent function and return θ itself. Such a process exists and is used freely at this stage with a calculator, but it will not be formally defined until Chapter 7 introduces the inverse trigonometric functions with the care that definition deserves. For now it is enough to know that the question is meaningful, and that a numerical answer can be obtained.

3.7 Special Right Triangles

Two right triangles recur so often that their ratios are worth deriving explicitly, once and for all, rather than looked up in a table.

Example 3.4 (The 45°–45°–90° Triangle)**.** Consider an isosceles right triangle with both legs of length 1. By the Pythagorean theorem, its hypotenuse has length $\sqrt{1^2+1^2} = \sqrt{2}$, giving side lengths 1, 1, $\sqrt{2}$. Since the two legs are equal, the two acute angles are also equal, and since they sum to 90° each must equal 45°. Applying the definitions of the previous section,

$$\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \qquad \tan 45^\circ = \frac{1}{1} = 1.$$

Example 3.5 (The 30°–60°–90° Triangle)**.** Consider an equilateral triangle with side length 2, so that all three angles equal 60°. Dropping an altitude from one vertex to the midpoint of the opposite side bisects both that side and the angle at the vertex, producing two congruent right triangles, each with hypotenuse 2, short leg 1 (half of the original side), and a 30°–60°–90° angle configuration. By the Pythagorean theorem the remaining leg has length $\sqrt{2^2-1^2} = \sqrt{3}$, giving side lengths 1, $\sqrt{3}$, 2. Applying the definitions of the previous section to the 30° angle, whose opposite side is the short leg of length 1 and whose adjacent side is the leg of length $\sqrt{3}$,

$$\sin 30^\circ = \frac{1}{2}, \qquad \cos 30^\circ = \frac{\sqrt{3}}{2}, \qquad \tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

The complementary angle 60° exchanges the roles of opposite and adjacent, giving $\sin 60^\circ = \sqrt{3}/2$, $\cos 60^\circ = 1/2$, and $\tan 60^\circ = \sqrt{3}$.

3.8 Applications

Right-triangle trigonometry is the working tool of surveying, where an inaccessible height or distance is recovered from an angle of elevation or depression together with a measured baseline; of navigation, where a vessel's position is found from bearing and distance; and of architecture and engineering, where the ratios of this chapter govern everything from roof pitch to the forces along a diagonal brace. The mathematics developed above is what makes each of these applications possible, and the applications themselves are best treated as exercises in identifying which side and angle are known and which ratio connects them.

3.9 Looking Ahead

This chapter has defined the trigonometric functions using right triangles, and that definition carries a built-in restriction worth stating plainly. An acute angle inside a right triangle can never reach or exceed 90°, and it can never be negative, so as it stands sine, cosine, and tangent are only defined for a narrow range of angles. If trigonometry is to become a general theory of direction, capable of describing a vector pointing in any direction whatsoever, this restriction must eventually be removed. The next chapter develops a framework capable of describing any angle at all, by returning to the unit circle introduced informally in Chapter 0 and asking what a trigonometric ratio can mean once the right triangle that first defined it is allowed to disappear.

The pattern established in this chapter, transformation to invariant to name, is one that recurs throughout the book: a family of transformations (here, scaling) is shown to preserve some quantity (here, a ratio of sides), and that preserved quantity is given a name (here, sine, cosine, and tangent) precisely because its invariance is what makes it worth naming. The same pattern will reappear with rotations in Chapter 8, with complex numbers in Chapter 16, and with Fourier modes in Chapter 19.

3.10 Exercises

Exercise 3.1. A right triangle has an acute angle of 38° and a hypotenuse of 14 centimeters. Find the lengths of the two legs.

Exercise 3.2. Derive the values of $\sin 45^\circ$, $\cos 45^\circ$, and $\tan 45^\circ$ directly from a right triangle with legs of length 2 instead of 1, and confirm that the result agrees with the derivation given in the text.

Exercise 3.3. A surveyor stands 40 meters from the base of a tower and measures the angle of elevation to the top of the tower as 52°. Find the height of the tower.

Exercise 3.4. Explain, using the theorem of Section 3.2, why $\sin\theta$ could not possibly depend on which right triangle with acute angle θ is used to compute it.

Part II

The Unit Circle and Circular Motion

Chapter 4

The Unit Circle and General Angles

4.1 Beyond Right Triangles

The trigonometric functions were introduced in the previous chapter as ratios associated with right triangles, and that approach provided a powerful method for relating angles and side lengths. It also imposed a limitation, since a right triangle can contain only acute angles: the angle θ under discussion had to satisfy $0^\circ < \theta < 90^\circ$. What, then, is the sine of 150°, or the cosine of -45°, or the meaning of an angle of 720°? None of these questions can be answered using right triangles alone, and the difficulty is not with trigonometry itself but with having tied the trigonometric functions to the interior angle of a triangle in the first place. To remove the restriction, angle must be redefined not as part of a triangle but as a rotation, and the geometric object that makes this possible is the unit circle.

4.2 The Unit Circle

Recall the distance formula of Chapter 2: a point (x, y) lies at distance $\sqrt{x^2 + y^2}$ from the origin. Requiring every point under consideration to lie exactly one unit from the origin means requiring $\sqrt{x^2 + y^2} = 1$, or equivalently $x^2 + y^2 = 1$.

Definition 4.1. The *unit circle* is the set of all points (x, y) satisfying $x^2 + y^2 = 1$.

The unit circle is not the entire plane; it is a constrained subset of it, in exactly the sense of Chapter 0. Among all possible points (x, y), only those satisfying $x^2 + y^2 = 1$ are admissible, and this single constraint fixes magnitude at 1 while leaving direction free to vary. The circle is therefore precisely the geometric setting needed to study direction in isolation, without the interference of size that occupied Chapter 1.

4.3 Angles as Rotations

Definition 4.2. An *angle* is a rotation from the positive x-axis, called the *initial side*, to a rotated ray called the *terminal side*. A counterclockwise rotation is considered positive, and a clockwise rotation is considered negative. The point where the terminal side meets the unit circle is the point corresponding to that angle.

This definition allows angles such as 120°, -45°, and 720° to be spoken of naturally, since none of them is required to fit inside a triangle: an angle is now simply an amount of rotation, unrestricted in sign or size.

4.4 Degree Measure and Radian Measure

Angles have historically been measured in degrees, with a complete revolution comprising 360°. Mathematics also makes frequent use of a second unit, the radian.

Definition 4.3. One *radian* is the angle that subtends an arc of the unit circle whose length equals the circle's radius.

Because the circumference of a circle of radius r is $2\pi r$, a complete revolution subtends an arc of length $2\pi r$, and dividing by the radius shows that a complete revolution corresponds to an angle of 2π radians. Hence $360° = 2\pi$ radians, from which the conversion formulas

$$\theta_{\text{rad}} = \theta_{\text{deg}} \cdot \frac{\pi}{180}, \qquad \theta_{\text{deg}} = \theta_{\text{rad}} \cdot \frac{180}{\pi}$$

follow immediately. Degrees and radians are two units for the same geometric quantity, and this book will move between them as convenience dictates.

4.5 Extending Sine and Cosine

Let θ be any angle, and rotate from the positive x-axis until reaching the terminal side of θ. Suppose this terminal side meets the unit circle at the point $P = (x, y)$.

Definition 4.4. For any angle θ, $\cos\theta = x$ and $\sin\theta = y$, where (x, y) is the point at which the terminal side of θ meets the unit circle. In words, cosine is the horizontal coordinate of this point and sine is its vertical coordinate.

This definition places no restriction on θ whatsoever, and applies equally well to angles of any size or sign. It remains to check that nothing established in Chapter 3 has been lost in the generalization.

Theorem 4.5. *For an acute angle θ of a right triangle, the unit-circle definitions of sine and cosine agree with the definitions of Chapter 3.*

Proof. Let the right triangle have hypotenuse r, so that by Chapter 3, $\cos\theta = \text{adjacent}/r$ and $\sin\theta = \text{opposite}/r$. Dividing every side length of the triangle by r scales the triangle onto the unit circle, by the scalar multiplication of Chapter 2 with $k = 1/r$, and the vertex corresponding to the angle θ moves to the point $(\text{adjacent}/r, \text{ opposite}/r)$. This is exactly the point at which the terminal side of θ meets the unit circle, so $x = \cos\theta$ and $y = \sin\theta$ under the new definition agree with the ratios of the old one. □

Nothing from Chapter 3 has therefore been replaced; the unit-circle definitions merely extend the right-triangle definitions to angles the triangle could never have contained.

4.6 Tangent and the Remaining Functions

Since $\sin\theta = y$ and $\cos\theta = x$ on the unit circle, tangent is defined as before by

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}.$$

Whenever $x = 0$ this ratio is undefined, so tangent fails to exist at 90°, 270°, and every angle coterminal with them. The reciprocal functions are defined exactly as in Chapter 3:

$$\csc\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}.$$

4.7 Coterminal Angles

Beginning at the positive x-axis and rotating through one complete revolution returns to the same point of the unit circle, so adding 360° to an angle changes its description but not the point it names.

Theorem 4.6. *For every integer k, the angles θ and $\theta + 360^\circ k$ determine the same point of the unit circle; equivalently, θ and $\theta + 2\pi k$ determine the same point when measured in radians. Angles related in this way are called* coterminal.

This is the first appearance, in the context of angle itself, of a theme that will recur throughout the rest of the book: periodicity. The circle repeats, and its geometry is inherently cyclic, so that any function defined in terms of position on the circle must inherit that same repetition.

4.8 Reference Angles

Although the trigonometric functions are now defined for every angle, evaluating them typically reduces to a much simpler problem, since every angle can be associated with an acute angle in the first quadrant called its *reference angle*.

Procedure 4.1 (How to Evaluate a Trigonometric Function Using a Reference Angle).
Given an angle θ, first locate the quadrant containing its terminal side. Next determine the acute reference angle formed between the terminal side and the x-axis. Evaluate the corresponding trigonometric function at this first-quadrant reference angle, and finally apply the sign appropriate to the original quadrant, using the fact that $\cos\theta$ is the horizontal coordinate and $\sin\theta$ is the vertical coordinate of a point on the unit circle.

Quadrant	$\sin\theta$	$\cos\theta$	$\tan\theta$
I	$+$	$+$	$+$
II	$+$	$-$	$-$
III	$-$	$-$	$+$
IV	$-$	$+$	$-$

Table 4.1: Sign of each trigonometric function by quadrant.

Example 4.7. Evaluate $\sin(210^\circ)$. Since $210^\circ = 180^\circ + 30^\circ$, the angle lies in Quadrant III with reference angle 30°. From Chapter 3, $\sin(30^\circ) = \frac{1}{2}$, and sine is negative in Quadrant III, so

$$\sin(210^\circ) = -\frac{1}{2}.$$

Practice now. Evaluate $\cos(135^\circ)$ using a reference angle.

Comparative Note: The Sexagenary Cycle and Simultaneous Periods.
A sharper illustration of periodicity than the calendar in general comes from the traditional Chinese sexagenary cycle, which names successive years, and historically days, by pairing one of ten symbols called heavenly stems with one of twelve symbols called earthly branches. Both sequences advance together, one position at a time, so that a given pairing recurs only once both cycles have returned to their starting point simultaneously. Since the stem cycle has length ten and the branch

cycle has length twelve, the combined cycle has length $\operatorname{lcm}(10, 12) = 60$, not the $10 \times 12 = 120$ pairings that might naively be expected, because the two cycles advance in lockstep rather than independently: only those pairings whose position differences are consistent with a single shared step actually occur, cutting the apparent count in half. This is ordinary modular arithmetic operating inside a human institution centuries before the vocabulary of cyclic groups existed to describe it. The unit circle exhibits the same structure in continuous form: two angles are coterminal exactly when their difference is a multiple of the circle's own period, 2π, and the interaction of two distinct periods, as will reappear when several oscillations of different frequencies are added together later in this book, likewise produces a combined period governed by a common multiple rather than a simple product of the two.

4.9 Exact Values on the Unit Circle

The special angles derived in Chapter 3 now acquire explicit geometric locations on the circle:

$$30^\circ \mapsto \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \qquad 45^\circ \mapsto \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \qquad 60^\circ \mapsto \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

Combined with the sign pattern of the reference-angle table, these three points generate the exact trigonometric value of every multiple of 30° and 45° around the entire circle, so that the unit circle functions as a complete map of the exact values a course in trigonometry is expected to know without a calculator.

4.10 Applications

The unit circle appears wherever a rotational phenomenon is studied: in the motion of wheels and gears, in the orbits of planets, in rotating machinery, and in the description of waves and oscillations. By fixing magnitude at 1 and allowing only direction to vary, the unit circle supplies a single, universal geometric language in which every one of these rotational systems can be described.

4.11 Looking Ahead

The unit circle allows the trigonometric functions to be defined for every real angle, and it has revealed periodicity, symmetry, and the exact geometric values underlying the special angles of Chapter 3. Everything done in this chapter, however, remains static: an angle is specified, a point is identified, a value is computed, and the process stops there. What happens when the angle changes continuously, as it does for a rotating wheel or an orbiting planet? What path does the corresponding point trace out over time, and how quickly does it move along that path? The next chapter begins the study of trigonometry as a theory of motion, taking up arc length, angular velocity, and the geometry of continuous rotation.

4.12 Exercises

Exercise 4.1. Find the reference angle for each of the following, and state the quadrant each lies in: 150°, -60°, 300°, $\frac{5\pi}{4}$.

Exercise 4.2. Evaluate $\tan(225^\circ)$ and $\sec(150^\circ)$ using reference angles.

Exercise 4.3. List three angles coterminal with 40°, including at least one negative angle, and explain why all four name the same point of the unit circle.

Exercise 4.4. Explain why $\sin\theta$ and $\cos\theta$ are defined for every real number θ, while $\tan\theta$ is not.

Chapter 5

Circular Motion and Continuous Change

5.1 When Angles Move

The previous chapter extended trigonometry beyond right triangles by defining sine and cosine on the unit circle: an angle determined a point on the circle, and the coordinates of that point supplied the values of the trigonometric functions. Everything studied there was nonetheless static — an angle was specified, a point was identified, a value was computed — and many of the phenomena that motivate trigonometry in the first place are not static at all. A wheel turns, a planet moves through its orbit, a gear rotates, and the hands of a clock sweep continuously through time; in each case the angle itself is changing, and it is more natural to treat it as a function of time, $\theta = \theta(t)$, than as a fixed quantity handed down in advance. The central question of this chapter is simple: what mathematics emerges once a point is allowed to move continuously around a circle? The answer will connect geometry, motion, and the trigonometric functions into a single framework.

5.2 Arc Length

Suppose a point travels along a circle of radius r, tracing a portion of the circumference called an *arc*. The distance traveled along the circle is the *arc length*, denoted s, and the relationship between s and the central angle θ depends on the unit in which θ is measured. When radians are used, that relationship turns out to be remarkably simple.

Recall from Chapter 4 that one radian is defined as the angle subtending an arc whose length equals the radius, so that $\theta = s/r$ is not a theorem but the very definition of radian measure. Solving for s gives $s = r\theta$ immediately, and the simplicity of this relationship is exactly why radians, rather than degrees, play the central structural role in the mathematics that follows: degrees are convenient for measurement, but radians are convenient for structure.

Theorem 5.1. *For a circle of radius r, the arc length subtended by a central angle θ, measured in radians, is $s = r\theta$.*

Proof. By the definition of radian measure, $\theta = s/r$. Multiplying both sides by r gives $s = r\theta$. □

5.3 Sector Area

A *sector* is the portion of a circle enclosed by two radii and the arc connecting them, and its area should evidently increase as either the radius or the angle increases. The formula is obtained by comparing a sector to the full circle: a complete revolution sweeps out 2π radians and encloses area πr^2, so a sector of angle θ should occupy the same fraction of the full area that θ occupies of 2π,

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}.$$

Theorem 5.2. *The area of a sector with radius r and central angle θ, measured in radians, is $A = \dfrac{1}{2}r^2\theta$.*

Proof. Multiplying both sides of $A/(\pi r^2) = \theta/(2\pi)$ by πr^2 gives $A = (\theta/2\pi)\pi r^2$, which simplifies to $A = \frac{1}{2}r^2\theta$. □

Example 5.3. A sprinkler rotates through an angle of $2\pi/3$ radians and sprays water to a distance of 6 meters. Find the arc length swept by the edge of the spray and the area watered. Using $s = r\theta$,

$$s = 6\left(\frac{2\pi}{3}\right) = 4\pi \text{ meters,}$$

and using $A = \frac{1}{2}r^2\theta$,

$$A = \frac{1}{2}(6^2)\left(\frac{2\pi}{3}\right) = 12\pi \text{ square meters.}$$

5.4 Angular Velocity

Introducing time allows the rate at which an angle changes to be measured directly.

Definition 5.4. The *angular velocity* of a rotating object, denoted ω, is the rate at which its angle changes. For uniform rotation, $\omega = \theta/t$.

Angular velocity is commonly expressed in radians per second, radians per minute, or revolutions per minute, and it measures how rapidly an angle changes regardless of the size of the object doing the rotating: a small gear and a large gear turning at the same angular velocity complete the same fraction of a revolution in the same time, even though a point on the larger gear travels much farther.

5.5 Linear Velocity

That last observation motivates a second, related quantity. A point on the rim of a bicycle wheel travels farther than a point near the hub even though both complete the same rotation in the same time, so angular velocity alone does not capture the actual distance traveled. Starting from $s = r\theta$ and dividing both sides by time t gives

$$\frac{s}{t} = r \cdot \frac{\theta}{t},$$

and recognizing s/t as the *linear velocity v* and θ/t as the angular velocity ω yields the relationship between them.

Theorem 5.5. *For uniform circular motion, $v = r\omega$. A larger radius produces a larger linear velocity even when the angular velocity remains unchanged.*

Procedure 5.1 (How to Relate Angular and Linear Velocity).

Given a point rotating about a center, identify the radius r from the center to the point and the angular velocity ω of the rotation. Apply $v = r\omega$ to obtain the linear velocity, and check that the units of r and ω are consistent (typically meters and radians per second) before reporting the result.

Example 5.6. A carousel rotates at 0.5 radians per second, and a rider sits 4 meters from the center. Using $v = r\omega$,

$$v = (4)(0.5) = 2 \text{ m/s},$$

so the rider travels two meters along the circular path every second.

Practice now. A wheel rotates at 3 radians per second and has radius 0.4 meters. Find the linear velocity of a point on the rim.

5.6 Uniform Circular Motion

This section arrives at the central idea of the chapter. Consider a point moving around the unit circle with constant angular velocity ω, beginning at angle 0. After time t the angle has become $\theta = \omega t$, and from Chapter 4 the point on the unit circle at angle θ has coordinates $(\cos\theta, \sin\theta)$. Substituting $\theta = \omega t$ gives the position at time t as $(\cos\omega t, \sin\omega t)$: the horizontal coordinate of the moving point changes according to a cosine function of time, and the vertical coordinate changes according to a sine function of time. Sine and cosine are, in other words, the projections of uniform circular motion onto two perpendicular axes, and this observation is among the most important in the entire subject, since it is what will eventually let an oscillation — a swinging pendulum, a vibrating string, a sound wave — be understood as the shadow of some circle turning at constant speed.

5.7 Coordinates of Uniform Circular Motion

The same idea extends immediately to motion beginning at an arbitrary starting angle θ_0 rather than at 0. After time t, the angle has become $\theta = \omega t + \theta_0$.

Theorem 5.7. *A point moving uniformly around the unit circle with angular velocity ω and initial angle θ_0 has coordinates*

$$x(t) = \cos(\omega t + \theta_0), \qquad y(t) = \sin(\omega t + \theta_0).$$

Proof. From Chapter 4, a point at angle θ has coordinates $(\cos\theta, \sin\theta)$. Uniform rotation with initial angle θ_0 gives $\theta = \omega t + \theta_0$, and substituting this into the coordinate formulas gives $x(t) = \cos(\omega t + \theta_0)$ and $y(t) = \sin(\omega t + \theta_0)$. □

5.8 Applications

Circular motion appears throughout science and engineering, from rotating machinery, gears, and flywheels to planetary orbits, electrical generators, sound waves, and signal processing. The formulas of this chapter reveal that every periodic oscillation can, in principle, be understood as the shadow of some circular motion projected onto an axis, an idea that will eventually lead to Fourier analysis in Chapter 19, where complicated signals are decomposed into collections of such circular motions added together. For now, the chapter offers a new interpretation of sine and cosine themselves: no longer merely ratios read off a triangle, but descriptions of motion unfolding in time.

5.9 Looking Ahead

This chapter introduced motion into trigonometry, developing formulas for arc length and sector area, relating angular velocity to linear velocity, and discovering that sine and cosine arise naturally as the coordinates of uniform circular motion. The functions themselves, however, remain in a sense still hidden: their role in describing motion around a circle has been established, but their shape, plotted on ordinary coordinate axes, has not yet been examined directly. What happens when $y = \sin\theta$ or $y = \cos\theta$ is plotted as a curve, with θ running along the horizontal axis instead of wrapping around a circle? The next chapter takes up the graphs of the trigonometric functions, and shows how the periodic motion studied here appears once it is unrolled and viewed through time.

5.10 Exercises

Exercise 5.1. A circle has radius 9 centimeters. Find the arc length and sector area corresponding to a central angle of $5\pi/6$ radians.

Exercise 5.2. A Ferris wheel has radius 15 meters and completes one revolution every 40 seconds. Find its angular velocity in radians per second, and the linear velocity of a rider at the rim.

Exercise 5.3. A point moves uniformly around the unit circle with angular velocity $\omega = \pi/3$ radians per second, starting at $\theta_0 = \pi/4$. Write its position $(x(t), y(t))$ as a function of t, and find its position at $t = 2$ seconds.

Exercise 5.4. Two gears of different radii are connected so that their rims move at the same linear velocity. Explain, using $v = r\omega$, why the gear with the smaller radius must have the larger angular velocity.

Chapter 6

Graphs of the Trigonometric Functions

6.1 From Motion to Graphs

The previous chapter showed that sine and cosine arise naturally from circular motion: a point moving around the unit circle with angular velocity ω has coordinates $x(t) = \cos(\omega t)$ and $y(t) = \sin(\omega t)$, equations that describe a position on a circle rather than a curve on a page. A natural question follows immediately. What happens if the circle itself is set aside and only one coordinate — say, the vertical one — is tracked through time? As the point rises and falls around the circle, its height changes continuously, and plotting angle along the horizontal axis against height along the vertical axis produces the familiar wave-shaped curve known as the graph of the sine function. That curve is therefore not an arbitrary construction handed down by convention; it is the geometric record of circular motion, unrolled onto a flat page.

6.2 Building the Sine Curve

Consider a point moving around the unit circle, and recall the special angles of Chapter 4. At $\theta = 0$ the point is $(1, 0)$, so $\sin(0) = 0$; at $\theta = \pi/2$ the point is $(0, 1)$, so $\sin(\pi/2) = 1$; at $\theta = \pi$ the point is $(-1, 0)$, so $\sin(\pi) = 0$; at $\theta = 3\pi/2$ the point is $(0, -1)$, so $\sin(3\pi/2) = -1$; and at $\theta = 2\pi$ the point returns to $(1, 0)$, so $\sin(2\pi) = 0$. Plotting these five points and connecting them smoothly produces the sine curve, a graph that records nothing more than the changing vertical position of the point rotating around the circle in the previous chapter.

6.3 Building the Cosine Curve

The cosine function records the horizontal coordinate of the same rotating point. At the same five special angles, $\cos(0) = 1$, $\cos(\pi/2) = 0$, $\cos(\pi) = -1$, $\cos(3\pi/2) = 0$, and $\cos(2\pi) = 1$. The resulting graph has the same wave shape as the sine curve but begins at its maximum value rather than at zero, and like the sine graph it is a direct consequence of the same circular motion, differing only in which coordinate is being tracked.

6.4 Periodic Functions

Both graphs reveal a striking property: after one complete revolution, the same pattern repeats exactly, a fact that is nothing more than the coterminal-angle theorem of Chapter 4 seen from a new angle.

Definition 6.1. A function f is *periodic* if there exists a positive number P such that $f(x+P) = f(x)$ for every x in the domain of f. The smallest such positive number is called the *period* of f.

Definition 6.2. The *amplitude* of a periodic function is the maximum distance from its midline to a peak or a trough. Amplitude measures the size of an oscillation, while period measures the length of one complete cycle.

6.5 Period and Amplitude of Sine and Cosine

The unit circle immediately supplies both quantities for sine and cosine.

Theorem 6.3. *The functions $y = \sin\theta$ and $y = \cos\theta$ have period 2π and amplitude 1.*

Proof. By the coterminal-angle theorem of Chapter 4, θ and $\theta + 2\pi$ determine the same point on the unit circle, so $\sin(\theta + 2\pi) = \sin\theta$ and $\cos(\theta + 2\pi) = \cos\theta$; the period is therefore 2π. Every point (x, y) on the unit circle satisfies $-1 \le x \le 1$ and $-1 \le y \le 1$, so the largest value either coordinate can attain is 1 and the smallest is -1. The distance from the midline $y = 0$ to either extreme is therefore 1, which is the amplitude. □

6.6 Symmetry

The graphs possess symmetries that follow directly from the geometry of the unit circle rather than from inspection of the curve alone.

Theorem 6.4. *The sine function is odd:* $\sin(-\theta) = -\sin\theta$.

Proof. Reflecting an angle across the x-axis reverses the sign of its vertical coordinate while leaving the horizontal coordinate unchanged. Since sine measures the vertical coordinate of a point on the unit circle, $\sin(-\theta) = -\sin\theta$. □

Theorem 6.5. *The cosine function is even:* $\cos(-\theta) = \cos\theta$.

Proof. Reflecting an angle across the x-axis leaves the horizontal coordinate unchanged. Since cosine measures the horizontal coordinate of a point on the unit circle, $\cos(-\theta) = \cos\theta$. □

These two properties are traditionally called odd symmetry and even symmetry, and each is a direct geometric consequence of what sine and cosine measure on the circle, rather than an independent fact requiring separate proof.

6.7 The Tangent Function

Recall from Chapter 4 that $\tan\theta = \sin\theta / \cos\theta$, undefined whenever $\cos\theta = 0$. This occurs at $\pi/2, 3\pi/2, 5\pi/2$, and every angle coterminal with them, so the graph of tangent contains a vertical asymptote at each of these values; unlike sine and cosine, the tangent graph is unbounded, stretching from $-\infty$ to ∞ between each pair of consecutive asymptotes.

Theorem 6.6. *The tangent function has period π.*

Proof. Adding π to an angle reverses the sign of both sine and cosine: $\sin(\theta + \pi) = -\sin\theta$ and $\cos(\theta + \pi) = -\cos\theta$. Therefore

$$\tan(\theta + \pi) = \frac{-\sin\theta}{-\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta,$$

so the graph of tangent repeats every π units, half the period of sine and cosine, precisely because the two sign changes that occur after adding only π cancel inside the ratio. □

6.8 Transformations of Trigonometric Graphs

Most applications involve a modified trigonometric function rather than the basic sine or cosine curve, typically written

$$y = A\sin(B\theta - C) + D.$$

Each parameter controls a distinct geometric feature of the graph. The factor A scales the graph vertically, so that the amplitude becomes $|A|$. The factor B compresses or stretches the graph horizontally, changing the period from 2π to $2\pi/|B|$, since a full cycle of the inner angle $B\theta - C$ now requires θ to advance through only $2\pi/|B|$ units rather than a full 2π. The quantity C/B shifts the graph horizontally and is called the *phase shift*, since setting $B\theta - C = 0$ shows that the graph's usual starting point has been delayed until $\theta = C/B$. Finally, the constant D moves the entire graph vertically, so that its new midline is $y = D$ rather than $y = 0$.

Procedure 6.1 (How to Graph a Trigonometric Function from Its Parameters). Given $y = A\sin(B\theta - C) + D$, first determine the amplitude $|A|$ and the period $2\pi/|B|$. Next determine the phase shift C/B, taking care with its sign, and the vertical shift D, which fixes the new midline. Use these four values to sketch one complete cycle of the curve using its key points, then extend the sketch periodically in both directions.

Example 6.7. Graph $y = 3\sin\left(2\theta - \frac{\pi}{2}\right) + 1$. The amplitude is 3, the period is $2\pi/2 = \pi$, the phase shift is $(\pi/2)/2 = \pi/4$ to the right, and the vertical shift is 1, giving a midline of $y = 1$. The graph therefore oscillates between 4 and -2, completing one full cycle every π units, with that cycle beginning $\pi/4$ units later than an unshifted sine curve would.

Practice now. Determine the amplitude, period, phase shift, and vertical shift of $y = 2\cos(3\theta + \pi) - 4$.

6.9 Applications

The graphs developed in this chapter model an enormous range of periodic phenomena, from seasonal temperature variation and ocean tides to sound waves, alternating electrical current, and biological rhythms. The same curves that emerge from a single point rotating around the unit circle in Chapter 5 turn out to describe some of the most important oscillatory processes in the natural and engineered world, which is exactly the payoff promised at the end of that chapter.

6.10 Looking Ahead

The graphs of sine and cosine reveal an important complication: a horizontal line typically intersects either graph at infinitely many points, so that an equation such as $\sin\theta = 1/2$ has infinitely many solutions rather than one. Sine and cosine, as they stand, therefore cannot be reversed uniquely across their entire domains, and if a trigonometric function is to be ”undone” at all, its domain must first be restricted to a piece on which it is one-to-one. The next chapter takes up this problem directly, developing the inverse trigonometric functions and showing how an angle can be recovered from a known ratio once the appropriate restriction has been made.

6.11 Exercises

Exercise 6.1. State the amplitude, period, phase shift, and vertical shift of $y = 4\sin\left(\frac{\theta}{2} - \pi\right) + 2$, and sketch one cycle.

Exercise 6.2. Explain, using the proof of the tangent period theorem, why the period of $\tan\theta$ is π while the period of $\sin\theta$ and $\cos\theta$ is 2π.

Exercise 6.3. Find all solutions of $\cos\theta = -\frac{\sqrt{2}}{2}$ in the interval $[0, 4\pi)$, and use the graph of cosine to explain why there is more than one solution.

Exercise 6.4. Write an equation of the form $y = A\sin(B\theta - C) + D$ for a curve with amplitude 5, period 4π, no phase shift, and midline $y = -3$.

Chapter 7

Inverse Trigonometric Functions

7.1 Why Ordinary Inverses Fail

The previous chapter closed on a difficulty worth stating plainly: the equation $\sin\theta = \frac{1}{2}$ does not have a single solution, since $\theta = \pi/6, 5\pi/6, 13\pi/6, 17\pi/6, \ldots$ all satisfy it. If several different inputs produce the same output, no inverse function can determine which input was originally intended, and sine, as a result, cannot be reversed uniquely across its full domain. The remedy is simple in principle: rather than working with the entire graph of sine, a portion of it is chosen on which every output occurs exactly once, and the inverse is defined only on that restricted portion.

7.2 One-to-One Functions and Invertibility

Definition 7.1. A function f is *one-to-one* if different inputs always produce different outputs; equivalently, $f(a) = f(b)$ implies $a = b$. A function has an inverse if and only if it is one-to-one on its domain.

Sine fails to be one-to-one on all of $\mathbb{R}$, since for instance $\sin(\pi/6) = \sin(5\pi/6) = \frac{1}{2}$, and consequently sine has no inverse on its full domain. If the domain is restricted to $[-\pi/2, \pi/2]$, however, every output value between -1 and 1 occurs exactly once, and on that interval sine becomes invertible.

7.3 Restricting the Domain of Sine

Theorem 7.2. *The sine function is one-to-one on* $[-\pi/2, \pi/2]$.

Proof. As θ moves from $-\pi/2$ to $\pi/2$, the corresponding point on the unit circle moves continuously from $(0, -1)$ to $(0, 1)$, and its vertical coordinate increases strictly throughout this motion. Since sine measures the vertical coordinate, distinct angles in this interval produce distinct sine values, so sine is one-to-one there. □

7.4 The Arcsine Function

Because sine is one-to-one on this restricted domain, it has an inverse defined there.

Definition 7.3. The *arcsine* function is defined by $\arcsin x = \theta$ if and only if $\sin\theta = x$ and $-\pi/2 \leq \theta \leq \pi/2$. Its domain is $[-1, 1]$ and its range is $[-\pi/2, \pi/2]$.

Thus $\arcsin(0) = 0$, $\arcsin(1) = \pi/2$, and $\arcsin(1/2) = \pi/6$. The last of these deserves emphasis: although many angles have sine equal to $1/2$, arcsine always returns the unique one of them lying in its restricted range.

7.5 Arccosine and Arctangent

The same idea extends to cosine and tangent, each restricted to whichever interval makes it one-to-one.

Definition 7.4. Cosine is one-to-one on $[0, \pi]$, and $\arccos x = \theta$ means $\cos\theta = x$ with $0 \le \theta \le \pi$.

Hence $\arccos(1) = 0$, $\arccos(0) = \pi/2$, and $\arccos(-1) = \pi$.

Definition 7.5. Tangent is one-to-one on $(-\pi/2, \pi/2)$, and $\arctan x = \theta$ means $\tan\theta = x$ with $-\pi/2 < \theta < \pi/2$.

Hence $\arctan(0) = 0$, $\arctan(1) = \pi/4$, and $\arctan(-1) = -\pi/4$. None of these three intervals was chosen arbitrarily: each is a maximal interval on which its function is monotonic and therefore one-to-one. Restricting the domain further than necessary would discard values that could otherwise be recovered, while enlarging it even slightly would destroy invertibility by reintroducing repeated outputs.

7.6 Composition Identities

Inverse functions obey a pair of composition rules, and the two directions of composition behave quite differently from one another.

Theorem 7.6. *For every $x \in [-1, 1]$, $\sin(\arcsin x) = x$.*

Proof. By definition, $\arcsin x$ is the angle whose sine equals x; applying sine to it simply recovers that value, x. □

The analogous statements $\cos(\arccos x) = x$ and $\tan(\arctan x) = x$ hold for the same reason. The reverse composition, however, is far more delicate.

Theorem 7.7. $\arcsin(\sin\theta) = \theta$ *holds only when* $-\pi/2 \le \theta \le \pi/2$*.*

Proof. By definition, arcsin must return a value lying in its restricted range $[-\pi/2, \pi/2]$. If θ already lies in that range, the returned value is θ itself. If θ lies outside that range, $\arcsin(\sin\theta)$ instead returns whichever angle inside $[-\pi/2, \pi/2]$ shares the same sine value as θ, which is a different number from θ whenever θ itself lies outside that interval. □

This asymmetry between the two composition directions is one of the most common sources of error in trigonometry, precisely because the forward composition $\sin(\arcsin x) = x$ always holds while the reverse composition does not, and it is worth working through carefully before relying on either.

Example 7.8. Evaluate $\arcsin\left(\sin\dfrac{3\pi}{4}\right)$. First, $\sin(3\pi/4) = \sqrt{2}/2$. The angle in $[-\pi/2, \pi/2]$ whose sine equals $\sqrt{2}/2$ is $\pi/4$, so

$$\arcsin\left(\sin\frac{3\pi}{4}\right) = \frac{\pi}{4},$$

not $3\pi/4$: the inverse function is required to return a value inside its restricted range, regardless of what angle originally produced the sine value inside the parentheses.

Procedure 7.1 (How to Evaluate an Inverse Trigonometric Expression).
Identify which inverse function is involved and recall its restricted output range. If the expression contains an inner trigonometric function applied to an angle, compute that inner value first, then find the unique angle in the restricted range that produces it. Before concluding, check explicitly whether a composition identity is being applied, and never assume $\arcsin(\sin\theta) = \theta$, or the analogous statement for cosine or tangent, without first verifying that θ already lies in the permitted range.

Practice now. Evaluate $\arccos\left(\cos\dfrac{5\pi}{4}\right)$.

7.7 Finding Angles from Ratios

Inverse trigonometric functions allow an unknown angle to be solved for directly rather than estimated. If $\tan\theta = 7/4$, applying arctan to both sides gives $\theta = \arctan(7/4)$, which a calculator evaluates to approximately 1.052 radians, or 60.3°. This formalizes, with a precise definition behind it, the process first used informally in Chapter 3 to recover an unknown angle from a known ratio of sides.

7.8 Right-Triangle Interpretation

An inverse trigonometric expression can also be read geometrically. Suppose $\theta = \arcsin(3/5)$, so that $\sin\theta = 3/5$. Interpreting this ratio as opposite/hypotenuse $= 3/5$ suggests a right triangle with sides 3, 4, 5, from which the remaining ratios follow at once: $\cos\theta = 4/5$ and $\tan\theta = 3/4$. This construction turns a single inverse trigonometric expression into a complete right triangle, from which every other trigonometric quantity associated with θ can be read off directly, without ever computing θ itself numerically.

7.9 Applications

Inverse trigonometric functions arise whenever an angle must be recovered from geometric information already in hand: determining a ship's bearing from coordinate data, computing an angle of elevation, analyzing forces in engineering, recovering a direction from vector components, and solving problems in navigation and surveying all rely on them. Together with the trigonometric functions themselves, they complete a cycle running from angle to ratio and back to angle again, closing the loop that Chapter 3 opened.

7.10 Looking Ahead

Inverse trigonometric functions solve equations of the simple forms $\sin\theta = a$, $\cos\theta = a$, and $\tan\theta = a$. Many equations that arise in practice are considerably more complicated, involving products, powers, multiple angles, or several trigonometric functions at once, and such equations cannot be solved systematically until a richer collection of identities is available. The next several chapters develop the geometry of rotation and the algebra of trigonometric identities that this richer toolkit requires, after which Chapter 11 returns to the problem of solving general trigonometric equations with the full apparatus in place.

7.11 Exercises

Exercise 7.1. Evaluate $\arcsin\left(\frac{\sqrt{3}}{2}\right)$.

Exercise 7.2. Evaluate $\arccos\left(-\frac{1}{2}\right)$.

Exercise 7.3. Evaluate $\arctan(1)$.

Exercise 7.4. Evaluate $\arcsin\left(\sin\frac{5\pi}{6}\right)$.

Exercise 7.5. Evaluate $\arccos\left(\cos\frac{7\pi}{6}\right)$.

Exercise 7.6. Find $\theta = \arctan\left(\frac{5}{3}\right)$ to the nearest tenth of a degree.

Exercise 7.7. If $\theta = \arcsin\left(\frac{8}{17}\right)$, find $\cos\theta$ and $\tan\theta$.

Exercise 7.8. Explain why $\sin(\arcsin x) = x$ holds for every $x \in [-1, 1]$, but $\arcsin(\sin\theta) = \theta$ does not hold for every θ.

Exercise 7.9. Construct a right triangle corresponding to $\theta = \arccos\left(\frac{12}{13}\right)$, and use it to determine $\sin\theta$ and $\tan\theta$.

Exercise 7.10. A surveyor measures the ratio height/horizontal distance $= 0.65$. Find the angle of elevation.

Part III

Rotation and Transformation

Chapter 8

Rotation Geometry

8.1 Combining Rotations

The previous chapter recovered angles from trigonometric ratios. A different kind of question now presents itself, concerning not a single angle but several angles acting in sequence. Suppose a point is rotated by an angle A, and the result is then rotated again by an angle B. What is the overall effect of the two rotations together? Does the outcome depend on the order in which they are performed, and can two successive rotations always be replaced by a single rotation? Questions of this kind move the discussion from individual angles to transformations acting on the plane, and, more importantly, they reveal the geometric origin of many identities that are too often presented as isolated formulas to be memorized. The goal of this chapter is to derive those identities from the geometry of rotation itself.

8.2 Rotations as Transformations

Recall from Chapter 4 that every point on the unit circle can be written as $(\cos\theta, \sin\theta)$, obtained by rotating the point $(1, 0)$ through angle θ. Consider now an arbitrary point (x, y), written in polar form as $x = r\cos\phi$ and $y = r\sin\phi$. Rotating this point by an angle θ changes only its angle, not its distance r from the origin, so the rotated point has coordinates $(r\cos(\phi+\theta), r\sin(\phi+\theta))$. The task of this section is to express these new coordinates in terms of x, y, and θ alone, without reference to ϕ or r.

Theorem 8.1. *Rotating a point (x, y) counterclockwise through an angle θ produces the point*

$$(x', y') = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta).$$

Proof. A rotation of the plane is a linear transformation fixing the origin: it carries sums of displacements to sums of the rotated displacements, and it preserves distance from the origin. Such a transformation is completely determined by where it sends the two points $(1, 0)$ and $(0, 1)$, since every point (x, y) is the combination $x(1, 0) + y(0, 1)$. By the definition of sine and cosine in Chapter 4, rotating $(1, 0)$, which lies at angle 0, by θ sends it to the point at angle θ, namely $(\cos\theta, \sin\theta)$. The point $(0, 1)$ lies a quarter turn ahead of $(1, 0)$, at angle $\pi/2$; a direct check across the four quadrants using the special angles of Chapter 4 shows that a quarter turn always sends a point (a, b) on the unit circle to $(-b, a)$, so rotating $(0, 1)$ by θ sends it to $(-\sin\theta, \cos\theta)$. Combining the two images by linearity,

$$(x, y) = x(1, 0) + y(0, 1) \longmapsto x(\cos\theta, \sin\theta) + y(-\sin\theta, \cos\theta) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta).$$

$\square$

8.3 Composition of Rotations

Rotations obey a strikingly simple composition law.

Theorem 8.2. *A rotation by angle A followed by a rotation by angle B is equivalent to a single rotation by angle $A + B$.*

Proof. Let a point initially have angular coordinate ϕ. After the first rotation its angular coordinate becomes $\phi + A$, and after the second it becomes $(\phi + A) + B$. By associativity of addition, $(\phi + A) + B = \phi + (A + B)$, so the final point is exactly the result of rotating once by angle $A + B$. □

This simple fact is the source of the most important identities in trigonometry.

8.4 Deriving the Sum Formulas

Consider the point $(\cos A, \sin A)$, obtained geometrically by rotating $(1, 0)$ through angle A. Rotating this point by an additional angle B and applying the rotation formula of Section 8.2 gives

$$x' = \cos A \cos B - \sin A \sin B, \qquad y' = \sin A \cos B + \cos A \sin B.$$

By the composition theorem just proved, however, rotating first by A and then by B is equivalent to rotating $(1, 0)$ once by $A + B$, so the resulting point must also equal $(\cos(A + B), \sin(A + B))$. Since both descriptions refer to the same point, corresponding coordinates must agree:

$$\boxed{\cos(A + B) = \cos A \cos B - \sin A \sin B}, \qquad \boxed{\sin(A + B) = \sin A \cos B + \cos A \sin B}.$$

These formulas are not arbitrary identities to be verified after the fact; they are the coordinate expression of rotation composition, true because two rotations performed in sequence must land on exactly the point that a single combined rotation would reach.

8.5 Difference Formulas

The difference formulas follow immediately from the sum formulas, without any further geometry. Replacing B with $-B$ in the sum formulas and using the symmetries proved in Chapter 6, $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$, gives

$$\boxed{\cos(A - B) = \cos A \cos B + \sin A \sin B}, \qquad \boxed{\sin(A - B) = \sin A \cos B - \cos A \sin B}.$$

Nothing new has been assumed here; the difference formulas are simply the sum formulas viewed through the even symmetry of cosine and the odd symmetry of sine established earlier in the book.

Example 8.3. Compute $\sin(75°)$ exactly. Writing $75° = 45° + 30°$ and applying the sum formula for sine,

$$\sin(75°) = \sin 45° \cos 30° + \cos 45° \sin 30° = \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{6} + \sqrt{2}}{4}.$$

Procedure 8.1 (How to Derive an Identity from Rotation Composition).

Identify a point obtained as the image of a known point under one rotation, then apply a second rotation to it using the rotation formula of Section 8.2 to obtain its coordinates algebraically. Separately, express the same final point using angle addition and the composition theorem. Since both expressions describe the identical point, equate corresponding coordinates and simplify the resulting equations; the result is the desired identity. Many of the identities appearing later in this book arise from precisely this strategy.

Practice now. Compute $\cos(15°)$ exactly, using $15° = 45° - 30°$.

Comparative Note: Arabic Root-and-Pattern Morphology.

Arabic and the other Semitic languages build words through a mechanism unusually close to the operator-and-object structure of this chapter. A word is generated from a consonantal root, typically three consonants carrying a core meaning, together with a separately specified pattern, a template of vowels and affixes into which the root's consonants are inserted. The root *f-'-l*, associated broadly with the idea of doing or acting, yields distinct words such as *fa'ala* (he did), *fa''ala* (he caused to do, intensively), *fā'ala* (he did reciprocally, with another), and *af'ala* (he caused to do), among others, depending only on which pattern is applied to the same fixed root. The root supplies the invariant content; the pattern supplies the transformation acting on it, and in each case the output can be described as Pattern(Root) = Word, a fixed object transformed by a chosen operator into a new but recognizably related object. This is precisely the structure a rotation exhibits in this chapter, where a fixed point is transformed by an operator $R(\theta)$ into a new point that is different in position yet identical in its distance from the origin: an invariant core, and a family of operators acting on it to produce variation without destroying what makes each output recognizable as a transformation of the same underlying thing.

8.6 Rotation Matrices

The rotation transformation of Section 8.2 can be written compactly as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

This array of numbers is called a matrix, and although a full study of matrices lies beyond the scope of this book, one fact is worth noting here: composing two rotations, as in Section 8.4, corresponds exactly to multiplying their two matrices together. The matrix form therefore encodes the same angle-addition structure derived earlier in this chapter, in a notation that will reappear in later mathematical study, particularly in linear algebra, physics, and computer graphics.

8.7 Applications

Rotations occur throughout science and technology: rotating objects in computer graphics, steering the joints of a robotic arm, controlling the orientation of a spacecraft, describing crystallographic symmetries, and analyzing mechanical linkages and gears all depend on the transformation studied in this chapter. The identities derived above are not merely algebraic conveniences to be looked up when needed; they are mathematical descriptions of how rotational transformations combine, and every one of these applications relies on that combination rule holding exactly.

8.8 Looking Ahead

The sum and difference formulas mark a turning point in the book. A large share of the identities encountered in the chapters that follow are direct consequences of them: the double-angle formulas of Chapter 10 arise simply by setting $A = B$ in the sum formulas derived here, the half-angle formulas emerge by solving those same equations in reverse, and the product-to-sum formulas follow from further algebraic manipulation of the same starting point. A large portion of trigonometric algebra therefore grows from a single geometric fact established in this chapter: rotations compose by adding angles. The next chapters develop this identity toolkit further, showing how increasingly complex trigonometric expressions can be simplified, transformed, and ultimately solved.

8.9 Exercises

Exercise 8.1. Use the sum formula to compute $\sin(105^\circ)$ exactly.

Exercise 8.2. Use the difference formula to compute $\cos(75^\circ)$ exactly.

Exercise 8.3. Use the sum formula to compute $\sin(15^\circ)$ exactly.

Exercise 8.4. Use the difference formula to compute $\cos(105^\circ)$ exactly.

Exercise 8.5. Starting from the sum formula for $\sin(A + B)$, derive the difference formula for sine.

Exercise 8.6. Starting from the sum formula for $\cos(A + B)$, derive the difference formula for cosine.

Exercise 8.7. Verify directly that two successive rotations of 40° and 50° produce the same result as a single rotation of 90°.

Exercise 8.8. Explain why a rotation cannot change the distance of a point from the origin.

Exercise 8.9. Write the rotation matrix corresponding to a rotation of 90°, and apply it to the point $(1, 0)$.

Exercise 8.10. Why must composing a rotation by A with a rotation by B produce the same result as a single rotation by $A + B$? What would break in the geometry of the circle if this were not true?

Chapter 9

Fundamental Identities

9.1 A Small Toolkit from the Unit Circle

The previous chapter discovered that the sum and difference formulas are not arbitrary rules to be memorized, but consequences of a single geometric fact: rotations compose by adding angles. This raises a broader question worth pursuing before the book moves any further, namely which other relationships among the trigonometric functions are similarly forced by geometry rather than imposed by convention. The answer is that a surprising share of the identities used throughout trigonometry follow from only a few underlying ideas already established in this book: the equation of the unit circle, the reciprocal relationships among the six trigonometric functions, the quotient definition of tangent, the symmetry properties of sine and cosine proved in Chapter 6, and the angle-addition formulas derived in Chapter 8. The purpose of this chapter is to gather these relationships into a compact toolkit that will support everything that follows.

9.2 The Pythagorean Identity

The most important identity in the toolkit comes directly from the definition of the unit circle itself. Recall from Chapter 4 that every point on the unit circle satisfies $x^2 + y^2 = 1$, while $x = \cos\theta$ and $y = \sin\theta$.

Theorem 9.1 (Pythagorean Identity). $\sin^2\theta + \cos^2\theta = 1$ *for every angle* θ*.*

Proof. Substituting $x = \cos\theta$ and $y = \sin\theta$ into $x^2 + y^2 = 1$ gives $\cos^2\theta + \sin^2\theta = 1$, which is the same statement with its two terms reordered. □

It is worth pausing to notice what has, and has not, just been established. This is not really a new fact about trigonometry; it is simply the equation of the unit circle, restated in trigonometric notation. No new geometry has been introduced here, only a familiar piece of geometry seen through new symbols.

9.3 Derived Pythagorean Identities

The remaining two Pythagorean identities follow from the first by ordinary algebra rather than by any further appeal to geometry.

Theorem 9.2. $1 + \tan^2\theta = \sec^2\theta$ *whenever* $\cos\theta \neq 0$*.*

Proof. Divide every term of $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$:

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}.$$

Recognizing $\tan\theta = \sin\theta/\cos\theta$ and $\sec\theta = 1/\cos\theta$ gives $\tan^2\theta + 1 = \sec^2\theta$. □

Theorem 9.3. $1 + \cot^2\theta = \csc^2\theta$ *whenever* $\sin\theta \neq 0$.

Proof. Divide every term of $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$:

$$1 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}.$$

Recognizing $\cot\theta = \cos\theta/\sin\theta$ and $\csc\theta = 1/\sin\theta$ gives $1 + \cot^2\theta = \csc^2\theta$. □

Neither of these two identities is an independent fact requiring separate memorization; each is the original Pythagorean identity, divided through by a different square.

9.4 Reciprocal and Quotient Identities

Several further identities restate definitions already introduced earlier in the book, gathered here for reference alongside their more substantial neighbors. The reciprocal identities record that

$$\csc\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta},$$

and the quotient identities record that

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

Each of these five equations is simply a definition from earlier chapters written in the form of an identity, and none requires its own proof.

9.5 Cofunction Identities

The difference formula of Chapter 8 immediately produces another useful family of identities.

Theorem 9.4. $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$.

Proof. Apply the difference formula $\sin(A - B) = \sin A\cos B - \cos A\sin B$ with $A = \pi/2$ and $B = \theta$:

$$\sin\left(\frac{\pi}{2} - \theta\right) = \sin\frac{\pi}{2}\cos\theta - \cos\frac{\pi}{2}\sin\theta.$$

Since $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$, this reduces to $\sin(\pi/2 - \theta) = \cos\theta$. □

The same substitution into the difference formula for cosine gives $\cos(\pi/2 - \theta) = \sin\theta$, and dividing the two results gives $\tan(\pi/2 - \theta) = \cot\theta$, with corresponding identities holding for secant and cosecant. These are called *cofunction identities* because each function is transformed into its complementary partner, and unlike the informal complementary-angle observations sometimes made about right triangles, each one here follows directly from the rotation-composition machinery of Chapter 8.

9.6 Even and Odd Identities

Chapter 6 established the symmetry properties of sine, cosine, and, by extension, tangent; those results are now incorporated into the toolkit of this chapter rather than re-derived. Cosine is even, $\cos(-\theta) = \cos\theta$, while sine and tangent are odd, $\sin(-\theta) = -\sin\theta$ and $\tan(-\theta) = -\tan\theta$. These identities are frequently useful when simplifying expressions that involve a negative angle, since they allow the negative sign to be moved outside the function entirely.

9.7 How to Prove a Trigonometric Identity

Unlike an ordinary equation, an identity is established not by solving for an unknown but by a chain of logically valid rewritings connecting one side to the other, and a systematic approach makes this process considerably easier to carry out reliably.

Procedure 9.1 (How to Prove a Trigonometric Identity).
Begin with whichever side of the proposed identity looks more complicated, and rewrite any tangent, cotangent, secant, or cosecant appearing in it in terms of sine and cosine whenever doing so seems useful. Apply the Pythagorean identities to replace expressions such as $1-\cos^2\theta$ with $\sin^2\theta$, or the reverse, whenever such a substitution simplifies the expression. Continue simplifying algebraically, working on one side of the proposed identity at a time rather than manipulating both sides simultaneously, and never cross-multiply across an equation that has not yet been established as true. Continue this process until the expression you started from matches the other side exactly.

Example 9.5. Prove that $\dfrac{1-\cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1+\cos\theta}$. Beginning with the left-hand side and multiplying numerator and denominator by $1+\cos\theta$,

$$\frac{1-\cos\theta}{\sin\theta} = \frac{(1-\cos\theta)(1+\cos\theta)}{\sin\theta(1+\cos\theta)} = \frac{1-\cos^2\theta}{\sin\theta(1+\cos\theta)}.$$

By the Pythagorean identity, $1-\cos^2\theta = \sin^2\theta$, so

$$\frac{1-\cos\theta}{\sin\theta} = \frac{\sin^2\theta}{\sin\theta(1+\cos\theta)} = \frac{\sin\theta}{1+\cos\theta},$$

which matches the right-hand side.

Practice now. Prove that $\dfrac{\cos\theta}{1-\sin\theta} = \dfrac{1+\sin\theta}{\cos\theta}$, by multiplying numerator and denominator of the left-hand side by $1+\sin\theta$.

Comparative Note: Chinese Syllable Structure and Admissible Space.
Standard Chinese phonology offers a sharper illustration of a constrained state space than a general appeal to phonotactics. A syllable can be described, to a first approximation, as a tuple of five independent slots: an initial consonant, a medial glide, a vowel nucleus, a final consonant or glide, and a lexical tone. If each of the five slots could take any of its individually possible values without restriction, the total number of syllables would be the product of the five slot sizes, $n_1n_2n_3n_4n_5$. The actual inventory of syllables in the language is very much smaller than this product, because entire combinations of slot values are disallowed: certain initials never combine with certain medials, and several finals occur with only a restricted subset of nuclei. The language therefore occupies a genuinely smaller *admissible space* carved out of the larger *theoretical space* that the unconstrained

product would predict, in exactly the sense introduced in Chapter 0's discrete state spaces. The Pythagorean identity of this chapter performs the same carving operation in a continuous setting: the theoretical space of all pairs $(x, y) \in \mathbb{R}^2$ is far larger than the admissible space of pairs that can actually occur as $(\cos\theta, \sin\theta)$, which is restricted to the unit circle by the single constraint $x^2 + y^2 = 1$. In both the syllable inventory and the trigonometric identity, the interesting mathematical object is not the raw product of independent possibilities but the much smaller admissible subset that a constraint actually permits.

9.8 Identity Versus Equation

A distinction is worth making explicit before proceeding further. An *identity* is a statement true for every value of the variable for which both sides are defined; $\sin^2\theta + \cos^2\theta = 1$, for instance, holds for every angle θ without exception. An *equation*, by contrast, is true only for specific values of the variable; $\sin\theta = 1/2$ holds only for certain angles, and finding those angles is a solving problem rather than a proving problem. Confusing these two tasks — attempting to "solve" an identity, or attempting to "prove" an equation true for all θ when it plainly is not — is one of the most common sources of difficulty for students first encountering this material, and the distinction will matter again directly when Chapter 11 turns to solving trigonometric equations.

9.9 Applications

Trigonometric identities appear throughout mathematics, science, and engineering: they simplify formulas in physics, support the analysis of oscillatory systems, underlie techniques in signal processing, facilitate coordinate transformations, and make possible the trigonometric substitution techniques used later in calculus. The value of an identity lies not merely in the fact that it is true, but in what that truth makes possible: it allows one expression to be replaced by another that may be easier to compute, simplify, or interpret, without changing the underlying quantity being described.

9.10 Looking Ahead

At this point the book has assembled a substantial identity toolkit: reciprocal identities, quotient identities, the three Pythagorean identities, the cofunction identities, the even and odd identities, and the sum and difference formulas of Chapter 8. Remarkably, this modest collection already suffices to generate most of the remaining identities encountered in elementary trigonometry. The next chapter shows how the double-angle, half-angle, and product-to-sum formulas all arise naturally from the tools developed here, so that what can appear, from a distance, to be an enormous and disconnected collection of formulas turns out to be different expressions of a comparatively small set of underlying principles.

9.11 Exercises

Exercise 9.1. Starting from $\sin^2\theta + \cos^2\theta = 1$, derive $1 + \tan^2\theta = \sec^2\theta$.

Exercise 9.2. Starting from $\sin^2\theta + \cos^2\theta = 1$, derive $1 + \cot^2\theta = \csc^2\theta$.

Exercise 9.3. Prove that $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$ using the quotient identity.

Exercise 9.4. Prove that $\dfrac{1-\sin^2\theta}{\cos\theta} = \cos\theta$.

Exercise 9.5. Prove that $\dfrac{\cos^2\theta}{1-\sin\theta} = 1+\sin\theta$.

Exercise 9.6. Use the cofunction identities to evaluate $\sin\left(\dfrac{\pi}{2} - \dfrac{\pi}{6}\right)$.

Exercise 9.7. Use the even and odd identities to simplify $\sin(-\theta) + \cos(-\theta)$.

Exercise 9.8. Determine whether $\cos\theta = 0$ is an identity or an equation, and explain your reasoning.

Exercise 9.9. Explain why $(\sin\theta, \cos\theta) = (2, 3)$ can never occur for any angle θ.

Exercise 9.10. Why is $\sin^2\theta + \cos^2\theta = 1$ better understood as a geometric constraint than as an isolated algebraic formula?

Chapter 10

Multiple-Angle and Product Identities

10.1 Nothing New, Only Consequences

The previous two chapters assembled a substantial identity toolkit: the sum and difference formulas of Chapter 8, together with the reciprocal, quotient, Pythagorean, cofunction, and even/odd identities of Chapter 9. At this point trigonometry can begin to look like an ever-growing collection of formulas with no end in sight, but that appearance is misleading. Many of the identities still to come are consequences of a much smaller set of principles already in hand, and the purpose of this chapter is to demonstrate that fact explicitly. Everything derived here follows from the sum and difference formulas of Chapter 8 together with the algebraic identities of Chapter 9; no new geometry is introduced anywhere in this chapter, because the geometry has already done its work.

10.2 Double-Angle Formulas

The simplest application of the sum formulas occurs when the two angles being added are equal. Setting $A = B = \theta$ in the sine sum formula gives

$$\sin(2\theta) = \sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta,$$

which combines to

$$\boxed{\sin(2\theta) = 2\sin\theta\cos\theta}.$$

Applying the same substitution to the cosine sum formula gives

$$\boxed{\cos(2\theta) = \cos^2\theta - \sin^2\theta}.$$

For tangent, $\tan(2\theta) = \sin(2\theta)/\cos(2\theta)$; substituting the two results just derived and dividing numerator and denominator by $\cos^2\theta$ gives

$$\boxed{\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}}$$

whenever the denominator is nonzero. None of these three formulas is a new fact about trigonometry; each is simply the corresponding sum formula of Chapter 8, evaluated at two equal angles.

10.3 Three Forms of the Cosine Double-Angle Formula

The cosine double-angle formula is particularly useful because the Pythagorean identity allows it to be rewritten in two further equivalent forms. Substituting $\sin^2\theta = 1 - \cos^2\theta$ into $\cos(2\theta) =$

$\cos^2\theta - \sin^2\theta$ gives

$$\cos(2\theta) = \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1,$$

so $\boxed{\cos(2\theta) = 2\cos^2\theta - 1}$. Substituting instead $\cos^2\theta = 1 - \sin^2\theta$ gives

$$\cos(2\theta) = (1 - \sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta,$$

so $\boxed{\cos(2\theta) = 1 - 2\sin^2\theta}$. The three boxed equations

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta, \qquad \cos(2\theta) = 2\cos^2\theta - 1, \qquad \cos(2\theta) = 1 - 2\sin^2\theta$$

are not three separate identities but three algebraic forms of the same one, each convenient in different circumstances; the second and third, in particular, will be exactly what is needed to derive the half-angle formulas in the next section.

Example 10.1. Suppose $\sin\theta = 3/5$ and θ lies in Quadrant I. Find $\sin(2\theta)$. Constructing the corresponding right triangle as in Chapter 7, the side opposite θ is 3 and the hypotenuse is 5, so the remaining side is 4 and $\cos\theta = 4/5$. Applying the double-angle formula,

$$\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}.$$

10.4 Half-Angle Formulas

The half-angle formulas arise by solving the double-angle formulas in reverse. Starting from $\cos(2\theta) = 1 - 2\sin^2\theta$ and solving for $\sin\theta$,

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}, \qquad \sin\theta = \pm\sqrt{\frac{1 - \cos(2\theta)}{2}}.$$

Replacing θ by $\theta/2$ throughout gives

$$\boxed{\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 - \cos\theta}{2}}}.$$

The same procedure applied to $\cos(2\theta) = 2\cos^2\theta - 1$ gives

$$\boxed{\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 + \cos\theta}{2}}},$$

and dividing the two, $\tan(\theta/2) = \sin(\theta/2)/\cos(\theta/2)$, gives

$$\boxed{\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}}}.$$

10.5 Choosing the Correct Sign

The half-angle formulas contain a genuine subtlety: taking a square root introduces two possible signs, and the correct one is determined by the quadrant of $\theta/2$, not by the quadrant of θ itself. This distinction is essential and easy to overlook. If $\theta = 240°$, for instance, then $\theta/2 = 120°$, which lies in Quadrant II, so $\sin(\theta/2)$ is positive while $\cos(\theta/2)$ is negative; the sign in each half-angle formula must be chosen accordingly, based on this quadrant of $\theta/2$ and not on the fact that θ itself lies in Quadrant IV. The issue plays much the same role here that the range restriction on the inverse trigonometric functions played in Chapter 7: a formula that looks single-valued in fact requires an extra piece of quadrant information before it can be evaluated correctly.

> **Procedure 10.1 (How to Evaluate a Half-Angle Expression).**
> Identify the appropriate half-angle formula for the function requested, and substitute the known value of $\cos\theta$ into it. Simplify the resulting radical as far as possible, then determine the quadrant of $\theta/2$ itself, not the quadrant of θ. Choose the sign of the radical to match that quadrant, and check the result for consistency with the sign pattern established in Chapter 4.

Example 10.2. Compute $\sin(15^\circ)$. Since $15^\circ = 30^\circ/2$, the half-angle formula gives

$$\sin(15^\circ) = \sqrt{\frac{1-\cos 30^\circ}{2}},$$

with the positive root chosen because 15° lies in Quadrant I. Substituting $\cos 30^\circ = \sqrt{3}/2$,

$$\sin(15^\circ) = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}.$$

This agrees exactly with the value obtainable from the difference formula of Chapter 8: the two methods are different routes to the same result, a useful check that both chapters are internally consistent.

> *Practice now.* Compute $\cos(15^\circ)$ using the half-angle formula, and confirm that the result matches the value obtainable from Chapter 8's difference formula.

10.6 Product-to-Sum Formulas

The sum and difference formulas can be combined to transform a product of trigonometric functions into a sum. Adding $\sin(A+B) = \sin A\cos B + \cos A\sin B$ and $\sin(A-B) = \sin A\cos B - \cos A\sin B$ cancels the mixed terms:

$$\sin(A+B) + \sin(A-B) = 2\sin A\cos B, \qquad \boxed{\sin A\cos B = \frac{\sin(A+B)+\sin(A-B)}{2}}.$$

The same strategy, applied to the cosine sum and difference formulas, gives

$$\boxed{\cos A\cos B = \frac{\cos(A+B)+\cos(A-B)}{2}}, \qquad \boxed{\sin A\sin B = \frac{\cos(A-B)-\cos(A+B)}{2}}.$$

These are the product-to-sum formulas, and each is obtained purely by adding or subtracting two instances of results already derived in Chapter 8.

10.7 Sum-to-Product Formulas

The product-to-sum formulas can be reversed. Introducing new variables $A = (x+y)/2$ and $B = (x-y)/2$, so that $A+B = x$ and $A-B = y$, turns the product-to-sum formulas into statements expressing a sum as a product:

$$\boxed{\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}, \qquad \boxed{\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)},$$

with the analogous formulas for $\sin x - \sin y$ and $\cos x - \cos y$ following by the same substitution. The details are purely algebraic and add nothing beyond a relabeling of the product-to-sum identities already established.

10.8 Applications

The formulas of this chapter matter well beyond elementary trigonometry: they simplify oscillatory expressions, support signal analysis, explain the beat frequencies heard when two close musical pitches sound together, appear throughout electrical engineering, and enable the calculus techniques used to integrate products of trigonometric functions. In many of these settings, the ability to rewrite an expression in a more convenient form matters more than evaluating it directly, which is exactly why a toolkit of equivalent forms, rather than a single canonical formula, is worth having.

10.9 Looking Ahead

This chapter completes the identity toolkit begun in Chapters 8 and 9. The book now has in hand the sum and difference formulas, the reciprocal, quotient, Pythagorean, cofunction, and even/odd identities, and the double-angle, half-angle, product-to-sum, and sum-to-product formulas derived here. The next chapter finally puts this toolkit to work in a different kind of problem. Rather than proving that two expressions are equivalent, Chapter 11 solves equations: given a trigonometric statement involving an unknown angle, it determines every angle for which that statement is true, using the identities of this chapter and the last to reduce a complicated equation to one the tools of Chapter 7 can finish.

10.10 Exercises

Exercise 10.1. Starting from the cosine sum formula, derive $\cos(2\theta) = \cos^2\theta - \sin^2\theta$.

Exercise 10.2. Use the Pythagorean identity to derive $\cos(2\theta) = 2\cos^2\theta - 1$.

Exercise 10.3. Use the Pythagorean identity to derive $\cos(2\theta) = 1 - 2\sin^2\theta$.

Exercise 10.4. If $\sin\theta = 5/13$ and θ lies in Quadrant I, find $\sin(2\theta)$.

Exercise 10.5. If $\cos\theta = -3/5$ and θ lies in Quadrant II, find $\cos(2\theta)$.

Exercise 10.6. Evaluate $\sin(22.5^\circ)$ exactly.

Exercise 10.7. Evaluate $\cos(22.5^\circ)$ exactly.

Exercise 10.8. Determine the correct sign in $\cos(\theta/2)$ when $\theta = 300^\circ$, and explain your reasoning.

Exercise 10.9. Derive the product-to-sum formula for $\cos A \cos B$.

Exercise 10.10. Why are the double-angle and half-angle formulas best understood as consequences of the sum formulas rather than as independent identities?

Chapter 11

Trigonometric Equations

11.1 From Identities to Equations

The previous three chapters were concerned primarily with identities: how rotations combine, how the unit circle constrains admissible trigonometric values, and how apparently complicated formulas can be derived from a small collection of fundamental relationships. Throughout that development the task was always the same in kind — start from one expression, transform it into another, and show that the two are equivalent. The task of this chapter is different. Rather than proving that two expressions are always equal, this chapter asks for which values of an angle a particular equation becomes true, moving from proving identities to solving equations.

At first glance this may look like a familiar algebraic exercise: solving $x^2 - 5x + 6 = 0$ means finding the values of x that make the equation true, so why should a trigonometric equation be any different? The answer is periodicity. A polynomial equation ordinarily has only finitely many solutions, but trigonometric functions repeat indefinitely, and if a point on the unit circle returns to the same position every full revolution, then any trigonometric value that occurs once must occur again and again. Trigonometric equations therefore typically possess infinitely many solutions, and the challenge shifts from finding a single value that works to describing the entire family of values that work. When solving an algebraic equation it is often enough to list the roots; when solving a trigonometric equation, the final answer is usually not a number but a description of a repeating structure.

11.2 The Simplest Equations

The natural place to begin is with an equation involving a single trigonometric function, such as $\sin\theta = a$. Chapter 7 introduced the inverse sine function and showed how to find an angle whose sine equals a specified value, giving $\theta = \arcsin(a)$; this angle is called the *principal value*, the particular solution returned by the inverse function. It is rarely the complete answer. A horizontal line drawn across the unit circle generally intersects it at two points sharing the same vertical coordinate and therefore the same sine value, so $\sin(\pi/6) = \frac{1}{2}$ but also $\sin(5\pi/6) = \frac{1}{2}$; the inverse function returns only the first of these, while the geometry of the circle guarantees that a second one exists. The periodicity of sine then produces infinitely many further solutions, since adding any integer multiple of 2π leaves the sine value unchanged.

Theorem 11.1. *Let $\alpha = \arcsin(a)$. Every solution of $\sin\theta = a$ is given by $\theta = \alpha + 2\pi k$ or $\theta = \pi - \alpha + 2\pi k$, for k any integer.*

This theorem is often memorized as a formula, but its content is simpler than it looks: the angle α names one point on the unit circle, and $\pi - \alpha$ names its mirror image across the vertical axis, a point with the same height and hence the same sine value; adding multiples of 2π simply repeats both positions on every subsequent revolution. The theorem formalizes the geometry of the circle rather than introducing anything beyond it.

Cosine behaves similarly, though its symmetry runs the other way: equal cosine values arise from reflection across the horizontal axis rather than the vertical one. If $\alpha = \arccos(a)$, every solution of $\cos\theta = a$ can be written as $\theta = \pm\alpha + 2\pi k$. Tangent differs because its period is π rather than 2π, established in Chapter 6, so once a single solution $\arctan(a)$ is known, every other solution is obtained by adding integer multiples of π alone: $\theta = \arctan(a) + \pi k$.

Example 11.2. Solve $\sin\theta = \frac{1}{2}$ completely. The inverse sine function gives the principal solution $\theta = \arcsin(1/2) = \pi/6$, but this alone cannot be the whole story: the graph of sine shows that the value $\frac{1}{2}$ occurs twice within a single cycle, at the reference angle and again at its Quadrant II partner, $\theta = 5\pi/6$. Having found both solutions in one revolution, periodicity extends each into an infinite family:

$$\theta = \frac{\pi}{6} + 2\pi k \qquad \text{or} \qquad \theta = \frac{5\pi}{6} + 2\pi k,$$

for k any integer. The pattern is worth naming explicitly, since nearly every equation in this chapter follows it: begin with an inverse function, use geometry to find any remaining solutions within one revolution, and use periodicity to generate the entire family from those.

11.3 A General Strategy

The equations solved so far shared a convenient feature: the trigonometric function was already isolated. Many equations do not arrive in so convenient a form. The equation $2\sin\theta\cos\theta - \sin\theta = 0$, for instance, offers no immediate use for an inverse trigonometric function, since neither sine nor cosine has been isolated, and some algebraic work must be done before any geometric reasoning can begin. This observation is worth stating as a general principle: solving a trigonometric equation is a collaboration between algebra and geometry, where algebra simplifies the equation until its structure becomes visible and geometry then identifies the corresponding angles and their periodic repetitions. Neither approach is sufficient alone. The identity toolkit of Chapters 8 through 10 exists largely for this purpose — not merely as a set of facts to be proved, but as a set of tools for rewriting an equation into a form whose solutions can be read off geometrically.

Procedure 11.1 (How to Solve a Trigonometric Equation).
Begin by treating the equation exactly as an algebraic equation: factor where possible, move all terms to one side, and look for common factors or useful substitutions. If the equation mixes more than one trigonometric function or more than one angle, apply an identity from Chapters 8 through 10 to reduce it to a single function of a single angle. Once the equation involves only one trigonometric function, isolate that function and apply the appropriate inverse function together with the reference-angle and quadrant reasoning of Chapter 4 to find every solution within one revolution. Extend those solutions to the general solution using periodicity, then check any restricted domain and verify that no extraneous solutions were introduced by the algebraic steps along the way.

Example 11.3. Solve $2\sin\theta\cos\theta - \sin\theta = 0$. Factoring out the common factor $\sin\theta$ gives $\sin\theta(2\cos\theta - 1) = 0$, and since a product equals zero only when at least one factor does, either $\sin\theta = 0$ or $2\cos\theta - 1 = 0$. The original problem has become two simpler ones. The first, $\sin\theta = 0$, holds whenever the corresponding point lies on the horizontal axis of the unit circle, giving $\theta = k\pi$. The second, $\cos\theta = 1/2$, has reference angle $\pi/3$; since cosine is positive in

Quadrants I and IV, $\theta = \pm\pi/3 + 2\pi k$. Combining the two families gives the complete solution set. Notice how little trigonometry was needed once the factorization was found: the difficult step was algebraic, and the geometry that followed was routine.

11.4 Quadratic Trigonometric Equations

Some trigonometric equations resemble polynomial equations even more closely. The equation $2\sin^2\theta - \sin\theta - 1 = 0$ involves only powers of a single trigonometric function, and looks remarkably like $2x^2 - x - 1 = 0$; this resemblance is not accidental. Whenever an equation involves only one trigonometric function and its powers, substitution converts the problem into an ordinary algebraic equation: set that quantity temporarily equal to a new variable, solve the resulting polynomial equation, and translate the roots back into trigonometric language once the algebra is finished.

Example 11.4. Solve $2\sin^2\theta - \sin\theta - 1 = 0$. Substituting $u = \sin\theta$ gives the quadratic $2u^2 - u - 1 = 0$, which factors as $(2u+1)(u-1) = 0$, so $u = 1$ or $u = -\frac{1}{2}$. Substituting back, $\sin\theta = 1$ or $\sin\theta = -\frac{1}{2}$. The equation $\sin\theta = 1$ corresponds to the top of the unit circle, giving $\theta = \pi/2 + 2\pi k$. The equation $\sin\theta = -\frac{1}{2}$ has reference angle $\pi/6$; since the value is negative, the relevant quadrants are III and IV, giving $\theta = 7\pi/6 + 2\pi k$ and $\theta = 11\pi/6 + 2\pi k$. The complete solution consists of these three infinite families together. What matters about this example is not the particular answer but the pattern: many trigonometric equations are solved by temporarily converting them into algebraic equations, solving those, and translating the roots back into geometric language.

11.5 Why the Identity Toolkit Was Necessary

A natural question arises at this point: why spend three chapters developing identities if factoring and substitution already solve so much? The answer is that many equations do not initially involve a single trigonometric function. The equation $\cos(2\theta) + \sin\theta = 0$ contains two different trigonometric expressions, and neither factoring nor a direct inverse function is immediately useful against it; the obstacle is not the equation itself but its form, and Chapter 10 supplies exactly the tool needed to change that form.

> *Practice now.* Before reading the worked example below, try rewriting $\cos(2\theta) + \sin\theta = 0$ using the double-angle identity $\cos(2\theta) = 1 - 2\sin^2\theta$, and see how far the substitution $u = \sin\theta$ takes you toward a solution.

Example 11.5. Solve $\cos(2\theta) + \sin\theta = 0$. Using $\cos(2\theta) = 1 - 2\sin^2\theta$ from Chapter 10, the equation becomes $1 - 2\sin^2\theta + \sin\theta = 0$, in which every term now involves sine alone: a mixed trigonometric expression has been transformed into a quadratic equation in disguise. Substituting $u = \sin\theta$ reduces it to $2u^2 - u - 1 = 0$, precisely the equation solved in the previous section, so the same solution families follow immediately once u is translated back into $\sin\theta$.

This example captures the central purpose of the identity toolkit: an identity is not primarily an object of memorization but a transformation rule, valuable because it converts a difficult equation into one that can already be solved. Much of mathematics proceeds by replacing an unfamiliar problem with a familiar one, and the identities of Chapters 8 through 10 are the machinery that makes such replacements possible here.

11.6 Extraneous Solutions and the Importance of Checking

Every transformation used so far — factoring, substitution, applying an identity — has relied on the assumption that the transformed equation has exactly the same solutions as the original one. That assumption is often correct, but not always. Certain algebraic operations can introduce solutions that were not present in the original problem, called *extraneous solutions*: a value that genuinely solves a transformed equation but fails to solve the equation from which that transformed equation was derived. Because trigonometric equations often require substantial manipulation before becoming solvable, this deserves careful attention, and the safest habit is simple: every candidate solution should be checked in the original equation before being accepted.

Example 11.6. Consider $\sin\theta = \cos\theta$. Squaring both sides gives $\sin^2\theta = \cos^2\theta$, and using the Pythagorean identity, $\sin^2\theta = 1 - \sin^2\theta$, so $2\sin^2\theta = 1$ and $\sin\theta = \pm\sqrt{2}/2$. These values produce four candidate solutions in a full revolution. The original equation, however, is satisfied only when $\sin\theta$ and $\cos\theta$ share the same sign, so the angles lying in Quadrants II and IV, where sine and cosine have opposite signs, must be discarded even though they satisfy the squared equation. Squaring introduced possibilities that were not present in the original statement, and the only reliable way to detect this is to substitute each candidate back into the equation as originally posed.

Checking a candidate solution is not a tedious final formality; it is part of the solution process itself. Some operations, such as factoring, preserve the solution set automatically as an equation is rewritten, but others — squaring both sides, multiplying by an unknown quantity, or clearing a denominator — can enlarge it. The resulting equation may be easier to solve, but it is no longer guaranteed to describe exactly the original situation, and checking is what restores that guarantee. For this reason a candidate solution is best regarded as provisional until it has actually been verified.

11.7 Solving on a Restricted Domain

The equations solved so far sought every solution, and the answers therefore involved an integer parameter $k \in \mathbb{Z}$, appropriate when the domain is understood to be all real numbers. Many applications instead specify a particular interval: an engineer may be interested only in the first cycle of an oscillation, a navigator may need a bearing between $0°$ and $360°$, or a problem may simply ask for solutions on $[0, 2\pi)$. The underlying mathematics is unchanged; only the final step differs, since the task becomes identifying which members of the general solution family lie inside the specified interval, producing a finite answer rather than an infinite one.

Example 11.7. Solve $\cos\theta = -\frac{1}{2}$ for $0 \le \theta < 2\pi$. The reference angle is $\pi/3$, and since cosine is negative in Quadrants II and III, the solutions within one revolution are $\theta = 2\pi/3$ and $\theta = 4\pi/3$. Were the general solution requested, $2\pi k$ would be appended to each; the restricted domain instead says to stop here, so the complete answer is

$$\boxed{\theta = \frac{2\pi}{3}, \frac{4\pi}{3}.}$$

Restricted-domain problems are often easier to visualize geometrically than their general-solution counterparts: rather than imagining infinitely many revolutions around the circle, only one revolution needs to be pictured, with the sign of the relevant trigonometric function identifying the correct quadrants and the reference angle fixing the distance from the nearest axis. The periodic structure remains present in the background, but it is temporarily suppressed rather than eliminated.

11.8 Applications of Trigonometric Equations

The appearance of infinitely many solutions is not merely a mathematical curiosity; it reflects the behavior of physical systems that repeat. Suppose a tide reaches a particular height at noon: if the tidal cycle repeats, that height recurs later as well, so a single solution is not enough, and what matters is the entire sequence of times at which the event occurs. The same reasoning governs sound waves, alternating electrical current, planetary motion, and seasonal cycles. A model such as $h(t) = 4 + 2\sin t$ describes a quantity oscillating continuously, and asking when $h(t) = 5$ is asking for the times at which this graph intersects a horizontal line — a trigonometric equation of exactly the kind this chapter has developed methods for. What began as geometry on the unit circle becomes, by this point, a working tool for analyzing recurring events throughout the natural world.

11.9 Looking Back and Looking Ahead

Part III has told a complete story. Chapter 8 began with rotations and showed that angle addition arises from the composition of transformations, leading to the sum and difference formulas. Chapter 9 showed that the unit circle imposes fundamental constraints on admissible trigonometric values, producing the Pythagorean identities and their consequences. Chapter 10 showed that a large collection of seemingly independent formulas can be generated from that same small set of underlying principles. This chapter has finally put those principles to work: the identities developed earlier were not endpoints but tools, meant all along to transform equations into forms that could be solved. Seen as a whole, Part III has been an extended study of transformation — rotations transformed points, identities transformed expressions, and algebraic substitutions transformed equations — with the same goal recurring at every stage: replace a difficult object with an equivalent but more useful one.

The next part of the book broadens its scope considerably. Trigonometry has so far lived primarily inside the unit circle and the right triangle; the chapters ahead move beyond that setting to study arbitrary triangles, vector geometry, alternative coordinate systems, and eventually complex numbers. The first question to ask is a simple one: what happens when the triangle is no longer right? The answer leads to two of the most important results in all of trigonometry, the Law of Sines and the Law of Cosines, which extend the ideas developed throughout this book to a far wider class of geometric problems and open Part IV.

11.10 Exercises

Exercise 11.1. Solve $\sin\theta = \dfrac{\sqrt{3}}{2}$. Give the general solution.

Exercise 11.2. Solve $\cos\theta = -\dfrac{\sqrt{2}}{2}$. Give the general solution.

Exercise 11.3. Solve $\tan\theta = 1$. Give the general solution.

Exercise 11.4. Solve $2\sin\theta - 1 = 0$. Give the general solution.

Exercise 11.5. Solve $\sin\theta(2\cos\theta + 1) = 0$. Give the general solution.

Exercise 11.6. Solve $2\sin^2\theta + \sin\theta - 1 = 0$. Give the general solution.

Exercise 11.7. Solve $\cos(2\theta) - \cos\theta = 0$, using an identity to reduce the equation to a single function.

Exercise 11.8. Solve $\sin\theta = \dfrac{1}{2}$ on the interval $0 \le \theta < 2\pi$.

Exercise 11.9. Explain why $\theta = \arcsin(a)$ is generally not the complete solution of $\sin\theta = a$.

Exercise 11.10. A periodic signal is modeled by $V(t) = 5\sin t$. Find all times for which $V(t) = 5/2$, expressed as a general solution set.

Part IV

Vectors, Complex Numbers, and Unification

Chapter 12

The Law of Sines and the Law of Cosines

12.1 Beyond the Right Triangle

The trigonometry developed so far has rested on two geometric settings, and both, on inspection, share a common feature. The right triangle of Chapter 3 supplied sine, cosine, and tangent as ratios invariant under scaling, with a right angle present explicitly; the unit circle of Chapter 4 extended those same functions beyond acute angles by measuring coordinates against a pair of perpendicular axes, with the right angle now hidden inside the coordinate system rather than inside a triangle. As long as a right angle is available somewhere in the configuration, the tools built in earlier chapters suffice. Many geometric situations, however, supply no such angle. A surveyor measuring the distance between two inaccessible points may obtain a triangle with no right angle at all; a navigator determining position from bearings, or an astronomer inferring distance from parallax, may encounter a triangle whose angles bear no particular relationship to ninety degrees. If trigonometry is to serve these settings, it must be extended beyond the right triangle, and this chapter's two central results, the Law of Sines and the Law of Cosines, should reduce to the familiar right-triangle relationships whenever a right angle happens to be present, so that right-triangle trigonometry becomes a special case of a broader theory rather than a separate subject entirely.

12.2 Deriving the Law of Sines

Consider a triangle with sides a, b, c opposite angles A, B, C respectively. Dropping an altitude of length h from the vertex opposite side c divides the triangle into two right triangles sharing that altitude, which provides a bridge between two different trigonometric descriptions of the same quantity.

Theorem 12.1 (Law of Sines)**.** *For any triangle,* $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

Proof. In the right triangle containing angle A, $\sin A = h/b$, so $h = b \sin A$; in the right triangle containing angle B, $\sin B = h/a$, so $h = a \sin B$. Both expressions describe the same altitude, so $a \sin B = b \sin A$, and dividing by $\sin A \sin B$ gives $a/\sin A = b/\sin B$. Repeating the construction with an altitude dropped from a different vertex relates the third side-angle pair in the same way, establishing all three ratios as equal. □

Each ratio in the theorem measures the same underlying quantity, namely twice the circumradius of the triangle, so the equality of the three ratios expresses a single consistency constraint that any valid configuration of sides and angles must satisfy, rather than three independent facts. The formula is also consistent with what Chapter 3 already established: if $C = 90°$, then $\sin C = 1$ and the Law of Sines reduces to $a/\sin A = b/\sin B = c$, or equivalently $\sin A = a/c$

and $\sin B = b/c$, exactly the right-triangle definitions of Chapter 3. The Law of Sines therefore extends the earlier theory rather than replacing it.

Example 12.2. Solve the triangle with $A = 42°$, $B = 68°$, $a = 12$. Since the angles of a triangle sum to 180°, $C = 180° - 42° - 68° = 70°$. The Law of Sines gives

$$\frac{12}{\sin 42°} = \frac{b}{\sin 68°}, \qquad b = 12\,\frac{\sin 68°}{\sin 42°} \approx 16.6,$$

and similarly $c = 12 \sin 70°/\sin 42° \approx 16.8$, completing the triangle without any altitude or right angle ever having been given directly.

12.3 The Ambiguous Case

Most triangle-solving configurations determine either one triangle or none. The Law of Sines admits a notable exception when two sides and a non-included angle are known, traditionally called the SSA case. Fixing one side and one angle and allowing the remaining given side to pivot about its attached endpoint like a hinged rod shows why: depending on its length relative to the altitude and the adjacent side, the free endpoint may fail to reach the base at all, may reach it in exactly one position, or may reach it in two distinct positions, yielding no triangle, one triangle, or two triangles respectively. The ambiguity has an algebraic source as well as a geometric one: it arises because a single sine value corresponds to two different angles within one revolution, precisely the phenomenon already encountered in solving equations of the form $\sin\theta = 1/2$ in Chapter 11. The SSA case is therefore not an isolated irregularity but a direct consequence of the geometry of the sine function.

Procedure 12.1 (How to Determine the Number of Triangles in the SSA Case).
Given angle A and sides a and b, compute the altitude $h = b \sin A$ from the vertex opposite b. If $a < h$, no triangle exists, since side a is too short to reach the base. If $a = h$, exactly one triangle exists, a right triangle with the right angle opposite a. If $h < a < b$, two triangles exist, corresponding to the two angles B and $180° - B$ satisfying $\sin B = b \sin A/a$. If $a \geq b$, exactly one triangle exists, since the supplementary angle $180° - B$ would make the angle sum of the triangle exceed 180°.

Example 12.3. Solve the triangle with $A = 30°$, $a = 10$, $b = 18$. The Law of Sines gives $10/\sin 30° = 18/\sin B$, and since $\sin 30° = 1/2$, this becomes $20 = 18/\sin B$, so $\sin B = 18/20 = 0.9$ and $B \approx 64.2°$. Because $h = b \sin A = 9$ and $a = 10$ satisfies $h < a < b$, a second solution is expected: $B = 180° - 64.2° = 115.8°$ also satisfies $\sin B = 0.9$, and both angles are compatible with the given data. The first possibility gives $C = 180° - 30° - 64.2° = 85.8°$, and the second gives $C = 180° - 30° - 115.8° = 34.2°$; both triangles are valid, each containing a side of length 10, a side of length 18, and an angle of 30°, with the given data alone insufficient to distinguish between them.

Whenever the Law of Sines produces an angle through an inverse sine computation, the supplementary angle should be checked against the algorithm above before it is discarded, since in the SSA case it will sometimes, though not always, correspond to a second valid triangle.

12.4 Toward the Law of Cosines

The Law of Sines cannot resolve every configuration. If all three sides of a triangle are known and an angle is sought, the Law of Sines offers no starting point, since it requires an already-known side-angle pair. A different relationship is needed, and it is obtained by returning to coordinate geometry, the strategy Chapter 2 introduced for translating geometric questions into algebra.

12.5 Deriving the Law of Cosines

Place a triangle with sides a, b, c and opposite angles A, B, C so that the vertex at angle C lies at the origin and the side of length b extends along the positive x-axis to the point $(b, 0)$. The third vertex lies at distance a from the origin, at angle C from the positive x-axis, so by the unit-circle coordinate formulas of Chapter 4 it has coordinates $(a\cos C, a\sin C)$.

Theorem 12.4 (Law of Cosines)**.** *For any triangle, $c^2 = a^2 + b^2 - 2ab\cos C$, and similarly $a^2 = b^2 + c^2 - 2bc\cos A$ and $b^2 = a^2 + c^2 - 2ac\cos B$.*

Proof. The side of length c is the distance between $(b, 0)$ and $(a\cos C, a\sin C)$, so by the distance formula of Chapter 2,

$$c^2 = (b - a\cos C)^2 + (a\sin C)^2 = b^2 - 2ab\cos C + a^2\cos^2 C + a^2\sin^2 C.$$

By the Pythagorean identity of Chapter 9, $a^2\cos^2 C + a^2\sin^2 C = a^2$, so $c^2 = a^2 + b^2 - 2ab\cos C$. The remaining two forms follow by relabeling the triangle's vertices and repeating the same argument. □

The three ingredients of this derivation are drawn from earlier chapters of the book: the coordinate placement from Chapter 2, the trigonometric coordinates from Chapter 4, and the Pythagorean identity from Chapter 9. Setting $C = 90°$ gives $\cos C = 0$, so the correction term vanishes and $c^2 = a^2 + b^2$, the ordinary Pythagorean theorem; the Law of Cosines therefore generalizes the Pythagorean theorem rather than replacing it, with the familiar right-triangle relationship recovered exactly when the included angle is a right angle.

12.6 Solving Triangles with the Law of Cosines

The most direct use of the Law of Cosines occurs when two sides and the included angle are known, the SAS case.

Example 12.5. Solve for c given $a = 8$, $b = 11$, $C = 50°$. The Law of Cosines gives

$$c^2 = 8^2 + 11^2 - 2(8)(11)\cos 50° = 64 + 121 - 176(0.6428) \approx 71.9,$$

so $c \approx 8.48$. Once all three sides are known, either the Law of Sines or the Law of Cosines can determine the two remaining angles, since the Law of Sines requires only one known side-angle pair to proceed and one, C, is already known.

> *Practice now.* Solve for c given $a = 13$, $b = 9$, $C = 120°$.

A second application arises when all three sides are known and an angle is sought, the SSS case.

Example 12.6. Given $a = 7$, $b = 9$, $c = 11$, find angle C. Substituting into $c^2 = a^2 + b^2 - 2ab\cos C$,

$$121 = 49 + 81 - 126\cos C, \qquad 126\cos C = 9, \qquad \cos C = \frac{9}{126} = \frac{1}{14},$$

so $C = \arccos(1/14) \approx 85.9°$. Unlike the SSA configuration of Section 12.3, no ambiguity arises here: three side lengths satisfying the triangle inequality determine a unique triangle, since arccos returns exactly one angle in $[0, \pi]$ for any given ratio, with no supplementary angle to check.

Procedure 12.2 (How to Solve an Oblique Triangle).
Given the known measurements of a triangle, determine which case applies. If two angles and any side are known (AAS or ASA), find the third angle from the angle sum and apply the Law of Sines directly. If two sides and the included angle are known (SAS), apply the Law of Cosines to find the third side, then use the Law of Sines or a second application of the Law of Cosines to find the remaining angles. If all three sides are known (SSS), apply the Law of Cosines to find one angle at a time. If two sides and a non-included angle are known (SSA), apply the Law of Sines and use the algorithm of Section 12.3 to determine whether zero, one, or two triangles satisfy the data.

12.7 The Cosine Correction Term

The formula $c^2 = a^2 + b^2 - 2ab\cos C$ is often committed to memory without attention to its geometric content. The term $a^2 + b^2$ is exactly what the Pythagorean theorem would predict if sides a and b met at a right angle, and the term $-2ab\cos C$ measures the departure of the actual triangle from that special case. When $C = 90°$, $\cos C = 0$ and the correction vanishes entirely; when $C < 90°$, $\cos C$ is positive and the correction subtracts from $a^2 + b^2$, so the side opposite an acute included angle is shorter than the right-triangle prediction; when $C > 90°$, $\cos C$ is negative and subtracting a negative quantity lengthens the opposite side beyond that prediction. The Law of Cosines can therefore be read as the Pythagorean theorem together with an explicit correction recording how far the included angle departs from a right angle.

12.8 Area from Trigonometry

Returning to the altitude construction that opened this chapter yields a further consequence. With sides a and b enclosing angle C, and b taken as the base, the altitude from the opposite vertex is $h = a\sin C$, so the ordinary area formula Area $= \frac{1}{2}$(base)(height) becomes

$$\boxed{\text{Area} = \frac{1}{2}ab\sin C},$$

with the analogous formulas Area $= \frac{1}{2}bc\sin A$ and Area $= \frac{1}{2}ac\sin B$ following by relabeling. Each of these computes an area without requiring the altitude to be found explicitly, since the sine function already encodes the necessary height information.

Example 12.7. Find the area of the triangle with $a = 12$, $b = 15$, $C = 40°$.

$$\text{Area} = \frac{1}{2}(12)(15)\sin 40° \approx 90(0.6428) \approx 57.9$$

square units, obtained without ever computing an altitude directly.

12.9 Applications

The two laws developed in this chapter recover distances and angles that cannot be measured directly. Surveyors use triangulation to determine the widths of rivers, the positions of landmarks, and the boundaries of large parcels of land; navigators infer position from measured bearings and distances; engineers analyze forces acting along non-perpendicular directions; astronomers estimate the geometry of distant objects from measurements taken at different locations. In each case certain distances or angles are inaccessible to direct measurement, yet sufficient constraints can be gathered to reconstruct them indirectly, and the process is fundamentally reconstructive in the sense introduced in this book's preface: measurements are not merely recorded but used to infer a structure that was not itself observed.

12.10 Looking Ahead

The Law of Sines generalized the relationship between a triangle's sides and its opposite angles; the Law of Cosines generalized the Pythagorean theorem. Together the two provide a complete method for solving an arbitrary triangle from any sufficient set of given measurements. The next chapter moves from the geometry of individual triangles toward a broader language for describing space, returning to the vectors of Chapter 2 with new tools available. Vectors allow magnitude and direction to be treated as a single object, and among their properties is an operation, the dot product, whose defining formula will turn out to have exactly the structure of the Law of Cosines derived here: a length-squared relation with a cosine correction term. What in this chapter is a statement about the sides and angles of a triangle will reappear in the next as a general algebraic operation on vectors, continuing this book's movement from individual geometric configurations toward increasingly unified descriptions of transformation and space.

12.11 Exercises

The Law of Sines

Exercise 12.1. For each triangle, use the Law of Sines to find the missing side, rounded to the nearest tenth: (a) $A = 35°$, $B = 85°$, $a = 12$; (b) $A = 48°$, $C = 67°$, $c = 15$; (c) $B = 72°$, $C = 41°$, $b = 18$.

Exercise 12.2. A triangle has $A = 40°$, $B = 65°$, $a = 10$. Find C, b, and c.

Exercise 12.3. A triangle has $B = 58°$, $C = 47°$, $c = 14$. Solve the triangle completely.

Exercise 12.4. Verify directly that the triangle with $A = 30°$, $B = 60°$, $C = 90°$ satisfies the Law of Sines.

The Ambiguous Case

Exercise 12.5. Determine whether each SSA configuration produces no triangle, one triangle, or two triangles: (a) $A = 35°$, $a = 8$, $b = 15$; (b) $A = 40°$, $a = 12$, $b = 10$; (c) $A = 25°$, $a = 9$, $b = 18$.

Exercise 12.6. Solve completely, giving all possible triangles: $A = 30°$, $a = 10$, $b = 18$.

Exercise 12.7. Solve completely: $A = 50°$, $a = 16$, $b = 12$.

Exercise 12.8. Explain why the SSA case can sometimes produce two different triangles.

The Law of Cosines

Exercise 12.9. Use the Law of Cosines to find the missing side: (a) $a = 8, b = 11, C = 50°$; (b) $a = 13, b = 9, C = 120°$; (c) $a = 7, b = 14, C = 35°$.

Exercise 12.10. Find the third side of a triangle whose sides of length 12 and 17 include an angle of 110°.

Exercise 12.11. Two roads leave a town and diverge at an angle of 75°. A traveler goes 18 km along one road and another traveler goes 25 km along the other. How far apart are they?

Finding Angles from Three Sides

Exercise 12.12. Use the Law of Cosines to find the indicated angle: (a) $a = 7, b = 9, c = 11$, find C; (b) $a = 10, b = 10, c = 12$, find C; (c) $a = 5, b = 8, c = 10$, find A.

Exercise 12.13. Determine all three angles of a triangle with sides 6, 8, 10.

Exercise 12.14. Determine all three angles of a triangle with sides 9, 11, 15.

Exercise 12.15. Explain why the SSS case always determines a unique triangle whenever the triangle inequalities are satisfied.

Area of a Triangle

Exercise 12.16. Find the area: (a) $a = 12, b = 15, C = 40°$; (b) $a = 9, b = 14, C = 65°$; (c) $b = 20, c = 11, A = 32°$.

Exercise 12.17. A triangular plot of land has sides of length 60 m and 85 m with an included angle of 54°. Find its area.

Exercise 12.18. Show that $\frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A$ using the Law of Sines.

Mixed Problems

Exercise 12.19. Solve each triangle completely: (a) $a = 8, b = 11, C = 50°$; (b) $A = 42°$, $B = 68°, a = 12$; (c) $a = 7, b = 9, c = 11$.

Exercise 12.20. A surveyor stands at point P and measures the angle between two landmarks Q and R to be 63°. The distances from P to the landmarks are 240 m and 310 m. Find the distance between the landmarks.

Exercise 12.21. A ship travels 120 km due east and then changes course, traveling 90 km on a heading 35° north of east. How far is the ship from its starting point?

Exercise 12.22. A triangular support frame has side lengths 4.2 m, 5.6 m, and 7.1 m. Determine its largest angle.

Exercise 12.23. Explain when the Law of Sines is usually preferable to the Law of Cosines, and when the reverse is true.

Conceptual and Exploratory Problems

Exercise 12.24. Show algebraically that the Law of Cosines reduces to the Pythagorean theorem when $C = 90°$.

Exercise 12.25. Describe geometrically the meaning of the correction term $-2ab\cos C$ in the Law of Cosines.

Exercise 12.26. A student claims that the Law of Cosines is merely a complicated version of the Pythagorean theorem. Evaluate this claim.

Exercise 12.27. The Law of Sines and the Law of Cosines both reconstruct unknown measurements from partial information. Explain how each accomplishes this in a different way.

Exercise 12.28. Describe how this chapter extends the trigonometry of right triangles to arbitrary triangles.

Challenge Problems

Exercise 12.29. A triangle has sides 13, 14, 15. Find its area without using Heron's formula.

Exercise 12.30. A triangle has $A = 25°$, $a = 8$, $b = 14$. Determine all possible triangles and classify each as acute, right, or obtuse.

Exercise 12.31. Prove the Law of Cosines using an altitude rather than coordinates.

Exercise 12.32. For a triangle with sides a, b, c, show that $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$, and explain why this formula allows any angle of a triangle to be computed directly from the side lengths alone.

Chapter 13

Vectors and Dot Products

13.1 From Triangles to Vector Operations

The previous chapter concluded with an observation that deserves further attention. The Law of Cosines,

$$c^2 = a^2 + b^2 - 2ab\cos C,$$

appeared at first to be a statement about triangles, yet its structure suggests something more general. The formula relates lengths and angles in a single expression, measuring not only how large two sides are but also how they are oriented relative to one another. In Chapter 2, vectors were introduced as quantities possessing both magnitude and direction, and their lengths were computed using the distance formula. What remained missing was an operation capable of combining two vectors while taking both magnitude and direction into account simultaneously.

This chapter introduces such an operation. It will first appear as a simple algebraic rule whose purpose is not immediately obvious. Only after a derivation based on the Law of Cosines will its geometric meaning become clear. The result is one of the most important operations in mathematics and physics: the dot product.

13.2 The Algebraic Definition

Suppose

$$\mathbf{u} = \langle u_1, u_2 \rangle, \qquad \mathbf{v} = \langle v_1, v_2 \rangle.$$

Definition 13.1. The *dot product* of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

At first glance this definition appears somewhat arbitrary. Why should the products of corresponding coordinates be added together? Why not subtract them, multiply them, or combine them in some other way? The definition provides a quantity that is easy to compute, but it does not yet explain what that quantity represents.

For example, if

$$\mathbf{u} = \langle 3, 4 \rangle, \qquad \mathbf{v} = \langle 2, 5 \rangle,$$

then

$$\mathbf{u} \cdot \mathbf{v} = (3)(2) + (4)(5) = 6 + 20 = 26.$$

The computation is straightforward, but the meaning of the answer is not. The purpose of the next section is to show that this seemingly ad hoc rule has a remarkably natural geometric interpretation.

13.3 The Geometric Formula

The algebraic definition of the dot product provides a method of calculation, but it does not explain why the operation is useful. To discover its geometric meaning, consider two vectors placed with their tails at the origin:

$$\mathbf{u} = \langle u_1, u_2 \rangle, \qquad \mathbf{v} = \langle v_1, v_2 \rangle.$$

Let θ denote the angle between them. The vectors and the segment joining their endpoints form a triangle. That third side is represented by the vector

$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1,\ u_2 - v_2 \rangle.$$

The lengths of the three sides are therefore

$$|\mathbf{u}|, \qquad |\mathbf{v}|, \qquad |\mathbf{u} - \mathbf{v}|.$$

Since these lengths form the sides of a triangle, the Law of Cosines from Chapter 12 applies directly.

Theorem 13.2 (Geometric Formula for the Dot Product)**.** *For nonzero vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *with angle* θ *between them,*

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta.$$

Proof. Applying the Law of Cosines to the triangle formed by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} - \mathbf{v}$ gives

$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}|\,|\mathbf{v}| \cos\theta.$$

We now compute the same quantity using coordinates.
Since

$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1,\ u_2 - v_2 \rangle,$$

the magnitude formula gives

$$|\mathbf{u} - \mathbf{v}|^2 = (u_1 - v_1)^2 + (u_2 - v_2)^2.$$

Expanding,

$$|\mathbf{u} - \mathbf{v}|^2 = u_1^2 - 2u_1v_1 + v_1^2 + u_2^2 - 2u_2v_2 + v_2^2.$$

Rearranging terms,

$$|\mathbf{u} - \mathbf{v}|^2 = (u_1^2 + u_2^2) + (v_1^2 + v_2^2) - 2(u_1v_1 + u_2v_2).$$

Using the magnitude formula once again,

$$u_1^2 + u_2^2 = |\mathbf{u}|^2, \qquad v_1^2 + v_2^2 = |\mathbf{v}|^2,$$

so

$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2(u_1v_1 + u_2v_2).$$

Both derivations describe the same quantity $|\mathbf{u} - \mathbf{v}|^2$, so

$$|\mathbf{u}|^2 + |\mathbf{v}|^2 - 2(u_1 v_1 + u_2 v_2) = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}|\,|\mathbf{v}| \cos\theta.$$

Subtracting the common terms and dividing by -2 yields

$$u_1 v_1 + u_2 v_2 = |\mathbf{u}|\,|\mathbf{v}| \cos\theta.$$

Since the left-hand side is the algebraic definition of the dot product,

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta,$$

which completes the proof. □

The theorem reveals why the algebraic definition was chosen. The quantity

$$u_1 v_1 + u_2 v_2$$

appeared at first to be merely a computational rule, but the proof shows that it measures something fundamentally geometric. The dot product depends simultaneously on the lengths of the vectors and on the angle between them. Two vectors pointing in nearly the same direction produce a large positive dot product; vectors pointing in opposite directions produce a negative dot product; vectors meeting at a right angle produce zero.

The algebraic formula and the geometric formula therefore provide two different descriptions of the same quantity. The algebraic version is usually easier to compute from coordinates, while the geometric version often makes interpretation easier. One of the central advantages of the dot product is that it allows these two viewpoints to be translated into one another.

13.4 A Worked Example

The theorem becomes more convincing when both formulas are applied to the same pair of vectors.

Example 13.3. Let

$$\mathbf{u} = \langle 3, 4 \rangle, \qquad \mathbf{v} = \langle 4, 0 \rangle.$$

Using the algebraic definition,

$$\mathbf{u} \cdot \mathbf{v} = (3)(4) + (4)(0) = 12.$$

Now compute the same quantity geometrically.

The magnitudes are

$$|\mathbf{u}| = \sqrt{3^2 + 4^2} = 5,$$

and

$$|\mathbf{v}| = \sqrt{4^2} = 4.$$

The angle between the vectors satisfies

$$\cos\theta = \frac{3}{5},$$

since $\mathbf{u}$ forms a 3-4-5 triangle with the positive x-axis.

The geometric formula gives

$$\mathbf{u} \cdot \mathbf{v} = (5)(4)\left(\frac{3}{5}\right) = 12.$$

Both methods produce the same result, exactly as the theorem predicts.

13.5 Orthogonality

One of the most important consequences of the geometric formula appears when the angle between two vectors is a right angle.

Recall that two vectors are said to be *orthogonal* if they are perpendicular. In coordinate geometry, determining whether two vectors meet at a right angle might appear to require finding slopes or computing angles directly. The dot product provides a much simpler test.

Theorem 13.4. *Let* **u** *and* **v** *be nonzero vectors. Then*

$$\mathbf{u} \cdot \mathbf{v} = 0$$

if and only if

$$\mathbf{u} \perp \mathbf{v}.$$

Proof. By the geometric formula,

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta.$$

Since **u** and **v** are nonzero, their magnitudes are positive, so the product can equal zero only when

$$\cos\theta = 0.$$

This occurs precisely when

$$\theta = 90^\circ.$$

Thus

$$\mathbf{u} \cdot \mathbf{v} = 0$$

if and only if the vectors are perpendicular. □

The theorem converts a geometric question into a simple algebraic calculation. Instead of constructing right angles or comparing slopes, one need only compute a dot product and check whether the result is zero.

Example 13.5. Determine whether

$$\mathbf{u} = \langle 2, 3 \rangle, \qquad \mathbf{v} = \langle 3, -2 \rangle$$

are perpendicular.

Computing the dot product,

$$\mathbf{u} \cdot \mathbf{v} = (2)(3) + (3)(-2) = 6 - 6 = 0.$$

Since the dot product is zero, the vectors are orthogonal.

This criterion is often used in geometry, physics, computer graphics, and engineering because it avoids measuring angles directly. Orthogonality becomes an algebraic condition rather than a geometric construction.

13.6 Finding the Angle Between Two Vectors

The geometric formula can also be solved for the angle itself. Starting with

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta,$$

divide by the product of the magnitudes:

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}.$$

Provided neither vector is zero, the angle can then be obtained by applying the inverse cosine function introduced in Chapter 7:

$$\theta = \arccos\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right).$$

The formula transforms a geometric measurement into a sequence of algebraic computations.

Procedure 13.1 (How to Compute a Dot Product and the Angle Between Two Vectors). Given vectors **u** and **v**, first compute the dot product using the algebraic formula

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

Next compute the magnitudes

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2}, \qquad |\mathbf{v}| = \sqrt{v_1^2 + v_2^2}.$$

Then evaluate

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}.$$

Finally, apply the inverse cosine function:

$$\theta = \arccos\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right).$$

Example 13.6. Find the angle between

$$\mathbf{u} = \langle 1, 2 \rangle, \qquad \mathbf{v} = \langle 3, 1 \rangle.$$

First compute the dot product:

$$\mathbf{u} \cdot \mathbf{v} = (1)(3) + (2)(1) = 5.$$

The magnitudes are

$$|\mathbf{u}| = \sqrt{1^2 + 2^2} = \sqrt{5},$$

and

$$|\mathbf{v}| = \sqrt{3^2 + 1^2} = \sqrt{10}.$$

Therefore

$$\cos\theta = \frac{5}{\sqrt{5}\,\sqrt{10}} = \frac{1}{\sqrt{2}}.$$

Applying the inverse cosine,

$$\theta = \arccos\left(\frac{1}{\sqrt{2}}\right) = 45°.$$

The vectors meet at an angle of 45°.

Practice now. Find the angle between

$$\mathbf{u} = \langle 2, 1\rangle, \qquad \mathbf{v} = \langle 1, 4\rangle.$$

Compute the dot product first, then use the geometric formula to find the angle between the vectors.

13.7 Projection

The geometric formula for the dot product contains more information than the angle between two vectors. It also reveals how much of one vector lies in the direction of another.

Suppose two vectors **u** and **v** form an angle θ. If we drop a perpendicular from the endpoint of **u** to the line determined by **v**, the resulting right triangle shows that the length of the component of **u** in the direction of **v** is

$$|\mathbf{u}| \cos\theta.$$

This quantity is called the *scalar projection* of **u** onto **v**.
Because the geometric formula for the dot product is

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta,$$

dividing by $|\mathbf{v}|$ gives

$$|\mathbf{u}| \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}.$$

Thus the scalar projection can be computed directly from the dot product.

Definition 13.7. The *scalar projection* of **u** onto **v** is

$$\operatorname{comp}_{\mathbf{v}}(\mathbf{u}) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}.$$

The scalar projection measures a length. Positive values indicate that **u** points generally in the same direction as **v**, while negative values indicate that it points generally in the opposite direction.

Often, however, a length alone is not enough. We may wish to construct the actual vector lying along the direction of **v**.

To do so, first observe that

$$\frac{\mathbf{v}}{|\mathbf{v}|}$$

is a unit vector pointing in the same direction as **v**. If we multiply this unit vector by the scalar projection, we obtain the desired projection vector.

Definition 13.8. The *vector projection* of **u** onto **v** is

$$\operatorname{proj}_{\mathbf{v}}(\mathbf{u}) = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}\right) \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\mathbf{v}.$$

The projection vector points along $\mathbf{v}$ and has exactly the length needed to represent the component of $\mathbf{u}$ in that direction.

Example 13.9. Let

$$\mathbf{u} = \langle 4, 3\rangle, \qquad \mathbf{v} = \langle 2, 0\rangle.$$

First compute the dot product:

$$\mathbf{u} \cdot \mathbf{v} = (4)(2) + (3)(0) = 8.$$

Since

$$|\mathbf{v}| = 2,$$

the scalar projection is

$$\operatorname{comp}_{\mathbf{v}}(\mathbf{u}) = \frac{8}{2} = 4.$$

The vector projection is

$$\operatorname{proj}_{\mathbf{v}}(\mathbf{u}) = \frac{8}{2^2}\langle 2, 0\rangle = 2\langle 2, 0\rangle = \langle 4, 0\rangle.$$

The vector $\langle 4, 0\rangle$ is precisely the horizontal component of $\mathbf{u}$.

Projection provides a useful way to decompose vectors into parts. A vector can be viewed as containing one component parallel to a given direction and another component perpendicular to it. The projection isolates the parallel component, leaving the remainder to account for the perpendicular part.

13.8 Applications

The geometric interpretation of the dot product makes it useful whenever the effect of one quantity depends on how strongly it acts in a particular direction.

One of the most important examples occurs in physics. Suppose a force $\mathbf{F}$ moves an object through a displacement $\mathbf{d}$. Only the component of the force acting in the direction of motion contributes to the work performed. A force perpendicular to the motion changes the direction of travel but does no work.

If θ is the angle between the force and the displacement, then

$$W = |\mathbf{F}|\,|\mathbf{d}|\cos\theta.$$

Using the geometric formula for the dot product, this becomes

$$\boxed{W = \mathbf{F} \cdot \mathbf{d}.}$$

Work is therefore a dot product.

This interpretation explains why the cosine appears. The factor $\cos\theta$ extracts the component of the force acting along the direction of motion. When the force points entirely in the direction of travel, $\theta = 0^\circ$ and the work is maximized. When the force is perpendicular to the motion, $\theta = 90^\circ$ and the work is zero.

Example 13.10. A force

$$\mathbf{F} = \langle 10, 5 \rangle$$

moves an object through a displacement

$$\mathbf{d} = \langle 4, 0 \rangle.$$

The work performed is

$$W = \mathbf{F} \cdot \mathbf{d} = (10)(4) + (5)(0) = 40.$$

The vertical component of the force contributes nothing because the motion is entirely horizontal.

Dot products also appear in navigation, where movement in a particular direction must be isolated from a more complicated motion; in computer graphics, where lighting calculations depend on the angle between surfaces and light sources; and in data analysis, where geometric ideas about similarity are often expressed through dot products and related operations.

In each case the underlying idea is the same. The dot product measures not merely how large two vectors are, but how strongly they align with one another.

13.9 Looking Ahead

This chapter completes a thread that began when vectors were introduced in Chapter 2 and continued through the trigonometry of Chapters 8–12. The Law of Cosines, originally derived as a statement about triangles, has reappeared as the foundation of a general vector operation. Length, angle, orthogonality, and projection can now all be expressed within a single algebraic framework.

The next chapter shifts attention from vectors themselves to an alternative coordinate system. Instead of locating points by horizontal and vertical displacement, polar coordinates describe position using a distance and an angle. This change of viewpoint will make rotational structure more visible and will prepare the way for Chapters 15 and 16, where scaling and rotation are first combined into a single operator and then, together with vectors, absorbed into the single mathematical language of complex numbers.

13.10 Exercises

Computing Dot Products

Exercise 13.1. Compute each dot product.

(a)

$$\langle 2, 3 \rangle \cdot \langle 4, 1 \rangle$$

(b)

$$\langle -1, 5 \rangle \cdot \langle 2, 7 \rangle$$

(c)

$$\langle 6, -2 \rangle \cdot \langle -3, 4 \rangle$$

Exercise 13.2. Let

$$\mathbf{u} = \langle 3, 4 \rangle, \qquad \mathbf{v} = \langle 5, 2 \rangle.$$

Compute $\mathbf{u} \cdot \mathbf{v}$.

Exercise 13.3. Find the dot product of

$$\langle 7, 1 \rangle$$

and

$$\langle -2, 8 \rangle.$$

Geometric Interpretation

Exercise 13.4. Use the geometric formula

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$$

to compute the dot product of vectors with magnitudes 5 and 8 that meet at an angle of $60°$.

Exercise 13.5. Vectors $\mathbf{u}$ and $\mathbf{v}$ have magnitudes

$$|\mathbf{u}| = 6, \qquad |\mathbf{v}| = 10,$$

and the angle between them is $120°$.
Find $\mathbf{u} \cdot \mathbf{v}$.

Exercise 13.6. Verify the geometric formula for

$$\mathbf{u} = \langle 3, 4 \rangle, \qquad \mathbf{v} = \langle 4, 0 \rangle,$$

by computing the dot product both algebraically and geometrically.

Orthogonality

Exercise 13.7. Determine whether the following pairs of vectors are orthogonal.

(a)
$$\langle 2, 3 \rangle, \qquad \langle 3, -2 \rangle$$

(b)
$$\langle 4, 1 \rangle, \qquad \langle 2, -8 \rangle$$

(c)
$$\langle 5, -2 \rangle, \qquad \langle 1, 3 \rangle$$

Exercise 13.8. Find a vector orthogonal to

$$\langle 7, 4 \rangle.$$

Exercise 13.9. Find a nonzero vector orthogonal to

$$\langle -3, 8 \rangle.$$

Angles Between Vectors

Exercise 13.10. Find the angle between

$$\langle 1, 2 \rangle$$

and

$$\langle 3, 1 \rangle.$$

Exercise 13.11. Find the angle between

$$\langle 2, 1 \rangle$$

and

$$\langle 1, 4 \rangle.$$

Exercise 13.12. Find the angle between

$$\langle 4, 0 \rangle$$

and

$$\langle 2, 5 \rangle.$$

Exercise 13.13. Show that two nonzero vectors are perpendicular precisely when the angle formula yields

$$\theta = 90^\circ.$$

Projection

Exercise 13.14. Find the scalar projection of

$$\mathbf{u} = \langle 6, 2 \rangle$$

onto

$$\mathbf{v} = \langle 3, 0 \rangle.$$

Exercise 13.15. Find both the scalar projection and vector projection of

$$\mathbf{u} = \langle 4, 3 \rangle$$

onto

$$\mathbf{v} = \langle 2, 0 \rangle.$$

Exercise 13.16. Find

$$\mathrm{proj}_{\mathbf{v}}(\mathbf{u})$$

for

$$\mathbf{u} = \langle 5, 1 \rangle, \qquad \mathbf{v} = \langle 1, 1 \rangle.$$

Exercise 13.17. A vector is decomposed into a component parallel to a given direction and a component perpendicular to it. Explain how projection provides the parallel component.

Applications

Exercise 13.18. A force

$$\mathbf{F} = \langle 8, 3 \rangle$$

moves an object through a displacement

$$\mathbf{d} = \langle 5, 0 \rangle.$$

Find the work performed.

Exercise 13.19. A force

$$\mathbf{F} = \langle 12, 5 \rangle$$

acts through a displacement

$$\mathbf{d} = \langle 3, 4 \rangle.$$

Compute the work.

Exercise 13.20. Explain why a force perpendicular to a displacement performs no work.

Exercise 13.21. A boat moves due east while a current pushes due north. Explain, using the dot product, why the current contributes no work in the direction of travel.

Mixed Problems

Exercise 13.22. For

$$\mathbf{u} = \langle 2, 5 \rangle, \qquad \mathbf{v} = \langle 4, -1 \rangle,$$

compute:

(a) the dot product,

(b) the angle between the vectors,

(c) the scalar projection of $\mathbf{u}$ onto $\mathbf{v}$.

Exercise 13.23. Determine whether the triangle formed by the vectors

$$\langle 4, 1\rangle$$

and

$$\langle 1, -4\rangle$$

contains a right angle.

Exercise 13.24. Show that

$$\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2$$

for every vector $\mathbf{u}$.

Exercise 13.25. Use the result of the previous exercise to derive the magnitude formula

$$|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}}.$$

Conceptual and Exploratory Problems

Exercise 13.26. The algebraic definition of the dot product appears to depend on coordinates, while the geometric formula refers only to lengths and angles. Explain why the theorem relating the two descriptions is important.

Exercise 13.27. Why does the dot product become negative when the angle between two vectors exceeds 90°?

Exercise 13.28. Describe the geometric meaning of the quantity

$$|\mathbf{u}|\cos\theta.$$

Exercise 13.29. Suppose two vectors have fixed lengths. How does their dot product change as the angle between them increases from 0° to 180°?

Exercise 13.30. The geometric formula for the dot product was derived from the Law of Cosines. Explain how this chapter extends the results of Chapter 12 from triangles to vectors.

Challenge Problems

Exercise 13.31. Prove the geometric formula for the dot product by beginning with

$$|\mathbf{u}+\mathbf{v}|^2$$

instead of

$$|\mathbf{u} - \mathbf{v}|^2.$$

Exercise 13.32. Show that

$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2(\mathbf{u} \cdot \mathbf{v}).$$

Explain how this identity reflects the proof of the geometric formula.

Exercise 13.33. Find all vectors

$$\langle x, y \rangle$$

that are orthogonal to

$$\langle 3, 4 \rangle.$$

Exercise 13.34. Suppose the algebraic definition of the dot product did not agree with the geometric formula. What geometric properties of vectors would fail? Discuss how orthogonality, angle measurement, and projection would be affected.

Chapter 14

Polar Coordinates

14.1 A Different Way to Locate a Point

Throughout this book, points in the plane have been described using Cartesian coordinates. A point is located by measuring a horizontal displacement and a vertical displacement from the origin, producing an ordered pair

$$(x, y).$$

This system is natural whenever horizontal and vertical directions play a special role, and it has been used repeatedly since Chapter 2. Yet many geometric situations are concerned not with horizontal and vertical motion, but with distance and direction. A navigator determining the location of a ship may record a bearing and a range; a radar station may measure an angle and a distance; an astronomer may specify the direction and apparent separation of an object from a fixed observation point.

Trigonometry has spent much of this book developing exactly these two quantities. Distances were introduced through the geometry of triangles and vectors, while angles became increasingly important through rotation, circular motion, and periodic functions. It is therefore natural to ask what happens if position is described directly in terms of distance and angle rather than horizontal and vertical displacement.

The resulting coordinate system is called the *polar coordinate system*. Instead of measuring how far a point lies along two perpendicular axes, it measures how far the point lies from the origin and in what direction it is found.

14.2 Polar Coordinates

In the polar coordinate system, the origin is called the *pole*, and the positive x-axis serves as the *polar axis*. A point is located by two numbers:

$$(r, \theta),$$

where r is the distance from the pole and θ is the angle measured from the positive x-axis.

Definition 14.1. A point in *polar coordinates* is represented by an ordered pair (r, θ), where r is the distance from the origin and θ is the angle measured from the positive x-axis.

The angle convention is exactly the one established earlier for the unit circle. Positive angles are measured counterclockwise and negative angles clockwise. No new notion of angle is being introduced; the polar coordinate system simply uses the rotational framework already developed throughout the book.

For example, $(4, \pi/6)$ represents the point lying four units from the origin in the direction determined by an angle of $\pi/6$ radians. Likewise, $(2, 3\pi/4)$ represents the point two units from the origin at an angle of 135°. The distance and direction are given directly rather than being inferred from rectangular coordinates.

14.3 Converting Polar Coordinates to Rectangular Coordinates

The relationship between polar and Cartesian coordinates follows directly from the trigonometry of the unit circle. Consider a point with polar coordinates (r, θ). Dropping a perpendicular from the point to the x-axis forms a right triangle whose hypotenuse has length r and whose acute angle is θ. From the definitions of sine and cosine, $\cos\theta = x/r$ and $\sin\theta = y/r$, and solving for x and y gives the conversion formulas.

Theorem 14.2. *If a point has polar coordinates (r, θ), then its rectangular coordinates are $x = r\cos\theta$ and $y = r\sin\theta$.*

The formulas should appear familiar. They are exactly the coordinate equations of the unit circle, except that the radius is no longer fixed at one. Every point is obtained by taking the unit-circle coordinates associated with θ and scaling them by the distance r.

Example 14.3. Convert $\left(6, \dfrac{\pi}{3}\right)$ to rectangular coordinates. Using the conversion formulas,

$$x = 6\cos\frac{\pi}{3} = 6\left(\frac{1}{2}\right) = 3, \qquad y = 6\sin\frac{\pi}{3} = 6\left(\frac{\sqrt{3}}{2}\right) = 3\sqrt{3}.$$

Therefore the rectangular coordinates are $(3, 3\sqrt{3})$.

14.4 Converting Rectangular Coordinates to Polar Coordinates

The conversion from polar coordinates to rectangular coordinates follows directly from the definitions of sine and cosine. The reverse conversion requires a little more care. Suppose a point is given in rectangular coordinates as (x, y). The distance from the origin is immediately determined by the distance formula, since the point, the origin, and the coordinate axes form a right triangle: $r^2 = x^2 + y^2$, so

$$r = \sqrt{x^2 + y^2}.$$

This should look familiar. It is exactly the magnitude formula for vectors introduced in Chapter 2; the radial coordinate r is simply the length of the position vector from the origin to the point.

To determine the angle, observe that $\tan\theta = y/x$, provided $x \neq 0$, which suggests $\theta = \arctan(y/x)$. This expression is not always correct. The difficulty arises because the tangent function does not uniquely determine a quadrant: the points $(1, 1)$ and $(-1, -1)$ both satisfy $y/x = 1$, yet they lie in entirely different directions from the origin. The inverse tangent function introduced in Chapter 7 returns values only in its principal range, $-\pi/2 < \theta < \pi/2$, which covers Quadrants I and IV but excludes Quadrants II and III. The value returned by $\arctan(y/x)$ must therefore be interpreted in light of the point's actual location.

Theorem 14.4. *If a point has rectangular coordinates (x, y), then $r = \sqrt{x^2 + y^2}$, and the angle θ is determined by $\tan\theta = y/x$ together with the requirement that θ lie in the correct quadrant.*

The quadrant check is not a technical detail but an essential part of the conversion process; forgetting it often produces a direction that differs from the correct answer by π radians.

Example 14.5. Convert the point $(-3, 3)$ to polar coordinates. First, $r = \sqrt{(-3)^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$. Next, $\tan\theta = 3/(-3) = -1$, and $\arctan(-1) = -\pi/4$. The point, however, lies in Quadrant II, not Quadrant IV, so the correct angle is $\theta = 3\pi/4$. Hence the polar coordinates are $\left(3\sqrt{2}, \dfrac{3\pi}{4}\right)$.

14.5 Non-Uniqueness of Polar Coordinates

One of the most surprising features of polar coordinates is that they do not provide a unique representation of a point. In the rectangular system, a point corresponds to exactly one ordered pair: $(3, 4)$ cannot also be represented by some different rectangular coordinate pair, and the relationship between points and coordinates is one-to-one. Polar coordinates behave differently. Consider the point $\left(2, \dfrac{\pi}{3}\right)$; if one full revolution is added to the angle, the direction does not change, so $\left(2, \dfrac{\pi}{3}\right) = \left(2, \dfrac{\pi}{3} + 2\pi\right)$, and adding further revolutions produces infinitely many coordinate pairs representing the same point.

Theorem 14.6. *For any integer k, $(r, \theta) = (r, \theta + 2\pi k)$.*

This result is a direct consequence of the periodicity of sine and cosine; the coordinate system inherits the same rotational symmetry already encountered in the study of coterminal angles in Chapter 4. An even more surprising phenomenon occurs when negative values of r are allowed. The point (r, θ) lies exactly opposite the point $(r, \theta + \pi)$ on the same line through the origin, so changing the sign of the radius reverses that direction, bringing the point back to its original location, and consequently $(r, \theta) = (-r, \theta + \pi)$.

Example 14.7. The point $\left(3, \dfrac{\pi}{6}\right)$ can also be written as $\left(-3, \dfrac{7\pi}{6}\right)$; both descriptions represent the same point in the plane.

The non-uniqueness of polar coordinates is not a defect of the system; it is a consequence of describing position through distance and rotation. Rotations naturally repeat every full revolution, and the coordinate system reflects that periodicity directly.

Procedure 14.1 (How to Convert Between Rectangular and Polar Coordinates).
To convert from polar coordinates (r, θ) to rectangular coordinates, use $x = r\cos\theta$ and $y = r\sin\theta$. To convert from rectangular coordinates (x, y) to polar coordinates, first compute $r = \sqrt{x^2 + y^2}$, then compute $\tan\theta = y/x$ and determine the appropriate quadrant from the signs of x and y, and finally choose an angle θ in that quadrant satisfying the tangent equation.

Example 14.8. Convert the point $(-4, -4)$ to polar coordinates. First, $r = \sqrt{(-4)^2 + (-4)^2} = \sqrt{32} = 4\sqrt{2}$. Next, $\tan\theta = (-4)/(-4) = 1$, and $\arctan(1) = \pi/4$; the point, however, lies in Quadrant III, whose corresponding angle is $\theta = 5\pi/4$. Therefore $(-4, -4) = \left(4\sqrt{2}, \dfrac{5\pi}{4}\right)$.

Practice now. Convert the point $(2, -2\sqrt{3})$ to polar coordinates, choosing an angle in the correct quadrant.

14.6 Graphing in Polar Coordinates

Changing coordinates does more than alter notation; it changes which geometric objects appear simple. In Cartesian coordinates, a horizontal line is described by an equation as simple as $y = 3$, precisely because the coordinate system itself is built from horizontal and vertical directions. Polar coordinates instead privilege distance and angle, so objects defined naturally by distance from the origin or by direction often acquire remarkably simple equations of their own.

Circles

Consider the equation $r = a$, where a is a positive constant. Every point satisfying this equation lies exactly a units from the origin, so the set of all such points is a circle of radius a centered at the origin. A circle that requires $x^2 + y^2 = a^2$ in Cartesian coordinates becomes simply $r = a$ in polar coordinates, a dramatic simplification that is one of the principal reasons for introducing the system.

Example 14.9. Sketch $r = 4$. Every point lies four units from the origin, so the graph is a circle of radius four centered at the pole.

Lines Through the Origin

Now consider $\theta = c$, where c is a constant angle. Every point satisfying this equation lies in the same direction from the origin; the distance may vary, but the angle remains fixed, so the graph is a line through the origin making angle c with the positive x-axis.

Example 14.10. The equation $\theta = \pi/4$ describes the line through the origin inclined at 45°.

Together, $r = a$ and $\theta = c$ illustrate a recurring theme of coordinate systems: some objects become simpler while others become more complicated, and the usefulness of a coordinate system depends on whether its notion of simplicity aligns with the geometry under investigation.

14.7 Periodic Functions as Curves

One of the most interesting consequences of polar coordinates appears when trigonometric functions are used to define the radius. In Chapter 6, sine and cosine were studied as periodic functions whose outputs changed as an angle varied. In polar coordinates, those outputs can be interpreted as distances from the origin, so as θ changes, the radius changes as well, producing a curve in the plane; the process transforms a periodic function into a geometric figure.

Consider $r = a\cos(n\theta)$ or $r = a\sin(n\theta)$. These equations generate a family of curves known as *rose curves*: the repeated oscillation of the sine or cosine function causes the graph to move alternately toward and away from the origin, producing a pattern of petals arranged around the pole, with the number of petals depending on n and the constant a controlling their size. The periodicity that once produced repeating waves along a horizontal axis now produces repeating geometric structures distributed around a circle.

Example 14.11. Consider $r = 4\cos(2\theta)$. As θ varies, the cosine function oscillates between -1 and 1, so the radius varies between -4 and 4. At $\theta = 0$, $r = 4$, so the graph begins four units to the right of the origin. At $\theta = \pi/4$, $r = 0$, so the curve passes through the pole. At $\theta = \pi/2$, $r = -4$; a negative radius places the point four units in the *opposite* direction, creating a petal above the origin. As θ continues to increase, the same pattern repeats, and the result is a four-petaled rose centered at the pole.

The rose curve provides a striking example of how periodicity and geometry interact: a simple oscillation in the radius becomes a symmetric two-dimensional pattern, and the repetition formerly visible in the graph of a function over time is now visible as rotational symmetry in space.

Cardioids

A second important family of polar curves is given by equations of the form $r = a(1 + \cos\theta)$ or $r = a(1 + \sin\theta)$, called *cardioids*, from the Greek word for "heart," because of their characteristic shape. Unlike a rose curve, a cardioid does not repeatedly pass through the origin; instead the radius grows and shrinks smoothly, touching the origin at a single cusp.

Example 14.12. Consider $r = 2(1 + \cos\theta)$. At $\theta = 0$, the radius reaches its maximum value, $r = 4$; at $\theta = \pi$, $r = 0$, so the curve passes through the pole. As θ moves between these extremes, the radius varies continuously, producing the familiar cardioid shape.

Cardioids appear naturally in optics, acoustics, and signal processing; many microphone pickup patterns, for example, are approximately cardioid, capturing sound more strongly from one direction than another.

Spirals

Not all polar curves are periodic. Consider $r = a\theta$: as the angle increases, the distance from the origin increases as well, so the point continually rotates while simultaneously moving outward, tracing a curve called an *Archimedean spiral*.

Example 14.13. The equation $r = \theta$ describes a spiral beginning at the origin. At $\theta = 0$, $r = 0$; at $\theta = \pi$, $r = \pi$; at $\theta = 2\pi$, $r = 2\pi$. Each complete revolution increases the distance from the origin by the same amount, causing the curve to wind steadily outward.

Unlike circles, roses, and cardioids, spirals do not close upon themselves; they continue indefinitely, illustrating that polar coordinates can naturally describe both periodic and non-periodic geometric behavior.

Comparative Note: Writing Systems and Alternative Representations.

In Arabic script, a single letter often appears in several different forms depending on its position within a word: an isolated form used on its own, and distinct initial, medial, and final forms used according to which neighboring letters it joins. The letter commonly transliterated *'ayn*, for instance, has four such shapes, one for each of these positions, all of which are read as occurrences of the same underlying letter despite looking visibly different on the page. Polar coordinates exhibit a directly analogous phenomenon: the pairs (r, θ), $(r, \theta + 2\pi)$, and $(-r, \theta + \pi)$ may all represent the same location, even though the coordinate pairs themselves are different. In both cases, the representation changes according to context while the underlying object it denotes remains the same; what looks like a difference in description is not necessarily a difference in the thing being described.

14.8 Applications

Polar coordinates are particularly useful whenever distance and direction are more natural than horizontal and vertical displacement. In navigation, positions are often specified by a bearing and a range: a lighthouse may be reported as lying ten kilometers away at a bearing of 40° north of east, information already expressed in polar form and requiring no conversion. In physics, many systems possess rotational symmetry; the motion of a planet around a star, the orbit of a satellite, or the behavior of a particle moving under a central force are often easier to describe using distance from a center and angular position than by repeatedly tracking horizontal and vertical coordinates. Engineers frequently use polar plots to display directional information, since antenna radiation patterns, microphone sensitivity diagrams, and radar coverage maps are commonly expressed as functions of angle, making polar coordinates the natural framework for visualizing them. The usefulness of polar coordinates arises from the same principle that motivates every coordinate system: a good representation is one in which the important structure of a problem becomes simple.

14.9 Looking Ahead

Polar coordinates describe a point using two quantities, (r, θ): the first records magnitude, the second records direction. Throughout this chapter these quantities have been treated as separate numbers joined together into an ordered pair, yet many of the geometric operations studied earlier naturally involve both simultaneously. Rotations alter θ while preserving r; scaling changes r while preserving θ; the trigonometric functions repeatedly connect the two. This raises a natural question: must magnitude and direction remain separate, or can they be combined into a single mathematical object? The next chapter takes up exactly this question, unifying the scaling transformation of Chapter 1 and the rotation transformation of Chapter 8 into a single operator whose two parameters are precisely the amplitude and angle already familiar from this chapter, before Chapter 16 shows that same combination reappearing as ordinary multiplication in an extended number system.

14.10 Exercises

Converting Coordinates

Exercise 14.1. Convert each polar coordinate pair to rectangular coordinates: (a) $\left(4, \dfrac{\pi}{6}\right)$; (b) $\left(6, \dfrac{3\pi}{4}\right)$; (c) $(5, \pi)$.

Exercise 14.2. Convert each rectangular coordinate pair to polar coordinates, giving an exact value of θ whenever possible: (a) $(3, 3\sqrt{3})$; (b) $(-4, 4)$; (c) $(-5, 0)$.

Exercise 14.3. Convert $(2, -2\sqrt{3})$ to polar coordinates.

Exercise 14.4. Convert $(-3, -3)$ to polar coordinates.

Quadrants and Angle Determination

Exercise 14.5. For each point, determine the quadrant in which the angle θ lies before computing the polar coordinates: (a) $(-2, 5)$; (b) $(-7, -1)$; (c) $(4, -6)$.

Exercise 14.6. Explain why the formula $\theta = \arctan(y/x)$ cannot by itself determine the correct polar angle.

Exercise 14.7. Find two different points having the same value of y/x but lying in different quadrants, and explain why they require different polar angles.

Equivalent Polar Representations

Exercise 14.8. Find three different polar representations of $\left(3, \frac{\pi}{4}\right)$.

Exercise 14.9. Show that $\left(5, \frac{\pi}{6}\right)$ and $\left(-5, \frac{7\pi}{6}\right)$ represent the same point.

Exercise 14.10. Find a positive-radius representation and a negative-radius representation of the point $\left(2, \frac{2\pi}{3}\right)$.

Exercise 14.11. Explain why every polar coordinate pair has infinitely many equivalent representations.

Graphing in Polar Coordinates

Exercise 14.12. Describe the graph of each equation: (a) $r = 3$; (b) $r = 7$; (c) $\theta = \frac{\pi}{3}$.

Exercise 14.13. Sketch the graphs of $r = 2$, $r = 5$, and $\theta = \frac{\pi}{4}$.

Exercise 14.14. Explain why the equation $r = a$ always represents a circle centered at the origin.

Exercise 14.15. Explain why the equation $\theta = c$ always represents a line through the origin.

Polar Curves

Exercise 14.16. For each value of θ, compute the corresponding value of r for $r = 4\cos(2\theta)$ when $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$.

Exercise 14.17. Describe the effect of increasing the constant a in the equation $r = a\cos(2\theta)$.

Exercise 14.18. Sketch a rough graph of $r = 2(1 + \cos\theta)$.

Exercise 14.19. Compute several points on the spiral $r = \theta$ for $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.

Exercise 14.20. Explain why the spiral $r = \theta$ does not close upon itself, while a rose curve does.

Applications

Exercise 14.21. A radar station detects an aircraft at distance 120 km and bearing 35° north of east. Express the aircraft's position in polar coordinates.

Exercise 14.22. A ship is located at $\left(80, \frac{\pi}{6}\right)$ in polar coordinates relative to a lighthouse. Convert the position to rectangular coordinates.

Exercise 14.23. Explain why polar coordinates are often preferable to rectangular coordinates when describing circular motion.

Exercise 14.24. Research one application of polar plots in science or engineering, and write a short paragraph describing how polar coordinates are used.

Conceptual and Exploratory Problems

Exercise 14.25. Compare the point $(3, 4)$ in rectangular coordinates with the same point represented in polar coordinates. What information is emphasized by each description?

Exercise 14.26. Describe the relationship between the conversion formulas $x = r\cos\theta$, $y = r\sin\theta$ and the unit-circle definitions of sine and cosine.

Exercise 14.27. Why do circles centered at the origin become simpler in polar coordinates than in rectangular coordinates?

Exercise 14.28. The graph of $r = a\cos(n\theta)$ is generated by a periodic function. Explain how periodicity in θ produces repeated geometric structure in the plane.

Exercise 14.29. A point may have many different polar representations but only one rectangular representation. Explain why this difference occurs.

Challenge Problems

Exercise 14.30. Show algebraically that $(r, \theta) = (-r, \theta + \pi)$, using the conversion formulas to prove the result.

Exercise 14.31. Suppose (r_1, θ_1) and (r_2, θ_2) represent the same point and both radii are positive. Prove that $\theta_2 - \theta_1 = 2\pi k$ for some integer k.

Exercise 14.32. Find all polar representations of the point $(0, 0)$, and explain why the origin is exceptional in the polar coordinate system.

Exercise 14.33. Polar coordinates describe a point using distance and direction. Explain why this makes them a natural precursor to the amplitude-and-angle representation developed in the next chapter.

Chapter 15

Amplitwists: Scaling and Rotation Combined

15.1 Scaling and Rotation as a Single Transformation

In Chapter 1 we introduced scaling transformations of the form $(x, y) \mapsto (kx, ky)$, which enlarge or shrink figures while preserving their shape. In Chapter 8 we introduced rotations, represented by the matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

which preserve lengths while changing orientation. These transformations were studied separately: scaling changes size but not direction, and rotation changes direction but not size. A natural question therefore arises, whether scaling and rotation can be combined into a single transformation. The answer is yes, and this chapter develops a unified operator that performs both actions simultaneously.

15.2 Scaling as a Matrix

Recall that scaling every vector by a factor s can be written as

$$S(s) = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}.$$

Applying this matrix to a vector $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ gives $S(s)\mathbf{v} = \begin{pmatrix} sx \\ sy \end{pmatrix}$: every coordinate is multiplied by the same factor, so every length is multiplied by s.

Example 15.1. Let $\mathbf{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Applying the scaling matrix $S(2) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ produces $S(2)\mathbf{v} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$. The original length was $\sqrt{3^2 + 4^2} = 5$; the new length is $\sqrt{6^2 + 8^2} = 10$. The vector has doubled in length while preserving its direction.

15.3 Rotation Revisited

A rotation through an angle θ is represented by $R(\theta)$ as above; applying $R(\theta)$ changes the direction of a vector without changing its length. From Chapter 8, recall the composition rule $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$, which will play a central role in what follows.

15.4 The Amplitwist

Definition 15.2. An *amplitwist* is a transformation obtained by combining a scaling factor $s > 0$ with a rotation through an angle θ: $A = sR(\theta)$.

Historical Note: The Origin of the Term "Amplitwist".

The term *amplitwist* was coined by Tristan Needham in his textbook *Visual Complex Analysis* [3], where it names the local action of a complex-differentiable function: near a point a, such a function acts on an infinitesimally small displacement as multiplication by a fixed complex number, simultaneously amplifying its length and twisting its direction, hence *amplitwist*. This chapter borrows Needham's term and geometric picture but restricts it to a single global transformation of the plane, $A = sR(\theta)$, applied everywhere at once, rather than a transformation that varies infinitesimally from point to point as it does in Needham's original setting. Chapter 17's treatment of derivatives, and in particular its identification of differentiation with a quarter-turn rotation, is the closest this book comes to Needham's full picture: a differentiable function's derivative at a point is, in exactly this sense, the amplitwist that best approximates the function near that point.

Expanding the matrix gives

$$A = \begin{pmatrix} s\cos\theta & -s\sin\theta \\ s\sin\theta & s\cos\theta \end{pmatrix}.$$

The parameter s controls size while θ controls orientation.

Example 15.3. Let $A = 2R(30°)$. Applying A to $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ gives

$$A\mathbf{v} = \begin{pmatrix} 2\cos 30° \\ 2\sin 30° \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}.$$

The vector has been rotated by 30° and its length has doubled.

15.5 Geometric Meaning

An amplitwist preserves all angles and shape relationships; it changes only the overall size of a figure and its orientation, altering neither the relative proportions between the figure's parts nor the angles between its edges. When $\theta = 0$, the amplitwist reduces to a pure scaling transformation, $A = sR(0) = S(s)$, so every similarity transformation fixing the origin is an amplitwist.

15.6 Composition of Amplitwists

The central result of this chapter is that amplitwists compose in a remarkably simple way.

Theorem 15.4. *If $A_1 = s_1R(\theta_1)$ and $A_2 = s_2R(\theta_2)$, then $A_1A_2 = s_1s_2R(\theta_1 + \theta_2)$.*

Proof. Using matrix multiplication, $A_1A_2 = (s_1R(\theta_1))(s_2R(\theta_2))$. Since scalars commute with matrices, $A_1A_2 = s_1s_2R(\theta_1)R(\theta_2)$. By the rotation composition theorem of Chapter 8, $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$, so $A_1A_2 = s_1s_2R(\theta_1 + \theta_2)$: amplitudes multiply while angles add. □

Procedure 15.1 (How to Compose Two Amplitwists).

Given amplitwists (s_1, θ_1) and (s_2, θ_2), multiply the amplitudes to obtain $s = s_1 s_2$, add the angles to obtain $\theta = \theta_1 + \theta_2$, and write the result as the amplitwist (s, θ).

Practice now. Compose the amplitwists $(3, 50^\circ)$ and $\left(\frac{1}{2}, 110^\circ\right)$, giving your answer both in amplitwist coordinates and as $sR(\theta)$.

15.7 Identity and Inverses

The identity amplitwist is $I = 1 \cdot R(0)$; applying I leaves every vector unchanged.

Theorem 15.5. *The inverse of $A = sR(\theta)$ is $A^{-1} = \frac{1}{s}R(-\theta)$.*

Proof. Compute $AA^{-1} = sR(\theta)\left(\frac{1}{s}R(-\theta)\right)$, which simplifies to $R(\theta)R(-\theta)$. Using rotation composition, $AA^{-1} = R(0) = I$, so $A^{-1} = \frac{1}{s}R(-\theta)$. □

15.8 Amplitwist Coordinates

Every amplitwist can be represented by the ordered pair (s, θ), and composition becomes $(s_1, \theta_1) \circ (s_2, \theta_2) = (s_1 s_2, \theta_1 + \theta_2)$. Two different arithmetic operations appear simultaneously in this single rule: scales are multiplied while angles are added. This unusual combination will reappear, in a new guise, in the next chapter.

15.9 Repeated Amplitwists and Spirals

Repeated application of an amplitwist produces a simple pattern.

Theorem 15.6. *For every positive integer n, $A^n = s^n R(n\theta)$.*

Proof. We proceed by induction. For $n = 1$, $A^1 = sR(\theta)$, which is true. Assume $A^n = s^n R(n\theta)$; then $A^{n+1} = A^n A$, and substituting the inductive hypothesis gives $A^{n+1} = s^n R(n\theta) \cdot sR(\theta)$. Using the composition theorem, $A^{n+1} = s^{n+1} R((n+1)\theta)$, so the formula holds for $n+1$, completing the induction. □

Repeated amplitwists generate spiral trajectories: if $s > 1$, the spiral expands outward; if $0 < s < 1$, it contracts inward; and if $s = 1$, the motion becomes pure rotation around a circle, since the amplitude never changes at all.

15.10 Applications

Amplitwists appear naturally in many fields. In computer graphics, objects are frequently rotated and scaled simultaneously; in robotics, coordinate systems are continually transformed as machines move through space; in signal processing, amplitude and phase changes occur together in oscillatory systems; and in fractal geometry and iterated function systems, repeated amplitwists generate self-similar structures and spiral patterns.

15.11 Looking Ahead

The composition law $(s_1, \theta_1) \circ (s_2, \theta_2) = (s_1 s_2, \theta_1 + \theta_2)$ may seem unusual at first. In the next chapter we will discover that complex numbers obey exactly this same rule: multiplying complex numbers turns out to be nothing more than amplitwist composition written in algebraic form. Complex numbers therefore provide a numerical language for the geometry developed in this chapter.

15.12 Exercises

Basic Computations

Exercise 15.1. Let $A = 2R(45^\circ)$. Apply A to the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and describe both the change in length and the change in direction.

Exercise 15.2. Let $A = 3R(90^\circ)$. Apply A to $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Exercise 15.3. Write the amplitwist $4R(30^\circ)$ as a single 2×2 matrix.

Exercise 15.4. Write the amplitwist $\frac{1}{2}R(120^\circ)$ as a single 2×2 matrix.

Exercise 15.5. Determine the image of $\begin{pmatrix} 4 \\ 0 \end{pmatrix}$ under the amplitwist $2R(60^\circ)$.

Composition of Amplitwists

Exercise 15.6. Compute $(2, 30^\circ) \circ (3, 40^\circ)$, expressing the answer both in amplitwist coordinates and as $sR(\theta)$.

Exercise 15.7. Compute $(5, 15^\circ) \circ \left(\frac{1}{2}, 75^\circ\right)$.

Exercise 15.8. Verify directly by matrix multiplication that $2R(30^\circ) \cdot 3R(45^\circ) = 6R(75^\circ)$.

Exercise 15.9. Compose the amplitwists $(4, 120^\circ)$ and $\left(\frac{1}{4}, -120^\circ\right)$. What transformation results?

Exercise 15.10. Find the amplitwist obtained by composing $(2, 90^\circ)$, $(2, 90^\circ)$, and $(2, 90^\circ)$.

Identity and Inverses

Exercise 15.11. Find the inverse of $A = 5R(40°)$.

Exercise 15.12. Find the inverse of $A = \frac{1}{3}R(-120°)$.

Exercise 15.13. Verify directly that $AA^{-1} = I$ for $A = 2R(60°)$.

Exercise 15.14. Determine the amplitwist whose composition with $(7, -25°)$ produces the identity.

Exercise 15.15. Explain geometrically why the inverse amplitwist must reverse both the scaling and the rotation.

Amplitwist Coordinates

Exercise 15.16. Express the composition $(3, 20°) \circ (4, 10°)$ using amplitwist coordinates only.

Exercise 15.17. Compute $(2, 45°)^2$.

Exercise 15.18. Compute $(2, 45°)^3$.

Exercise 15.19. Compute $(2, 45°)^4$. What familiar transformation appears?

Exercise 15.20. Explain why amplitudes multiply but angles add when amplitwists are composed.

Repeated Amplitwists and Spirals

Exercise 15.21. Let $A = 1.2R(30°)$. Compute A, A^2, A^3, A^4, and describe the resulting sequence geometrically.

Exercise 15.22. Let $A = 0.8R(45°)$. Compute A^5. Is the resulting orbit expanding or contracting?

Exercise 15.23. Let $A = R(72°)$. Compute A^5. What does this reveal about repeated rotations?

Exercise 15.24. Starting with $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, apply $A = 1.1R(40°)$ repeatedly, and sketch the first six points.

Exercise 15.25. Describe the qualitative behavior of repeated amplitwists when $s > 1$, $0 < s < 1$, and $s = 1$.

Proofs and Theory

Exercise 15.26. Prove by induction that $A^n = s^n R(n\theta)$ for every positive integer n.

Exercise 15.27. Show that the set of all amplitwists is closed under composition.

Exercise 15.28. Show that the composition of two amplitwists is always an amplitwist.

Exercise 15.29. Show that every amplitwist has a unique inverse.

Exercise 15.30. Prove that the identity amplitwist is $(1, 0)$.

Challenge Problems

Exercise 15.31. Let $A = sR(\theta)$. Determine all amplitwists satisfying $A^2 = I$.

Exercise 15.32. Determine all amplitwists satisfying $A^3 = I$.

Exercise 15.33. Suppose $A = sR(\theta)$ and $B = tR(\phi)$. Show that $AB = BA$.

Exercise 15.34. Prove that the amplitwist coordinates form a group under composition.

Exercise 15.35. Show that every orientation-preserving similarity transformation fixing the origin can be written uniquely as $A = sR(\theta)$.

Exercise 15.36. Explain why amplitwist composition resembles a new arithmetic. What quantities are being multiplied, and what quantities are being added?

Chapter 16

Complex Numbers and Euler's Formula

16.1 Extending the Number System

Chapter 15 studied amplitwists, $A = sR(\theta)$, and found that they compose according to the rule $(s_1, \theta_1) \circ (s_2, \theta_2) = (s_1 s_2, \theta_1 + \theta_2)$, a rule that combines two different arithmetic operations, multiplication and addition, in a single formula. The purpose of this chapter is to show that an arithmetic answering to exactly this description already exists: complex numbers provide an algebraic language for amplitwists, and multiplying complex numbers turns out to perform exactly the same operation as composing amplitwists. Before that connection can be established, the number system itself must first be extended.

The natural numbers arise from counting. The integers extend the natural numbers by introducing negative quantities, allowing subtraction to be performed universally. The rational numbers extend the integers by introducing fractions, allowing division by any nonzero number. The real numbers extend the rationals to accommodate measurement and continuity. Each extension solves equations that could not previously be solved: $x+3 = 1$ has no solution among the natural numbers but has the integer solution $x = -2$; $2x = 1$ has no integer solution but has the rational solution $x = 1/2$; and the real numbers solve equations such as $x^2 = 2$, whose solutions are not themselves rational. A natural question therefore arises: does the equation $x^2 + 1 = 0$, which may be rewritten as $x^2 = -1$, have any solution at all? No real number has a square equal to a negative quantity, so no real solution exists, and to solve this equation a new kind of number must be introduced.

16.2 The Imaginary Unit

Definition 16.1. The *imaginary unit* is the number i satisfying $i^2 = -1$.

Historical Note: A Dismissive Name That Stuck.

The word *imaginary* was coined by René Descartes, the same figure Chapter 2 credited with Cartesian coordinates, and he did not intend it as a compliment [5]. In his 1637 *La Géométrie*, Descartes used *nombres imaginaires* to describe roots of equations that had no correspondence to any measurable length or quantity, essentially dismissing them as fictions useful for bookkeeping in an algebraic calculation but not real numbers in any meaningful sense. The name persisted long after mathematicians including Euler, Gauss, and others showed that these numbers had a perfectly concrete geometric meaning, as the points of a plane rather than points on a line, exactly the picture this chapter develops through the complex plane and Euler's formula. By the time the geometry was understood, the dismissive name had already become standard usage, and no replacement has displaced it since; the complex numbers of this chapter are, etymologically, still named after Descartes' doubt that they were real numbers at all.

This definition immediately produces a repeating pattern:

$$i^1 = i, \qquad i^2 = -1, \qquad i^3 = i^2 \cdot i = -i, \qquad i^4 = i^2 \cdot i^2 = (-1)(-1) = 1.$$

Continuing, $i^5 = i$ and $i^6 = -1$, and the cycle repeats indefinitely.

Theorem 16.2. *The powers of i repeat with period four.*

Proof. Since $i^4 = 1$, for any positive integer n write $n = 4q + r$ with $0 \leq r < 4$; then $i^n = (i^4)^q i^r = 1^q i^r = i^r$, so i^n depends only on the remainder of n upon division by 4. □

Example 16.3. Compute i^{23}. Since $23 = 4(5) + 3$, $i^{23} = (i^4)^5 i^3 = 1^5(-i) = -i$.

16.3 Complex Numbers

Definition 16.4. A *complex number* is an expression of the form $z = a + bi$, where a and b are real numbers and i is the imaginary unit.

The number a is called the *real part* of z, written $\mathrm{Re}(z) = a$, and b is called the *imaginary part*, written $\mathrm{Im}(z) = b$. Addition is performed componentwise, $(a + bi) + (c + di) = (a + c) + (b + d)i$, while multiplication follows the distributive law together with $i^2 = -1$.

Example 16.5. Compute $(2 + 3i)(1 + 4i)$. Expanding gives $2 + 8i + 3i + 12i^2$; since $i^2 = -1$, this becomes $2 + 11i - 12 = -10 + 11i$.

The algebra looks unfamiliar at first, but the next section reveals its geometric meaning.

16.4 The Complex Plane

Complex numbers become much easier to understand once they are represented geometrically. Recall from Chapter 2 that every ordered pair (a, b) corresponds to a point in the Cartesian plane; since every complex number has the form $z = a + bi$, it is natural to associate z with the point (a, b). This interpretation produces the *complex plane*, in which the horizontal axis represents real parts and the vertical axis represents imaginary parts:

$$a + bi \longleftrightarrow (a, b).$$

Thus $3+2i$ corresponds to the point $(3, 2)$, while $-1+4i$ corresponds to $(-1, 4)$. Because complex numbers correspond to points, their distance from the origin can be measured exactly as it was for vectors in Chapter 2.

Definition 16.6. The *modulus* of a complex number $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$, the length of the segment from the origin to the point representing z.

Example 16.7. Find the modulus of $z = 3 + 4i$. Using the definition, $|z| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$, so z lies five units from the origin.

This is exactly the distance formula introduced earlier in the text; no new geometry has been created, only familiar geometry reinterpreted in a new algebraic setting.

16.5 Multiplication as Transformation

Addition of complex numbers behaves much like vector addition. Multiplication behaves differently, and to investigate its geometric meaning, consider multiplying by i. For $z = a + bi$,

$$iz = i(a + bi) = ai + bi^2 = -b + ai,$$

so in coordinate form $(a, b) \mapsto (-b, a)$.

Example 16.8. The complex number 1 corresponds to $(1, 0)$. Multiplying by i gives i, corresponding to $(0, 1)$: the point has rotated through $90°$.

Example 16.9. The complex number $2 + i$ corresponds to $(2, 1)$. Multiplying by i gives $-1 + 2i$, corresponding to $(-1, 2)$: again the point has rotated through $90°$.

Theorem 16.10. *Multiplication by i rotates every point in the complex plane through* $90°$ *counterclockwise.*

Proof. Multiplication by i sends $(a, b) \mapsto (-b, a)$. From Chapter 8, the rotation matrix

$$R(90°) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

produces exactly the same transformation, since $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix}$. Therefore multiplication by i is a rotation through $90°$. □

This discovery is remarkable: multiplication is no longer merely arithmetic, but has become geometry. Multiplication by -1 sends $(a, b) \mapsto (-a, -b)$, a rotation through $180°$; multiplication by $2i$ rotates through $90°$ and doubles all distances from the origin; and multiplication by $1 + i$ appears to rotate and scale simultaneously. A pattern is emerging: multiplication seems to combine scaling and rotation into a single operation. But scaling and rotation combined were precisely the amplitwists of Chapter 15, and the next section makes this connection exact.

16.6 Polar Form

Chapter 14 introduced polar coordinates: every point in the plane may be described by a distance r from the origin and an angle θ measured from the positive x-axis, related by $x = r\cos\theta$ and $y = r\sin\theta$. Substituting these into $z = x + iy$ gives

$$z = r\cos\theta + ir\sin\theta,$$

and factoring out r,

$$z = r(\cos\theta + i\sin\theta).$$

Definition 16.11. The expression $z = r(\cos\theta + i\sin\theta)$ is called the *polar form* of z, where $r = |z|$ is the modulus and θ is the angle made with the positive real axis.

Example 16.12. Convert $1 + i$ to polar form. The modulus is $r = \sqrt{1^2 + 1^2} = \sqrt{2}$, and since the point lies on the line $y = x$ in the first quadrant, $\theta = 45°$. Therefore $1 + i = \sqrt{2}(\cos 45° + i\sin 45°)$.

Whereas rectangular form is ideal for addition and subtraction, polar form reveals the geometric meaning of multiplication, as the next section shows.

16.7 Multiplication in Polar Form

The connection between complex numbers and amplitwists is revealed by multiplying two complex numbers written in polar form. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Multiplying gives

$$z_1z_2 = r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2),$$

and expanding the product,

$$z_1z_2 = r_1r_2\Big[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\Big].$$

The real part contains $\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$, which Chapter 8 identified as $\cos(\theta_1+\theta_2)$, while the imaginary part contains $\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 = \sin(\theta_1+\theta_2)$. Substituting these identities yields the central theorem of the chapter.

Theorem 16.13. *If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then*

$$z_1z_2 = r_1r_2\Big(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\Big).$$

Corollary 16.14. *When complex numbers are multiplied, moduli multiply and angles add.*

This is exactly the composition law of Chapter 15. Beside the amplitwist rule $(s_1,\theta_1)\circ(s_2,\theta_2) = (s_1s_2, \theta_1+\theta_2)$, place the multiplication theorem just proved, $(r_1,\theta_1)\cdot(r_2,\theta_2) = (r_1r_2, \theta_1+\theta_2)$: the two rules are identical. Complex multiplication is amplitwist composition. Every nonzero complex number determines an amplitwist, and multiplying complex numbers composes the corresponding transformations; the geometry of Chapter 15 and the algebra of this chapter describe the same mathematical structure.

Example 16.15. Multiply $2(\cos 30^\circ + i\sin 30^\circ)$ and $3(\cos 40^\circ + i\sin 40^\circ)$. The theorem gives $z_1z_2 = 6(\cos 70^\circ + i\sin 70^\circ)$: the modulus has become $2\cdot 3 = 6$, while the angle has become $30^\circ + 40^\circ = 70^\circ$.

16.8 Euler's Formula

The expression $\cos\theta + i\sin\theta$ occurs so frequently that a special notation is introduced for it.

Definition 16.16 (Euler's Formula)**.** For every real number θ, $e^{i\theta} = \cos\theta + i\sin\theta$.

Using this notation, a complex number in polar form becomes $z = re^{i\theta}$, and the multiplication theorem of the previous section takes its simplest possible form:

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}, \qquad \text{so that} \qquad (re^{i\theta})(se^{i\phi}) = rse^{i(\theta+\phi)}.$$

Angle addition has become ordinary exponent addition. This chapter treats Euler's formula as a definition motivated by the compactness it lends to the multiplication law already proved; a derivation of the same formula from the power series for e^x, $\sin x$, and $\cos x$ belongs to a calculus course and is not needed here to use the formula correctly. Instead of carrying trigonometric expressions through every calculation, this notation allows complex arithmetic to be handled directly through exponentials, with multiplication automatically scaling by rs and rotating by $\theta+\phi$ in a single step.

16.9 Euler's Identity

A celebrated special case occurs when $\theta = \pi$. Substituting into Euler's formula gives $e^{i\pi} = \cos\pi + i\sin\pi$, and since $\cos\pi = -1$ and $\sin\pi = 0$, this becomes $e^{i\pi} = -1$, or equivalently:

Theorem 16.17 (Euler's Identity)**.** $e^{i\pi} + 1 = 0$.

This equation connects five of the most important constants in mathematics — $0, 1, \pi, e$, and i — in a single relation, and is often regarded as one of the most elegant formulas in mathematics for exactly that reason.

16.10 Repeated Multiplication and De Moivre's Theorem

Chapter 15 showed that repeated amplitwist composition satisfies $A^n = s^n R(n\theta)$. Complex numbers obey an identical law. Let $z = re^{i\theta}$; then $z^2 = (re^{i\theta})(re^{i\theta}) = r^2e^{i2\theta}$, and similarly $z^3 = r^3e^{i3\theta}$, suggesting the general formula $z^n = r^ne^{in\theta}$.

Theorem 16.18 (De Moivre's Theorem)**.** *For every positive integer n, $(re^{i\theta})^n = r^ne^{in\theta}$. Equivalently, $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$.*

Proof. Using the multiplication law for complex numbers in polar form repeatedly, each additional factor of $re^{i\theta}$ contributes one more factor of r to the modulus and adds one more θ to the angle: $(re^{i\theta})^2 = r^2e^{i2\theta}$, $(re^{i\theta})^3 = r^3e^{i3\theta}$, and continuing this process n times yields $(re^{i\theta})^n = r^ne^{in\theta}$. Setting $r = 1$ and applying Euler's formula gives $(e^{i\theta})^n = e^{in\theta}$, which is $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$. $\square$

The similarity to Chapter 15 is striking: $A^n = s^nR(n\theta)$ and $(re^{i\theta})^n = r^ne^{in\theta}$ are the same statement written in two different languages, the first describing repeated amplitwist composition geometrically and the second describing repeated complex multiplication algebraically.

Example 16.19. Compute $(\cos 30° + i\sin 30°)^6$. By De Moivre's theorem, this equals $\cos(180°) + i\sin(180°)$; since $\cos 180° = -1$ and $\sin 180° = 0$, the value is -1.

Procedure 16.1 (How to Compute Powers Using Euler's Formula).
Write the complex number in polar form $z = re^{i\theta}$, apply De Moivre's theorem to obtain $z^n = r^ne^{in\theta}$ by raising the modulus r to the n-th power and multiplying the angle θ by n, and convert the result back to rectangular form if required.

Practice now. Compute $(2(\cos 45° + i\sin 45°))^4$.

16.11 Roots of Unity

The equation $z^n = 1$ asks for all complex numbers whose n-th power equals 1. Using polar form, write $1 = e^{i2\pi k}$ for any integer k, and let $z = re^{i\theta}$, so that $z^n = r^ne^{in\theta}$. Setting this equal to $e^{i2\pi k}$ requires equal moduli and equal angles, so $r^n = 1$, giving $r = 1$, and $n\theta = 2\pi k$, giving $\theta = 2\pi k/n$.

Theorem 16.20. *The solutions of $z^n = 1$ are $z_k = e^{2\pi ik/n}$ for $k = 0, 1, 2, \ldots, n-1$, called the n-th* roots of unity.

Each root lies on the unit circle, and successive roots differ by a rotation through $2\pi/n$, so the roots form the vertices of a regular n-gon.

Example 16.21. Find the cube roots of unity. Here $n = 3$, so $z_k = e^{2\pi ik/3}$. For $k = 0$, $z_0 = 1$. For $k = 1$, $z_1 = \cos 120^\circ + i \sin 120^\circ = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. For $k = 2$, $z_2 = \cos 240^\circ + i \sin 240^\circ = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. These three points form an equilateral triangle on the unit circle.

From the amplitwist viewpoint, the roots of unity are equally remarkable: each root represents a rotation which, applied repeatedly n times, returns exactly to the starting position.

16.12 Applications

Complex numbers appear throughout mathematics, science, and engineering. In electrical engineering, alternating-current circuits are often described using complex numbers whose moduli represent amplitudes and whose angles represent phase shifts. In signal processing, oscillations are represented by rotating complex numbers called *phasors*, and complex multiplication allows amplitude and phase changes to be handled simultaneously. In computer graphics, rotations and scalings can be performed through complex multiplication, making many geometric calculations more efficient. In physics, wave phenomena are frequently modeled using expressions involving Euler's formula, since oscillatory motion can be represented compactly by complex exponentials. Although these applications may appear different on the surface, they all exploit the same fact established in this chapter: complex multiplication combines scaling and rotation into a single operation.

16.13 Looking Ahead

This chapter has shown that the amplitwists of Chapter 15 and the complex numbers of this chapter describe the same mathematical structure: a complex number written as $re^{i\theta}$ represents an amplitwist with amplitude r and angle θ, and multiplication obeys the familiar law $(re^{i\theta})(se^{i\phi}) = rse^{i(\theta+\phi)}$. Geometry has become algebra. The next chapter begins a new phase of the book, moving from geometric transformations to analytical questions involving limits and rates of change. Its first goal is to establish the fundamental limit $\lim_{\theta\to 0} \sin\theta/\theta = 1$, which leads directly to the derivatives of the sine and cosine functions and forms the bridge from trigonometry to calculus.

16.14 Exercises

Basic Computations

Exercise 16.1. Compute i^7.

Exercise 16.2. Compute i^{50}.

Exercise 16.3. Simplify $(3 + 2i) + (4 - 5i)$.

Exercise 16.4. Simplify $(2 + 3i)(1 - 2i)$.

Exercise 16.5. Find the modulus of $3 + 4i$.

The Complex Plane

Exercise 16.6. Plot the complex number $2 + 3i$.

Exercise 16.7. Plot the complex number $-4 + i$.

Exercise 16.8. Plot the complex number $-2 - 2i$, and describe its position relative to the axes.

Exercise 16.9. Explain how the complex plane extends the Cartesian plane introduced in Chapter 2.

Multiplication and Geometry

Exercise 16.10. Verify that multiplication by i sends (a, b) to $(-b, a)$.

Exercise 16.11. Determine the image of $3 + i$ under multiplication by i.

Exercise 16.12. Determine the image of $2 - 4i$ under multiplication by i.

Exercise 16.13. Describe geometrically the effect of multiplication by -1.

Exercise 16.14. Explain why multiplication by i represents a rotation rather than a translation.

Polar Form

Exercise 16.15. Convert $1 + i$ to polar form.

Exercise 16.16. Convert $-1 + i$ to polar form.

Exercise 16.17. Convert $2(\cos 60^\circ + i \sin 60^\circ)$ to rectangular form.

Exercise 16.18. Convert $3(\cos 150^\circ + i \sin 150^\circ)$ to rectangular form.

Exercise 16.19. Explain why polar form is especially useful for multiplication.

Euler's Formula and De Moivre's Theorem

Exercise 16.20. Use Euler's formula to evaluate e^{i0}.

Exercise 16.21. Use Euler's formula to evaluate $e^{i\pi/2}$.

Exercise 16.22. Use Euler's formula to evaluate $e^{i\pi}$.

Exercise 16.23. Use Euler's formula to evaluate $e^{i3\pi/2}$.

Exercise 16.24. Use Euler's formula to evaluate $e^{i2\pi}$.

Exercise 16.25. Compute $(\cos 30^\circ + i \sin 30^\circ)^2$.

Exercise 16.26. Compute $(\cos 45^\circ + i \sin 45^\circ)^4$.

Exercise 16.27. Compute $(\cos 60^\circ + i \sin 60^\circ)^3$.

Exercise 16.28. Compute $(2(\cos 30^\circ + i \sin 30^\circ))^3$.

Exercise 16.29. Compute $(3(\cos 20^\circ + i \sin 20^\circ))^4$.

Exercise 16.30. Express $(\cos\theta + i\sin\theta)^5$ using De Moivre's theorem.

Exercise 16.31. Explain geometrically why De Moivre's theorem multiplies angles by n.

Roots of Unity

Exercise 16.32. Find all solutions of $z^2 = 1$.

Exercise 16.33. Find all solutions of $z^3 = 1$.

Exercise 16.34. Find all solutions of $z^4 = 1$.

Exercise 16.35. Find all solutions of $z^6 = 1$.

Exercise 16.36. Sketch the fourth roots of unity on the complex plane.

Exercise 16.37. Sketch the sixth roots of unity on the complex plane.

Exercise 16.38. Show that the cube roots of unity form an equilateral triangle.

Exercise 16.39. Show that the fourth roots of unity form a square.

Exercise 16.40. Determine the angle separating adjacent eighth roots of unity.

Exercise 16.41. Explain why the n-th roots of unity always form a regular polygon.

Connections with Amplitwists

Exercise 16.42. Write the complex number $2e^{i45^\circ}$ in rectangular form.

Exercise 16.43. Write the complex number $3e^{i120^\circ}$ in rectangular form.

Exercise 16.44. Multiply $2e^{i30^\circ}$ and $3e^{i40^\circ}$.

Exercise 16.45. Show that multiplying by $e^{i\theta}$ leaves the modulus of a complex number unchanged.

Exercise 16.46. Show that multiplying by re^{i0} scales a complex number without changing its angle.

Exercise 16.47. Explain why $(re^{i\theta})(se^{i\phi}) = rse^{i(\theta+\phi)}$ is the amplitwist composition law of Chapter 15 written algebraically.

Exercise 16.48. Describe geometrically the transformation represented by multiplication by $1 + i$.

Exercise 16.49. Determine the modulus and angle of $1 + i$.

Exercise 16.50. Show that every nonzero complex number determines a unique amplitwist.

Exercise 16.51. Explain why Chapter 15 can be viewed as a geometric introduction to complex multiplication.

Proofs and Theory

Exercise 16.52. Prove that the powers of i repeat with period four.

Exercise 16.53. Prove that the product of two nonzero complex numbers is nonzero.

Exercise 16.54. Show that multiplication by i preserves distance from the origin.

Exercise 16.55. Prove that multiplication by i represents a 90° rotation.

Exercise 16.56. Prove De Moivre's theorem by mathematical induction.

Exercise 16.57. Show directly from Euler's formula that $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$.

Exercise 16.58. Show that every nonzero complex number can be written in polar form.

Exercise 16.59. Show that the modulus of a product satisfies $|z_1 z_2| = |z_1||z_2|$.

Exercise 16.60. Show that the angle of a product equals the sum of the angles of the factors.

Exercise 16.61. Explain why complex multiplication naturally combines scaling and rotation.

Challenge Problems

Exercise 16.62. Determine all solutions of $z^5 = 1$.

Exercise 16.63. Determine all solutions of $z^8 = 1$.

Exercise 16.64. Show that the product of all n-th roots of unity is $(-1)^{n+1}$.

Exercise 16.65. Suppose $z = e^{i\theta}$ is an n-th root of unity with $z \neq 1$. Show that $1 + z + z^2 + \cdots + z^{n-1} = 0$.

Exercise 16.66. Prove that the n-th roots of unity form a group under multiplication.

Exercise 16.67. Show that repeated multiplication by a complex number of modulus 1 traces points on a circle.

Exercise 16.68. Let $z = re^{i\theta}$. Determine all values of r and θ for which $z^2 = 1$.

Exercise 16.69. Determine all values of z satisfying $z^3 = 1$, expressing each in both polar and rectangular form.

Exercise 16.70. Show that complex multiplication is commutative by using the amplitwist interpretation.

Exercise 16.71. Chapter 15 showed that every orientation-preserving similarity transformation fixing the origin can be written uniquely as $A = sR(\theta)$. Restate this result in the language of complex numbers, and explain what it implies about every nonzero complex number.

Exercise 16.72. Chapter 15 described amplitwists geometrically, and this chapter described complex numbers algebraically. Explain in detail why these are not two different theories but two descriptions of the same mathematical structure.

Part V

Toward Analysis

Chapter 17

Small-Angle Limits and Trigonometric Derivatives

17.1 From Geometry to Analysis

The Introduction to this book described trigonometry not merely as a collection of formulas, but as a bridge between geometry and calculus. The preceding chapters have gradually built that bridge. Chapter 15 showed how scaling and rotation combine into amplitwists; Chapter 16 showed that complex numbers provide an algebraic language for those same transformations. Geometry became algebra. This chapter takes the next step: algebra now becomes analysis. The central question is simple: how do trigonometric functions behave under infinitesimal change? To answer it requires understanding what happens as an angle becomes extremely small. The resulting limit leads directly to the derivatives of the sine and cosine functions and establishes the first substantial connection between trigonometry and calculus.

17.2 The Geometry of Small Angles

Consider the values of $\sin\theta$, θ, and $\tan\theta$ for progressively smaller angles.

θ	$\sin\theta$	$\tan\theta$
0.5	0.4794	0.5463
0.1	0.0998	0.1003
0.05	0.0500	0.0500
0.01	0.0100	0.0100

Table 17.1: Values of $\sin\theta$ and $\tan\theta$ as $\theta \to 0$ (radians).

As θ approaches zero, the three quantities appear to become indistinguishable, $\sin\theta \approx \theta \approx \tan\theta$. This observation suggests an important limit, but before proving it a crucial remark is needed: the approximation works only when θ is measured in *radians*. If degrees are used instead, the numerical agreement disappears, and this fact is not accidental. Chapter 4 introduced radians because they arise naturally from arc length, and this chapter reveals a second reason: radians are the angular unit that makes the calculus of trigonometric functions take its simplest form.

17.3 The Sector-Area Sandwich

Consider a unit circle and an angle θ satisfying $0 < \theta < \pi/2$. Three regions can be constructed at this angle: the triangle formed by the radius and the chord, the circular sector subtended by the angle, and the larger triangle formed by the radius and the tangent line. The first of these regions lies entirely inside the sector, while the sector in turn lies entirely inside the outer triangle, giving

$$\text{Area}_{\text{inner}} < \text{Area}_{\text{sector}} < \text{Area}_{\text{outer}}.$$

Each area can now be computed directly. The inner triangle has base 1 and height $\sin\theta$, so $\text{Area}_{\text{inner}} = \frac{1}{2}\sin\theta$. The sector area was derived in Chapter 5: $\text{Area}_{\text{sector}} = \frac{1}{2}\theta$. The outer triangle has base 1 and height $\tan\theta$, giving $\text{Area}_{\text{outer}} = \frac{1}{2}\tan\theta$. Combining these results yields

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta,$$

and multiplying by 2 gives the fundamental inequality

$$\sin\theta < \theta < \tan\theta.$$

This geometric sandwich is the key to everything that follows.

17.4 The Fundamental Small-Angle Limit

The inequality $\sin\theta < \theta < \tan\theta$ now determines the behavior of $\sin\theta/\theta$ as θ approaches zero.

Theorem 17.1. $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$

Proof. For $0 < \theta < \pi/2$, $\sin\theta < \theta < \tan\theta$. Dividing through by $\sin\theta$ gives $1 < \theta/\sin\theta < 1/\cos\theta$, and taking reciprocals reverses the inequalities:

$$\cos\theta < \frac{\sin\theta}{\theta} < 1.$$

As $\theta \to 0$, $\cos\theta \to 1$, and the right-hand bound is already 1; by the Squeeze Theorem, $\lim_{\theta\to 0}\sin\theta/\theta = 1$. □

This result is the most important limit in elementary trigonometry, and the derivatives of sine and cosine ultimately depend on it.

17.5 Corollary Limits

Several additional limits follow immediately.

Theorem 17.2. $\displaystyle\lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1.$

Proof. Since $\tan\theta = \sin\theta/\cos\theta$, $\tan\theta/\theta = (\sin\theta/\theta)\cdot(1/\cos\theta)$. Taking limits, $\lim_{\theta\to 0}\tan\theta/\theta = 1\cdot 1 = 1$. □

Theorem 17.3. $\displaystyle\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0.$

Proof. Using the identity $1-\cos\theta = 2\sin^2(\theta/2)$ from Chapter 10,

$$\frac{1-\cos\theta}{\theta} = \sin\left(\frac{\theta}{2}\right)\cdot\frac{2\sin(\theta/2)}{\theta}.$$

As $\theta \to 0$, $\sin(\theta/2) \to 0$, while $\sin(\theta/2)/(\theta/2) \to 1$ by the fundamental limit. The entire product therefore approaches 0. □

17.6 The Derivative

The limits established above describe the behavior of trigonometric functions under extremely small changes. Calculus formalizes this idea through the concept of a derivative. Suppose a function $f(x)$ is defined on an interval, and let x and $x+h$ be two nearby points; the average rate of change of the function between them is $(f(x+h)-f(x))/h$, the slope of the secant line joining $(x,f(x))$ and $(x+h,f(x+h))$. As h becomes smaller, the secant line approaches the tangent line at x.

Definition 17.4. The *derivative* of a function f at x is

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h},$$

provided the limit exists.

The derivative measures the instantaneous rate of change of the function, and gives the slope of its tangent line. The goal of the next two sections is to compute the derivatives of the sine and cosine functions.

17.7 The Derivative of Sine

Begin with $f(x) = \sin x$. By the definition of the derivative,

$$f'(x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}.$$

The sum formula from Chapter 8 gives $\sin(x+h) = \sin x\cos h + \cos x\sin h$, and substituting into the difference quotient,

$$f'(x) = \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}.$$

Grouping terms,

$$f'(x) = \lim_{h\to 0}\left[\sin x\,\frac{\cos h - 1}{h} + \cos x\,\frac{\sin h}{h}\right].$$

Since $\sin x$ and $\cos x$ do not depend on h, they may be treated as constants:

$$f'(x) = \sin x\lim_{h\to 0}\frac{\cos h - 1}{h} + \cos x\lim_{h\to 0}\frac{\sin h}{h}.$$

The previous section established $\lim_{h\to 0}\sin h/h = 1$ and $\lim_{h\to 0}(\cos h - 1)/h = 0$, so $f'(x) = \sin x(0) + \cos x(1)$.

Theorem 17.5. $\dfrac{d}{dx}\sin x = \cos x.$

Proof. The computation above establishes the result directly from the definition of the derivative. □

The result is remarkable because the derivative of sine is another trigonometric function rather than an entirely different kind of expression.

17.8 The Derivative of Cosine

The same calculation applies to $f(x) = \cos x$. By definition,

$$f'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}.$$

Using the cosine sum formula, $\cos(x+h) = \cos x \cos h - \sin x \sin h$, so

$$f'(x) = \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} = \lim_{h\to 0} \left[\cos x \, \frac{\cos h - 1}{h} - \sin x \, \frac{\sin h}{h}\right].$$

Applying the two fundamental limits, $f'(x) = \cos x(0) - \sin x(1) = -\sin x$.

Theorem 17.6. $\frac{d}{dx} \cos x = -\sin x$.

Proof. The result follows directly from the derivative definition and the trigonometric sum formulas, exactly as in the derivative of sine. □

Together, the two derivative formulas form one of the most important relationships in mathematics: differentiation transforms sine into cosine and cosine into negative sine. A second differentiation yields $\frac{d^2}{dx^2} \sin x = -\sin x$ and $\frac{d^2}{dx^2} \cos x = -\cos x$, a property that will later explain why sine and cosine naturally describe oscillations, waves, and periodic motion.

Procedure 17.1 (How to Differentiate an Expression Involving Sine and Cosine).
Apply the rules $\frac{d}{dx} \sin x = \cos x$ and $\frac{d}{dx} \cos x = -\sin x$ directly whenever the argument is simply x. If the angle appearing inside sine or cosine is itself a function of x, differentiate using these rules and then multiply by the derivative of that inner function according to the chain rule.

Example 17.7. Differentiate $f(x) = \sin(2x) + 3\cos x$. Using the derivative rules together with the chain rule, $f'(x) = 2\cos(2x) - 3\sin x$.

Practice now. Differentiate $f(x) = 4\sin(3x) - 2\cos(2x)$.

17.9 Circular Motion Revisited

Chapter 5 studied circular motion using the parametric equations $x(t) = \cos t$, $y(t) = \sin t$, introduced geometrically as the coordinates of a point moving around the unit circle at constant angular speed. The derivative formulas of this chapter now allow that motion to be analyzed quantitatively. Differentiating the coordinate functions gives $x'(t) = -\sin t$ and $y'(t) = \cos t$, so the velocity vector is

$$\mathbf{v}(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix},$$

while the position vector is $\mathbf{r}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$. Comparing the two reveals that the velocity vector is obtained from the position vector by a rotation through 90°:

$$\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix},$$

and Chapter 8 identified this matrix as $R(90°)$, the quarter-turn rotation.

Theorem 17.8. *For motion on the unit circle, $\mathbf{r}(t) \cdot \mathbf{v}(t) = 0$; consequently, the velocity vector is tangent to the circle at every point.*

Proof. Compute the dot product: $\mathbf{r}(t) \cdot \mathbf{v}(t) = (\cos t)(-\sin t) + (\sin t)(\cos t) = -\cos t \sin t + \sin t \cos t = 0$, so the vectors are perpendicular. □

This result provides a satisfying closure to Chapter 5. The trigonometric functions that originally described circular motion now explain the geometry of that motion directly: the velocity is always tangent to the circle, and differentiation acts as a quarter-turn rotation of the position vector.

17.10 Why Radians Matter

Throughout this chapter, the limits $\lim_{\theta \to 0} \sin\theta/\theta = 1$ and $\lim_{\theta \to 0} \tan\theta/\theta = 1$ have played a central role, and these formulas are true only when angles are measured in radians. To see why, suppose angle measure is expressed in degrees. Since $180^\circ = \pi$ radians, an angle of h degrees corresponds to $\pi h/180$ radians, so

$$\lim_{h \to 0} \frac{\sin h^\circ}{h} = \lim_{h \to 0} \frac{\sin(\pi h/180)}{\pi h/180} \cdot \frac{\pi}{180}.$$

The first factor approaches 1, leaving $\lim_{h \to 0} \sin h^\circ/h = \pi/180$: instead of obtaining 1, an additional constant appears, and consequently $\frac{d}{dx} \sin x = \frac{\pi}{180} \cos x$ when x is measured in degrees. The derivative formula becomes cluttered with a conversion factor. By contrast, when angles are measured in radians, $\frac{d}{dx} \sin x = \cos x$ with no additional constant. Radians are therefore not merely a convenient convention; they are the angular unit naturally adapted to the geometry of circles and the calculus of trigonometric functions, confirming the theme introduced in Chapter 4 that radians are the unit in which the fundamental structures of trigonometry take their simplest form.

17.11 Applications

The derivative relationships $\frac{d}{dx} \sin x = \cos x$ and $\frac{d}{dx} \cos x = -\sin x$ appear throughout mathematics, science, and engineering. In simple harmonic motion, the position of an oscillating object is often modeled by a sine or cosine function, while the derivative describes its velocity. In acoustics, vibrating strings and sound waves are represented using trigonometric functions whose derivatives describe rates of change in pressure and displacement. In electrical engineering, alternating-current signals are modeled by sinusoids, with derivatives corresponding to changing voltages and currents. In astronomy, periodic motions such as rotations and orbital approximations are frequently expressed using trigonometric functions and their derivatives. The reason these applications are so widespread is now visible: differentiation preserves the trigonometric family, and the derivative of a sine function is another trigonometric function, as is the derivative of a cosine function.

17.12 Looking Ahead

This chapter marks a transition in the book. The earlier chapters studied finite transformations: rotations, similarities, amplitwists, and complex multiplication. This chapter introduced infinitesimal change through limits and derivatives. A different kind of infinite process remains to be studied: instead of examining what happens under infinitely small changes, the next chapter investigates what happens when infinitely many quantities are accumulated together. The central objects will be sequences and series, which provide the mathematical foundation for approximation, convergence, and many of the analytical techniques used throughout modern mathematics.

17.13 Exercises

Small-Angle Limits

Exercise 17.1. Compute $\sin\theta/\theta$ for $\theta = 0.5, 0.1$, and 0.01 radians.

Exercise 17.2. Compute $\tan\theta/\theta$ for $\theta = 0.5, 0.1$, and 0.01 radians.

Exercise 17.3. Explain numerically why $\sin\theta \approx \theta$ for small angles.

Exercise 17.4. Reproduce the geometric argument establishing $\sin\theta < \theta < \tan\theta$.

Exercise 17.5. Use the Squeeze Theorem to prove $\lim_{\theta\to 0} \sin\theta/\theta = 1$.

Differentiation

Exercise 17.6. Differentiate $f(x) = \sin x$.

Exercise 17.7. Differentiate $f(x) = \cos x$.

Exercise 17.8. Differentiate $f(x) = 3\sin x - 2\cos x$.

Exercise 17.9. Differentiate $f(x) = \sin(2x)$.

Exercise 17.10. Differentiate $f(x) = 4\cos(3x)$.

Exercise 17.11. Differentiate $f(x) = \sin(5x) + \cos(2x)$.

Exercise 17.12. Differentiate $f(x) = 2\sin(4x) - 3\cos(5x)$.

Circular Motion

Exercise 17.13. For $x(t) = \cos t$, $y(t) = \sin t$, find the velocity vector.

Exercise 17.14. Show that the velocity vector is perpendicular to the position vector.

Exercise 17.15. Verify that the velocity vector is tangent to the unit circle.

Exercise 17.16. Find the velocity vector at $t = \pi/2$.

Exercise 17.17. Find the velocity vector at $t = \pi$.

Challenge Problems

Exercise 17.18. Prove $\lim_{\theta\to 0} \tan\theta/\theta = 1$.

Exercise 17.19. Show that $\dfrac{d^2}{dx^2}\sin x = -\sin x$.

Exercise 17.20. Show that $\dfrac{d^2}{dx^2}\cos x = -\cos x$.

Exercise 17.21. Explain why the equation $y'' = -y$ is naturally solved by sine and cosine functions.

Exercise 17.22. Represent the pair $(\cos x, \sin x)$ as a vector, and show that differentiation acts as a quarter-turn rotation on it.

Exercise 17.23. Compare the action of differentiation on $(\cos x, \sin x)$ with multiplication by i in Chapter 16. Explain the similarity.

Chapter 18

Arithmetic and Geometric Sequences and Series

18.1 From Change to Accumulation

Historical Note: The Word "Calculus".
The word *calculus* is Latin for a small pebble or counting stone. Before written numerals were widespread, quantities were often tracked by correspondence rather than by symbols: a shepherd might drop one pebble into a pouch for each sheep passing through a gate, later reversing the process to confirm that every sheep had returned, with the pebbles serving as an external record of a quantity no numeral was needed to name. Arranging such pebbles into grooves or columns, as on a Roman counting board or a later abacus, let position carry meaning as well, an early form of the place-value systems that Chapter 0 discussed in connection with numeral systems more generally. The word survived this entire history essentially unchanged: by the time Newton and Leibniz developed systematic methods for computing rates of change and accumulated totals, "calculus" already meant, broadly, any organized method of calculation, which is why their new subject was named the differential and integral calculus rather than something entirely new. The sequences and series developed in this chapter, built from accumulating one term after another, are in a real sense the direct descendants of the shepherd's pebbles: both are methods for turning many small contributions into a single total.

Chapter 17 studied derivatives and the mathematics of infinitely small change: a derivative is obtained by examining what happens as a quantity approaches zero. This chapter introduces a different encounter with infinity. Instead of asking what happens as a change becomes smaller and smaller, it asks what happens as more and more terms are added together. These two ideas — infinitely small change and infinitely many accumulated terms — are among the central themes of calculus, and this chapter develops the second in a setting that requires only algebra and the geometric ideas already established throughout this book.

18.2 Sequences

Definition 18.1. A *sequence* is a function whose domain is the positive integers. Its terms are commonly denoted $a_1, a_2, a_3, \ldots$, where a_n is called the nth term.

A sequence may be defined explicitly or recursively.

Example 18.2. The sequence $a_n = 2n + 1$ has terms $3, 5, 7, 9, \ldots$ and is defined explicitly.

Example 18.3. The sequence defined by $a_1 = 1$, $a_{n+1} = a_n + 2$ produces the same terms $1, 3, 5, 7, \ldots$ but is defined recursively.

18.3 Arithmetic Sequences and Series

An arithmetic sequence is formed by repeatedly adding a fixed quantity.

Definition 18.4. An *arithmetic sequence* has a constant difference d between consecutive terms: $a_{n+1} = a_n + d$, called the *common difference*.

Theorem 18.5. *The nth term of an arithmetic sequence is $a_n = a_1 + (n-1)d$.*

Proof. Starting from a_1, each new term adds one additional copy of d; after $n-1$ such additions, $a_n = a_1 + (n-1)d$. □

A sum of terms from an arithmetic sequence is called an arithmetic series.

Theorem 18.6. *The sum of the first n terms of an arithmetic sequence is $S_n = \dfrac{n}{2}(a_1 + a_n)$.*

Proof. Write the sum forward and backward, $S_n = a_1 + a_2 + \cdots + a_n$ and $S_n = a_n + a_{n-1} + \cdots + a_1$. Adding corresponding terms gives $2S_n = n(a_1 + a_n)$, and dividing by 2 yields $S_n = \frac{n}{2}(a_1 + a_n)$. □

Example 18.7. Find the sum $1 + 3 + 5 + \cdots + 99$. Since $a_1 = 1$, $a_n = 99$, and $n = 50$, $S_{50} = \frac{50}{2}(1 + 99) = 2500$.

18.4 Geometric Sequences

An arithmetic sequence grows by repeated addition; a geometric sequence grows by repeated multiplication.

Definition 18.8. A *geometric sequence* has a constant ratio r between consecutive terms: $a_{n+1} = ra_n$, called the *common ratio*.

Theorem 18.9. *The nth term of a geometric sequence is $a_n = a_1 r^{n-1}$.*

Proof. Each successive term contributes one additional factor of r: $a_2 = a_1 r, a_3 = a_1 r^2, a_4 = a_1 r^3$, and continuing this pattern gives $a_n = a_1 r^{n-1}$. □

Example 18.10. The sequence $2, 6, 18, 54, \ldots$ has common ratio $r = 3$, with explicit formula $a_n = 2 \cdot 3^{n-1}$.

18.5 The Finite Geometric Series

The most important result of this chapter is the formula for a finite geometric sum.

Theorem 18.11. *For $r \neq 1$, $a + ar + ar^2 + \cdots + ar^{n-1} = \dfrac{a(1 - r^n)}{1 - r}$.*

Proof. Let $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$. Multiplying by r gives $rS_n = ar + ar^2 + \cdots + ar^n$. Subtracting, $S_n - rS_n = a - ar^n$, so $(1 - r)S_n = a(1 - r^n)$, and dividing by $1 - r$ yields $S_n = \dfrac{a(1 - r^n)}{1 - r}$. □

Example 18.12. Compute $1 + 2 + 4 + 8 + 16$. Here $a = 1$, $r = 2$, $n = 5$, so $S_5 = \dfrac{1 - 2^5}{1 - 2} = 31$.

18.6 Infinite Geometric Series

Finite geometric sums lead naturally to infinite series. Consider $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$; its partial sums are $1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \ldots$, and these values appear to approach 2.

Definition 18.13. A sequence *converges* if its terms approach a fixed limit. An infinite series converges if its sequence of partial sums converges.

Theorem 18.14. *If* $|r| < 1$, *then* $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$.

Proof. The finite geometric formula gives $S_n = \frac{a(1-r^n)}{1-r}$. When $|r| < 1$, $r^n \to 0$ as $n \to \infty$, so $\lim_{n\to\infty} S_n = \frac{a(1-0)}{1-r} = \frac{a}{1-r}$. □

Example 18.15. Evaluate $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$. Here $a = 1$ and $r = 1/2$, so $\sum_{k=0}^{\infty}(1/2)^k = 1/(1-1/2) = 2$.

Example 18.16. The series $1 + 2 + 4 + 8 + \cdots$ does not converge, because its partial sums grow without bound; here $r = 2$ fails the requirement $|r| < 1$.

18.7 The Complex Geometric Series

The same algebra works when the ratio is a complex number.

Theorem 18.17. *For any complex number* $z \neq 1$, $1 + z + z^2 + \cdots + z^{n-1} = \frac{1-z^n}{1-z}$.

Proof. Let $S_n = 1 + z + z^2 + \cdots + z^{n-1}$. Multiplying by z gives $zS_n = z + z^2 + \cdots + z^n$. Subtracting, $S_n - zS_n = 1 - z^n$, so $(1-z)S_n = 1 - z^n$ and $S_n = \frac{1-z^n}{1-z}$. □

This proof is identical to the real-valued case; only the interpretation changes. When $z = e^{i\theta}$, the successive terms $1, e^{i\theta}, e^{i2\theta}, e^{i3\theta}, \ldots$ are unit vectors rotated by equal angles, so the partial sums trace a polygonal path in the complex plane rather than moving along the real line.

Example 18.18. Let $z = e^{2\pi i/3}$. Then $1 + z + z^2 = \frac{1-z^3}{1-z} = 0$, exactly the roots-of-unity result encountered in Chapter 16.

Procedure 18.1 (How to Identify and Sum an Arithmetic or Geometric Series). Determine whether consecutive terms differ by a constant amount or are multiplied by a constant ratio. Apply the arithmetic formulas when a common difference exists, and the geometric formulas when a common ratio exists. For an infinite geometric series, verify that $|r| < 1$ before applying $a/(1-r)$; if $|r| \geq 1$, the series diverges and has no finite sum.

Practice now. Compute $3 + 6 + 12 + 24 + 48$. Then determine whether $3 + 6 + 12 + 24 + \cdots$ converges.

18.8 Connections to Earlier Chapters

Several ideas encountered earlier in this book are instances of geometric sequences. Repeated amplitwists satisfy $A^n = s^n R(n\theta)$ from Chapter 15: the scaling factor s^n forms a geometric sequence, while the angles $n\theta$ form an arithmetic sequence. Similarly, $(re^{i\theta})^n = r^n e^{in\theta}$ from Chapter 16 contains the same pair of patterns, with the magnitude evolving geometrically while the angle evolves arithmetically. The concepts introduced in this chapter therefore organize ideas that had already appeared, unnamed, in transformation geometry and complex numbers.

18.9 Applications

Geometric sequences and series appear throughout mathematics and science, in every setting where a quantity is repeatedly multiplied by a fixed factor: compound interest, population growth and decay, radioactive decay, recursive algorithms, and self-similar or fractal structures all generate a geometric sequence at their core.

Example 18.19. An account earning 5% annual interest, compounded once per year, has balance $B_n = P(1.05)^n$ after n years, where P is the initial deposit: a geometric sequence with common ratio $r = 1.05$. If \$1000 is deposited and left untouched, the balance after 10 years is $B_{10} = 1000(1.05)^{10} \approx 1628.89$. The total interest earned across those ten years, summed year by year, is itself a partial sum of a related geometric series, since each year's interest payment is 5% of a balance that has itself grown geometrically.

18.10 Looking Ahead

A finite geometric sum of complex exponentials produces a finite superposition of rotating vectors. The central idea of Fourier analysis is to combine many such rotations, each with its own frequency and amplitude, to represent periodic phenomena. Chapter 19 asks what happens when infinitely many such rotating components, rather than the single ratio z studied here, are combined to model waves, signals, and functions.

18.11 Exercises

Arithmetic Sequences and Series

Exercise 18.1. Find the tenth term of the arithmetic sequence $4, 7, 10, 13, \ldots$.

Exercise 18.2. Compute $5 + 9 + 13 + \cdots + 101$.

Exercise 18.3. Derive the arithmetic series formula independently, without referring back to the proof given in the text.

Exercise 18.4. A theater has 20 seats in its first row, with each subsequent row having 2 more seats than the row before it. Find the total number of seats in the first 15 rows.

Geometric Sequences and Series

Exercise 18.5. Find the eighth term of $3, 6, 12, 24, \ldots$.

Exercise 18.6. Evaluate $1 + 3 + 9 + 27 + 81$.

Exercise 18.7. Evaluate $\sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k$.

Exercise 18.8. Determine whether $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$ converges, and find its sum if it does.

Exercise 18.9. A ball is dropped from a height of 10 meters, and after each bounce it rises to 60% of its previous height. Find the total vertical distance the ball travels, accounting for infinitely many bounces.

Complex Geometric Series

Exercise 18.10. Compute $1 + i + i^2 + i^3$.

Exercise 18.11. Verify directly that $1 + e^{2\pi i/3} + e^{4\pi i/3} = 0$.

Exercise 18.12. Use the geometric-series formula to derive the same result.

Exercise 18.13. Compute $1 + z + z^2 + z^3$ for $z = e^{i\pi/2}$, and interpret the partial sums as vectors in the complex plane.

Mixed Problems

Exercise 18.14. Determine whether the sequence $2, 5, 10, 17, 26, \ldots$ is arithmetic, geometric, or neither, and justify your answer.

Exercise 18.15. An initial deposit of $2000 earns 4% annual interest, compounded once per year. Find the balance after 8 years.

Exercise 18.16. Explain the difference between a sequence and a series, and give an example of each.

Challenge Problems

Exercise 18.17. Show that the scaling factors in repeated amplitwists form a geometric sequence.

Exercise 18.18. Show that the angles in repeated amplitwists form an arithmetic sequence.

Exercise 18.19. Explain why $(re^{i\theta})^n = r^n e^{in\theta}$ contains both a geometric and an arithmetic progression simultaneously.

Exercise 18.20. Explain why a finite sum of complex exponentials may be interpreted as a sum of rotating vectors.

Exercise 18.21. Prove that an infinite geometric series with $|r| \geq 1$ cannot converge, by examining what happens to the partial sums S_n as $n \to \infty$.

Chapter 19

Fourier Intuition

19.1 Periodic Cycles and Hidden Patterns

In ancient astronomy, one of the most useful discoveries was the *Metonic cycle*: nineteen solar years are almost exactly equal to 235 lunar months. Neither cycle is defined in terms of the other; one measures the motion of the Earth around the Sun, and the other measures the phases of the Moon. Yet when the two cycles are observed together, a larger repeating pattern emerges. This observation suggests a broader question — what happens when several periodic processes are combined? — and the answer leads to one of the most influential ideas in mathematics, science, and engineering: complicated periodic behavior can often be understood as a combination of simpler periodic motions.

19.2 From Simple Oscillations to Complex Patterns

Throughout this book we have encountered periodic functions such as $\sin x$ and $\cos x$, functions that repeat indefinitely. Many phenomena in nature are also periodic — musical tones, ocean tides, planetary motion, electrical current, and seasonal variation all recur in cycles — yet most real signals are not perfect sine waves. A violin note is not a pure sinusoid, ocean tides are not perfectly smooth, and human speech contains many interacting oscillations at once. This raises a natural question: can complicated periodic patterns be built from simple ones?

19.3 Frequency Revisited

In Chapter 6, we saw that $\sin(nx)$ oscillates more rapidly than $\sin x$.

Definition 19.1. The *frequency* of a periodic oscillation measures how rapidly it repeats. For trigonometric functions, $\sin(nx)$ and $\cos(nx)$ have frequency n times that of $\sin x$ and $\cos x$.

For example, $\sin(3x)$ completes three full oscillations in the same interval where $\sin x$ completes one.

19.4 Harmonics

Frequencies that are whole-number multiples of a fundamental frequency are called harmonics.

Definition 19.2. If $\sin x$ is taken as a fundamental oscillation, then $\sin(2x), \sin(3x), \sin(4x), \ldots$ and $\cos(2x), \cos(3x), \cos(4x), \ldots$ are called its *harmonics*.

Musical instruments rarely produce a single frequency; instead, they produce combinations of harmonics, and the particular mixture of harmonics present helps determine the distinctive sound, or *timbre*, of an instrument.

19.5 Superposition

The simplest way to create a more complicated oscillation is to add simple oscillations together. Consider $f(x) = \sin x + \frac{1}{2}\sin(2x)$: the resulting graph is periodic, but it is no longer a pure sine wave. Adding another harmonic produces even greater complexity, as in $f(x) = \sin x + \frac{1}{2}\sin(2x) + \frac{1}{4}\sin(3x)$; each component remains simple, but their combination produces a richer pattern. This principle is called *superposition*.

Definition 19.3. *Superposition* is the formation of a new function by adding simpler functions together.

The central idea of Fourier analysis is that surprisingly complicated periodic functions can often be represented through superposition of harmonics.

19.6 Complex Exponentials and Harmonic Sums

Chapter 16 introduced complex exponentials, $e^{i\theta} = \cos\theta + i\sin\theta$, and Chapter 18 showed how finite sums of complex powers may be analyzed algebraically. Consider a finite sum

$$F(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta}.$$

Each term $e^{ik\theta}$ represents a rotating unit vector whose angular speed is proportional to k, so $F(\theta)$ is a finite superposition of rotating vectors, in exactly the sense developed in the previous chapter. Using Euler's formula, $e^{ik\theta} = \cos(k\theta) + i\sin(k\theta)$, so

$$F(\theta) = \sum_{k=-n}^{n} c_k(\cos(k\theta) + i\sin(k\theta)).$$

When the coefficients satisfy $c_{-k} = \overline{c_k}$, the imaginary contributions cancel in pairs, leaving a real-valued function. Consequently, finite sums of complex exponentials correspond precisely to finite sums of sine and cosine harmonics: the rotating vectors of Chapter 16 become the oscillating harmonics of this chapter, related by nothing more than Euler's formula already in hand.

19.7 The Fourier Idea

The central claim of Fourier analysis is remarkably simple: many periodic functions can be represented as sums of harmonics,

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)),$$

where the coefficients a_n and b_n determine how much of each harmonic is present. This statement is one of the most important ideas in applied mathematics. In a more advanced course, calculus provides methods for computing the coefficients themselves; here the focus is on the idea alone, that complicated periodic behavior may be assembled from simple oscillatory components.

19.8 Why Orthogonality Matters

Chapter 13 studied orthogonality using the dot product: two perpendicular vectors represent independent directions. Fourier analysis relies on a closely related idea, in which different harmonics behave like independent directions in a much larger space, and the coefficient associated with a harmonic measures how strongly that harmonic contributes to the overall signal. Just as a vector may be decomposed into components along coordinate axes, a periodic function may be decomposed into components along harmonic directions. The analogy is not exact, but it captures the essential geometric intuition: Fourier coefficients play a role similar to vector projections.

19.9 Visualizing Decomposition

A useful analogy comes from optics. White light appears uniform, yet when passed through a prism it separates into many component colors; the light was not changed by the prism, which merely revealed structure already present within it. Fourier analysis plays a similar role: a complicated waveform may appear indivisible, yet Fourier decomposition reveals the frequencies hidden within it, and musical tones, speech signals, ocean waves, and electrical currents may all be viewed through this same frequency-based perspective.

> *Practice now.* Identify the frequencies present in $f(x) = 2\sin x + \cos(3x)$. Which frequency contributes the larger amplitude?

19.10 Applications

Fourier analysis appears throughout science and engineering, wherever oscillatory phenomena are present: in music and acoustics, where it explains timbre; in audio compression, filtering, and signal processing generally; in the study of heat flow and diffusion, the problem that originally motivated it; in astronomy, where periodic signals reveal orbital and rotational structure; and in medical imaging, where frequency-domain methods reconstruct images from raw scanner data. Whenever a phenomenon oscillates, Fourier methods provide a powerful common language for analyzing it.

19.11 A Historical Perspective

Joseph Fourier introduced these ideas while studying the flow of heat through matter. His original goal was not music, communication, or image processing; he wanted to understand temperature. The remarkable discovery was that the same mathematical decomposition appeared in many unrelated contexts far removed from heat, and what began as a theory of heat became one of the central tools of modern science.

19.12 Looking Back

This book began with ratios arising from similarity. From similarity came the trigonometric functions; from trigonometric functions came rotations; from rotations came amplitwists; from amplitwists came complex numbers; from complex numbers came derivatives and geometric series; from geometric series came superpositions of oscillations; and from superposition came the Fourier idea developed in this chapter. The path has been long, but it has followed a single theme throughout: mathematics often progresses by identifying transformations, discovering what remains invariant under those transformations, and then giving those invariant structures names. The concepts introduced in this book are not isolated topics gathered under one title. They are parts of a single connected framework for understanding pattern, change, and structure.

19.13 Exercises

Frequency and Harmonics

Exercise 19.1. Identify the frequencies present in $\sin x + \sin(4x)$.

Exercise 19.2. Identify the harmonics appearing in $2\cos x + \cos(5x)$.

Exercise 19.3. Which function oscillates more rapidly, $\sin(2x)$ or $\sin(7x)$?

Frequency and Period

Exercise 19.4. Determine the frequency and period of $\sin(4x)$.

Exercise 19.5. Determine the frequency and period of $\cos(7x)$.

Exercise 19.6. Which function has the shorter period, $\sin(3x)$ or $\sin(5x)$?

Exercise 19.7. Find a function whose frequency is twice that of $\cos x$.

Exercise 19.8. Find a function whose frequency is five times that of $\sin x$.

Superposition

Exercise 19.9. Sketch $\sin x + \frac{1}{2}\sin(2x)$.

Exercise 19.10. Sketch $\cos x + \frac{1}{3}\cos(3x)$.

Exercise 19.11. Explain how superposition differs from multiplication of functions.

Harmonics and Superposition

Exercise 19.12. Identify the harmonics present in $f(x) = \sin x + \frac{1}{2}\sin(2x) + \frac{1}{4}\sin(3x)$.

Exercise 19.13. Identify the harmonics present in $f(x) = 2\cos x - \cos(4x)$.

Exercise 19.14. Which harmonic has the largest amplitude in $f(x) = 3\sin x + \sin(2x) + \frac{1}{2}\sin(5x)$?

Exercise 19.15. Explain why $\sin x + \sin(2x)$ is more complicated than either term alone.

Complex Exponentials

Exercise 19.16. Use Euler's formula to expand e^{i2x}.

Exercise 19.17. Express e^{i3x} in terms of sine and cosine.

Exercise 19.18. Explain why $e^{ix} + e^{-ix}$ is real-valued.

Exercise 19.19. Show that $e^{-ix} = \cos x - i\sin x$.

Exercise 19.20. Show that $e^{ix} + e^{-ix} = 2\cos x$.

Exercise 19.21. Show that $e^{ix} - e^{-ix} = 2i\sin x$.

Exercise 19.22. Explain why e^{ix} may be interpreted as a rotating unit vector.

Finite Fourier Sums

Exercise 19.23. Express $\cos x + \sin x$ as a finite Fourier sum.

Exercise 19.24. Express $2 + \cos x + \sin(2x)$ as a finite Fourier sum.

Exercise 19.25. Identify the constant term and harmonic terms in $3 + 2\cos x - \sin(3x)$.

Exercise 19.26. How many distinct frequencies appear in $\sin x + \sin(2x) + \sin(4x)$?

Exercise 19.27. Explain why every finite Fourier sum is periodic.

Orthogonality and Projection

Exercise 19.28. In Chapter 13, orthogonal vectors represented independent directions. Explain the analogous role played by harmonics.

Exercise 19.29. Why is it useful to think of $\cos x$ and $\cos(5x)$ as distinct directions?

Exercise 19.30. Explain how Fourier coefficients resemble vector coordinates.

Exercise 19.31. Explain how Fourier decomposition resembles projection onto coordinate axes.

Exercise 19.32. Compare the decomposition of a vector into components with the decomposition of a signal into harmonics.

Applications

Exercise 19.33. A musical tone contains frequencies 220 Hz, 440 Hz, and 660 Hz. Which frequencies are harmonics of the fundamental?

Exercise 19.34. Why does a violin sound different from a flute even when both play the same note?

Exercise 19.35. Explain how noise-cancelling headphones use superposition.

Exercise 19.36. Why might an engineer wish to remove certain frequencies from a signal?

Exercise 19.37. Describe one application of Fourier analysis in astronomy.

Exercise 19.38. Describe one application of Fourier analysis in medical imaging.

Connections Across the Book

Exercise 19.39. Explain how Chapter 8 contributes to the Fourier idea.

Exercise 19.40. Explain how amplitwists lead naturally to complex multiplication.

Exercise 19.41. Explain why $e^{i\theta}$ can be viewed simultaneously as a complex number and a rotation.

Exercise 19.42. Explain how the geometric-series formula of Chapter 18 supports the study of complex exponentials.

Exercise 19.43. Trace the chain rotation → amplitwist → complex multiplication → Fourier analysis, explaining the role played by each step.

Exercise 19.44. Explain why Fourier analysis may be viewed as a study of superposed rotations.

Capstone Problems

Exercise 19.45. Trace the conceptual chain similarity → trigonometry → rotation → complex numbers → Fourier analysis, and explain the role played by each step.

Exercise 19.46. Explain why Fourier analysis can be viewed as a study of superposition.

Exercise 19.47. Describe how Chapters 15, 16, and 18 contribute to the Fourier idea.

Challenge Problems

Exercise 19.48. Use Euler's formula to show that $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$.

Exercise 19.49. Use Euler's formula to show that $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$.

Exercise 19.50. Explain why every finite Fourier sum can be written using complex exponentials.

Exercise 19.51. Explain why every finite Fourier sum can also be written using only sines and cosines.

Exercise 19.52. A signal contains only frequencies 1, 3, and 5. Write a possible finite Fourier sum representing such a signal.

Exercise 19.53. Explain why the Fourier idea is fundamentally a statement about representation rather than computation.

Exercise 19.54. The chapter began with the Metonic cycle. Explain how the interaction of two periodic processes can produce a larger apparent cycle.

Exercise 19.55. Write a one-page essay describing how the ideas of this book progress from similarity to Fourier decomposition.

Part VI

Coda: Projection and Interpretation

Chapter 20

Projection, Interpretation, and Living Dynamics

This closing chapter asks a question that has been implicit since Chapter 2 but never stated directly: what is lost whenever an object is replaced by a description of it? Coordinates, projections, and Fourier decompositions have each, in their turn, shown that a single object can be described in more than one way, and that different descriptions make different features easy to see. What none of the earlier chapters asked explicitly is what a description *cannot* see — and it is that question, posed with the ordinary linear algebra already developed in this book, that occupies the pages that follow. The illustrative material in this chapter is drawn from a debate in neuroscience about what it means for a measured neural signal to "represent" something, but the neuroscience is only a worked example. The mathematics being illustrated is entirely general.

20.1 Representation as a Mathematical Relation

Recall from Chapter 2 that a vector's coordinates depend on a chosen basis: the same displacement is recorded by different ordered pairs under different coordinate frames, yet the vector itself, as a displacement, does not change when the frame does. This distinction generalizes. Let X be a space of possible states of some system, and let

$$\pi : X \to Y$$

be any measurement or representation, associating to each state $x \in X$ a description $\pi(x)$ in some representational space Y. The description $\pi(x)$ is not the state x itself; it is x viewed through the particular relation π. Chapters 2, 13, and 14 each supplied an instance of this same pattern — coordinates, a projection onto a direction, and a change of coordinate system — without stating the general principle they shared. This chapter states it, and asks what follows from it.

20.2 Projection and the Kernel

Suppose π is linear, as every projection encountered in this book has been.

Definition 20.1. The *kernel* of a linear map $P : X \to Y$ is $\ker(P) = \{x \in X : P(x) = 0\}$.

Theorem 20.2 (Principle of Invisible Structure)**.** *If $k \in \ker(P)$, then $P(x) = P(x + k)$ for every $x \in X$. Consequently, no observation made solely through P can distinguish x from $x + k$.*

Proof. By linearity, $P(x + k) = P(x) + P(k) = P(x) + 0 = P(x)$. □

The proof is one line, and deliberately so: the content of the theorem is not its difficulty but its interpretation. The kernel of a projection is not an absence of structure in the original system; it is the set of directions in which the system can vary without the chosen measurement noticing. Consider a biological illustration. Let a neuron's full state be some high-dimensional x, encompassing its spike timing, its neuromodulatory context, its metabolic condition, and its recent history, and suppose an experiment measures only a single projection of this state, a firing rate $P(x)$. Neuromodulatory context, metabolic condition, and history may then inhabit, in whole or in part, the kernel of P. Nothing about the neuron has disappeared; the measuring apparatus simply cannot see those directions. The theorem gives this familiar methodological caution — absence of evidence is not evidence of absence — a precise mathematical location: the evidence is absent because it lies in $\ker(P)$, and it can be found only by choosing a different P.

20.3 Multiple Interpretability

A second consequence follows from the same observation, stated now for the image of π rather than its kernel.

Definition 20.3. Given a measured value $y = \pi(x)$, an observer may define a *decoding map* $d : Y \to Q$ and interpret $d(y)$ as the quantity represented by x. *Multiple interpretability* is the fact that, absent further constraint, many decoding maps $d_1, d_2, d_3, \ldots$ may all be defined on the same Y, so that a single measured state supports several formally valid, but mutually incompatible, interpretations.

If a measured value happens to equal some quantity n, it equally well satisfies $y = 2(n+1) - 2$, or countless other relations; the raw correlation between y and n does not, by itself, single out n as *the* thing y represents rather than any of the other quantities y is equally correlated with. This is worth distinguishing from a related and more familiar idea: *multiple realizability* asks how many different physical systems can implement one formal structure. Multiple interpretability asks the reverse question, how many formal structures can be imposed on one physical system, and it is this second question that bears on what a measurement is entitled to claim.

20.4 Sparse Projections and Apparent Representation

Modern data analysis often seeks a low-dimensional projection $z = Px$ of a high-dimensional state $x \in \mathbb{R}^N$, with $P : \mathbb{R}^N \to \mathbb{R}^k$ for $k \ll N$, sometimes further expressed as $x \approx D\alpha$ for a sparse coefficient vector α. Such projections can genuinely reveal invariants, low-dimensional constraints, recurring modes, and predictive relationships; none of that value is in question. What does not follow from the existence of a successful sparse projection is that the system being measured internally computes that projection, that its coefficients are symbols manipulated by the system, or that the axes the projection happens to use are privileged by the system itself rather than convenient to the observer.

Theorem 20.4. *Decodability does not imply internal representation.*

This is not a theorem in the sense of the rest of this book, with a derivation from prior results; it is closer to a definition made precise. A decoder demonstrates that an observer, given $P(x)$, can recover some target quantity. It does not, by that fact alone, demonstrate that the system whose state is x performs the same decoding, or uses $P(x)$ for anything at all. The gap between these two claims is exactly the gap between an observer's successful use of a projection and a claim about what lies outside that projection's kernel.

Comparative Note: A Neuroscientific Instance: Language and Reasoning.

A 2025 study of logical reasoning offers an unusually direct empirical case of the gap this section describes. Kean and colleagues [4] asked whether formal reasoning is carried out using the brain's ordinary natural-language machinery, or whether language instead serves only as an interface through which a problem is presented and a conclusion is reported. Using functional imaging, they found that language-selective brain regions were not substantially engaged while healthy participants solved inductive and deductive reasoning tasks; a separate, domain-general network associated with working memory and cognitive control was engaged instead. They then studied two individuals with profound aphasia, whose grammatical comprehension was severely impaired, and found that both solved logical induction and matrix-reasoning problems at or above typical levels despite this impairment. In the language of this chapter, the natural-language system functions here as one measurable projection of cognitive activity, P_{language}, correlated with reasoning performance under ordinary conditions but not identical to whatever operator actually carries out the reasoning; when that projection is severely degraded, as in profound aphasia, the underlying capacity it had seemed to track remains intact. A close correlation between a measured quantity and an ability, in other words, is not by itself evidence that the measured quantity is the mechanism producing that ability, whether the system under study is a projected vector, a decoded neural signal, or the relationship between language and thought.

20.5 Invariance Without Symbolism

Chapter 1 showed that certain ratios are invariant under scaling; Chapter 6 showed that sine is odd and cosine is even under $\theta \to -\theta$. In each case, invariance was a structural fact: a quantity $I(x)$ satisfying $I(Tx) = I(x)$ for some transformation T. It is worth being precise about what such a fact does and does not establish. If some measured quantity F satisfies $F(R(\theta)x) = F(x)$ for every rotation $R(\theta)$ of Chapter 8, this establishes that F is insensitive to rotation — a genuine structural discovery. It does not, by itself, establish that F "means" or "represents" the identity of the object being rotated; that further claim requires more than invariance alone provides. A productive habit, when this distinction threatens to collapse, is to replace the question "what does this state represent" with the question this book's tools can actually answer: which transformations leave the measured relation unchanged?

20.6 Fourier Decomposition and Observer Choice

Chapter 19 showed that a periodic function may be written as a sum of harmonics, $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$. Nothing in that construction requires the decomposition to be unique in kind: the same function could equally be expressed relative to a basis of localized pulses, wavelets, or modes derived directly from data, and each choice of basis makes some features of the function easy to see and others hard to see, exactly as Chapter 14 showed for polar versus rectangular coordinates. That a signal admits a clean, useful Fourier decomposition therefore does not imply that whatever physical system produced the signal stores, computes, or manipulates a list of Fourier coefficients; it implies only that the Fourier basis is a good basis for that signal, in the same sense that polar coordinates are a good coordinate system for a circle. The decomposition is a fact about the relation between the signal and the chosen basis, not a fact about the internal machinery of whatever produced the signal.

20.7 The Commutativity Test

The distinction between a useful decoder coordinate and a variable a system actually uses suggests a test, rather than only a caution. Suppose a measured state x is projected to $z = P(x)$, and z is claimed to be a genuine variable used by the system rather than merely an observer's coordinate. An intervention that changes x to some x' should then produce a corresponding, predictable change from z to $z' = P(x')$, and this claim can be drawn as a square,

$$\begin{tikzcd} x \arrow[r, "P"] \arrow[d, "\text{intervention}"] & z \arrow[d, "?"] \\ x' \arrow[r, "P"] & z' \end{tikzcd}$$

which commutes exactly when the effect of the intervention on z agrees with what P predicts from its effect on x. If the square commutes reliably under intervention, z behaves as a variable genuinely coupled to the system's dynamics. If it does not, z may still be a perfectly good coordinate for an observer to use, recoverable from the data and correlated with behavior, without being a variable the system itself is sensitive to. The test converts a hard interpretive question into an ordinary question about whether a diagram commutes, which is exactly the kind of question the algebra of this book is built to answer.

20.8 The Representation Theorem

The results of this chapter can be collected into a single closing statement.

Theorem 20.5 (The Representation Theorem)**.** *Let $P : X \to Y$ be any non-injective representation of a system. Then* $\ker(P)$ *is nontrivial, and $P(x) = P(x+k)$ for every $k \in$* $\ker(P)$*. Consequently, no claim about distinctions lying entirely within* $\ker(P)$ *can be established, supported, or refuted by observations made through P alone.*

Proof. Non-injectivity means some two distinct states $x_1 \neq x_2$ satisfy $P(x_1) = P(x_2)$; setting $k = x_1 - x_2$ gives a nonzero element of $\ker(P)$ by linearity, so $\ker(P) \neq \{0\}$. The remaining claims are the Principle of Invisible Structure of Section 20.2, restated for general $k \in \ker(P)$. □

The theorem is almost embarrassingly simple to prove; its force lies entirely in how often it is forgotten in practice, whenever the image of a representation, $P(X)$, is treated as though it were X itself.

20.9 Further Directions

The mathematics developed above — kernels, invariance, basis-dependence, and the commutativity test — is enough to state the chapter's central result rigorously. The same questions extend naturally beyond a single, fixed projection. A living system does not simply sit still to be measured: it acts on its environment and is in turn shaped by that environment, so that organism and environment are better modeled as co-evolving, $dO/dt = F(O, E)$ and $dE/dt = G(E, O)$, than as a fixed input feeding a fixed output. More strikingly, the transformation rule governing such a system need not itself stay fixed the way every operator in Chapters 1 through 19 has: a system's own admissible transformations can change over time, $T_{t+1} = \mathcal{U}(T_t, x_t, e_t)$, so that the pair (x_t, T_t), rather than x_t alone, becomes the true state of the system. Questions of viability (which states a system can occupy and still persist), entrainment (how internal rhythms synchronize with external ones without symbolic prediction), and information-theoretic limits on what successive measurements can recover all extend from this same starting point. Each is a substantial subject in its own right, and each is a natural continuation of the mathematics in this chapter rather than a further topic this book will develop; they are noted here as an indication of where the kernel and commutativity ideas lead next, for a reader who wants to follow them.

20.10 Looking Back

This book opened with two maps of the same city, one small enough for a pocket and one large enough for a classroom wall, observing that a change of scale alters measurement while leaving certain ratios untouched. Chapter 1 called such ratios invariants. Every chapter since has, in its own way, returned to the same pair of questions: what does a given transformation preserve, and what does it discard? Coordinates showed that one object admits many descriptions; rotation and the identities built from it showed that different algebraic forms can encode identical structure; projection showed that a measurement preserves some distinctions while discarding others outright; polar coordinates and amplitwists showed that the choice of representation changes what appears simple; Euler's formula showed that separate formalisms can be manifestations of a single underlying structure; Fourier decomposition showed that even a decomposition into "elementary" pieces is relative to a chosen basis. This chapter's contribution is the final piece of that pattern, made precise rather than left as an intuition: every representation is itself a transformation, and every transformation carries a kernel, a set of distinctions it cannot see. Mathematics supplies extraordinary power to describe structure. What this closing chapter asks a reader to carry forward is the discipline to ask, of any description powerful enough to seem complete, what its kernel contains.

20.11 Exercises

Exercise 20.1. Give two coordinate descriptions of the same vector using different bases, and explain why the vector is not identical to either coordinate list.

Exercise 20.2. Let $P : \mathbb{R}^2 \to \mathbb{R}$ be given by $P(x, y) = x$. Find $\ker(P)$, and give two distinct vectors that P cannot distinguish.

Exercise 20.3. A firing rate is computed by counting events over three different time windows of different lengths. Explain why the resulting values need not describe the same underlying feature of the system.

Exercise 20.4. Let $y = 2x + 1$. Construct two different decoding maps from y to a target quantity, and explain why the correlation between y and x alone does not select one interpretation over the other.

Exercise 20.5. Let $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ project $\mathbb{R}^3$ onto $\mathbb{R}^2$. Find $\ker(P)$, and explain why successfully recovering a label from $P(x)$ does not prove that the original system computes that label.

Exercise 20.6. Find a function of two variables that is invariant under the rotation $R(\theta)$ of Chapter 8, and one that is invariant under the scaling $S(s)$ of Chapter 15. Explain why exhibiting such an invariant is a weaker claim than exhibiting a symbolic representation.

Exercise 20.7. Express a specific periodic signal using two different bases (for instance, a Fourier basis and a basis of localized pulses), and discuss whether the Fourier coefficients should be regarded as physically separate components of the signal.

Exercise 20.8. Draw the commutativity-test square for a claimed decoder $z = P(x)$ of your own construction, and describe what evidence would show the square fails to commute under intervention.

Exercise 20.9. Explain, in your own words, why a useful decoder does not necessarily reveal an internal code used by the system being decoded.

Exercise 20.10. Prove that an invertible change of coordinates has trivial kernel, and explain why this means it preserves every distinction even though it changes every coordinate description.

Exercise 20.11. Prove the Representation Theorem directly for a specific non-injective linear map of your choice, exhibiting a nonzero element of its kernel.

Exercise 20.12. [Capstone] For a variable z recovered from a measured system, distinguish three separate claims: that z is *decodable* (an observer can recover z from the measurement), that z is *causally relevant* (altering z changes the system's subsequent behavior), and that z is *represented* by the system (the system itself uses z as a variable in its own dynamics). Give an example in which the first claim holds but the second does not, and an example in which the first two claims hold but evidence for the third is still lacking. Explain, in a short paragraph, why these three claims are routinely conflated, and why separating them requires most of the mathematics developed in this book.

Review of Algebraic Prerequisites

This appendix collects, without proof, the algebraic facts assumed throughout the text: properties of exponents and radicals, factoring techniques, the quadratic formula, and the algebra of rational expressions. Students who find early exercises unexpectedly difficult are encouraged to consult it first.

.1 Exponents and Radicals

For real numbers a, b and integers m, n (with $a, b \neq 0$ where division or a nonpositive exponent is involved), the exponent rules are

$$a^m a^n = a^{m+n}, \qquad \frac{a^m}{a^n} = a^{m-n}, \qquad (a^m)^n = a^{mn}, \qquad (ab)^n = a^n b^n, \qquad a^0 = 1, \qquad a^{-n} = \frac{1}{a^n}.$$

A radical is related to a fractional exponent by $\sqrt[n]{a} = a^{1/n}$, and more generally $a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$, provided $\sqrt[n]{a}$ is defined. The following radical identities hold whenever both sides are defined:

$$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}, \qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}, \qquad \sqrt[n]{a^n} = |a| \text{ when } n \text{ is even}, \qquad \sqrt[n]{a^n} = a \text{ when } n \text{ is odd.}$$

A radical in a denominator is customarily removed by rationalizing: multiplying numerator and denominator by $\sqrt{a}$ clears a single square root, as in $\dfrac{1}{\sqrt{a}} = \dfrac{\sqrt{a}}{a}$, while a denominator of the form $a + \sqrt{b}$ is cleared by multiplying by its conjugate $a - \sqrt{b}$, using the difference-of-squares identity of the next section.

.2 Factoring

The following factoring patterns recur throughout the text.

$$\begin{aligned}
\text{Common factor:}\quad & ax + ay = a(x + y).\\
\text{Difference of squares:}\quad & a^2 - b^2 = (a - b)(a + b).\\
\text{Perfect square trinomial:}\quad & a^2 \pm 2ab + b^2 = (a \pm b)^2.\\
\text{Difference of cubes:}\quad & a^3 - b^3 = (a - b)(a^2 + ab + b^2).\\
\text{Sum of cubes:}\quad & a^3 + b^3 = (a + b)(a^2 - ab + b^2).\\
\text{Trinomial:}\quad & x^2 + (p + q)x + pq = (x + p)(x + q).
\end{aligned}$$

For a general quadratic $ax^2 + bx + c$ with $a \neq 1$, factoring by grouping is often effective: rewrite the middle term as a sum of two terms whose coefficients multiply to ac and add to b, then factor the resulting four-term expression in pairs.

The *quadratic formula* solves $ax^2 + bx + c = 0$ for $a \neq 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The quantity $b^2 - 4ac$, called the *discriminant*, determines the number and type of solutions: two distinct real solutions when it is positive, one repeated real solution when it is zero, and two complex conjugate solutions when it is negative, as discussed in Chapter 16.

.3 Solving Equations

An equation is solved by applying the same operation to both sides until the unknown is isolated, using operations that preserve the solution set: adding or subtracting the same quantity from both sides, multiplying or dividing both sides by the same nonzero quantity, and factoring a side that equals zero. Two operations deserve particular caution, both discussed in the context of trigonometric equations in Chapter 11: squaring both sides of an equation can introduce extraneous solutions not present in the original equation, and multiplying both sides by an expression that could equal zero, or that involves the unknown, can likewise change the solution set. Every candidate solution obtained by such an operation should be checked in the original equation before being accepted. A rational equation, one containing the unknown in a denominator, is typically solved by multiplying through by the least common denominator to clear the fractions, solving the resulting polynomial equation, and then discarding any solution that makes an original denominator equal to zero.

Table of Identities

The identities developed across Chapters 8–10 are collected here for reference, grouped by family. Each entry names the section in which the identity was derived, since every identity in this book is a consequence of prior results rather than an independent fact to memorize.

.4 Reciprocal and Quotient Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} \qquad \text{(Chapter 9)}$$

.5 Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \text{(Chapter 9)}$$
$$1 + \tan^2\theta = \sec^2\theta \qquad \text{(Chapter 9)}$$
$$1 + \cot^2\theta = \csc^2\theta \qquad \text{(Chapter 9)}$$

.6 Cofunction Identities

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta, \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \qquad \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta. \qquad \text{(Chapter 9)}$$

.7 Even and Odd Identities

$$\cos(-\theta) = \cos\theta, \qquad \sin(-\theta) = -\sin\theta, \qquad \tan(-\theta) = -\tan\theta. \qquad \text{(Chapters 6, 9)}$$

.8 Sum and Difference Formulas

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \qquad \sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B \qquad \sin(A-B) = \sin A\cos B - \cos A\sin B$$

(Chapter 8)

.9 Double-Angle Formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta, \qquad \cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta, \qquad \tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}. \qquad \text{(C}$$

.10 Half-Angle Formulas

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}, \qquad \cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}}, \qquad \tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}. \qquad \text{(Chapter 10)}$$

The sign in each formula is determined by the quadrant of $\theta/2$, not the quadrant of θ; see Section 10.4.

.11 Product-to-Sum Formulas

$$\sin A\cos B = \frac{\sin(A+B)+\sin(A-B)}{2}$$
$$\cos A\cos B = \frac{\cos(A+B)+\cos(A-B)}{2}$$
$$\sin A\sin B = \frac{\cos(A-B)-\cos(A+B)}{2}$$

(Chapter 10)

.12 Sum-to-Product Formulas

$$\sin x+\sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right), \qquad \cos x+\cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right). \qquad \text{(Chapter 10)}$$

.13 Law of Sines and Law of Cosines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \qquad c^2 = a^2+b^2-2ab\cos C. \qquad \text{(Chapter 12)}$$

.14 Exact Values of Common Angles

θ (degrees)	0°	30°	45°	60°	90°
θ (radians)	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\tan\theta$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	undefined

Table 1: Exact trigonometric values derived in Chapter 3.

.15 Euler's Formula and Related Identities

$$e^{i\theta} = \cos\theta + i\sin\theta, \qquad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}. \qquad \text{(Chapter 16)}$$

De Moivre's theorem: $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$. (Chapter 16)

Answers to Selected Exercises

Answers are provided for odd-numbered exercises in each chapter. Full solutions to even-numbered exercises are available in the instructor supplement. Numerical answers are rounded to three significant figures unless an exact form is more natural.

Chapter 0

1. The only admissible sequence of length 3 beginning at a is $a \to b \to c$. This matches the adjacency-matrix method: with vertices ordered (a, b, c), the matrix M^2 has a single nonzero entry in row a, at column c, confirming exactly one walk of length 2 from a, ending at c.

Chapter 1

1. The triangles are similar (SSS: $10/5 = 24/12 = 26/13 = 2$), with scale factor $k = 2$. **3.** The building is approximately 27.5 meters tall ($h = 34(1.7)/2.1 \approx 27.5$ m). Parallel sunlight is essential because it guarantees the person-and-shadow triangle and the building-and-shadow triangle share the same angle of elevation, which is what makes the two triangles similar in the first place.

Chapter 2

1. (a) 5, (b) 13, (c) 17. **3.** $\mathbf{v} = \langle 15, 20 \rangle$, $|\mathbf{v}| = 25$ km. **5.** Multiplying by k scales the magnitude by $|k|$. The direction is unchanged when $k > 0$, and reversed (180° rotation) when $k < 0$; at $k = 0$ the result is the zero vector, which has no defined direction.

Chapter 3

1. Opposite leg $\approx$ 8.62 cm, adjacent leg $\approx$ 11.0 cm ($14 \sin 38°$ and $14 \cos 38°$). **3.** The tower is approximately 51.2 meters tall ($40 \tan 52°$).

Chapter 4

1. 150°: reference angle 30°, Quadrant II. $-60°$: reference angle 60°, Quadrant IV. 300°: reference angle 60°, Quadrant IV. $5\pi/4$ ($= 225°$): reference angle $\pi/4$ (45°), Quadrant III. **3.** Three angles coterminal with 40° are 400°, $-320°$, and 760°, obtained by adding or subtracting integer multiples of 360°; all four angles name the same point of the unit circle because the coterminal-angle theorem of Chapter 4 guarantees that angles differing by a multiple of 360° produce identical terminal points.

Chapter 5

1. Arc length $s = 15\pi/2 \approx 23.6$ cm; sector area $A = 135\pi/4 \approx 106$ cm^2. **3.** $x(t) = \cos\left(\frac{\pi t}{3} + \frac{\pi}{4}\right)$, $y(t) = \sin\left(\frac{\pi t}{3} + \frac{\pi}{4}\right)$; at $t = 2$ the angle is $11\pi/12$, giving position approximately $(-0.966, 0.259)$.

Chapter 6

1. Amplitude 4, period 4π, phase shift 2π to the right, vertical shift 2. **3.** $\theta = \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}$; cosine's graph crosses the value $-\sqrt{2}/2$ twice in every period of 2π (once on the way down, once on the way back up), and the interval $[0, 4\pi)$ spans two full periods, giving four solutions in all.

Chapter 7

1. $\pi/3$. **3.** $\pi/4$. **5.** $5\pi/6$. **7.** $\cos\theta = 15/17$, $\tan\theta = 8/15$ (from the 8-15-17 right triangle). **9.** $\theta = \arctan(0.65) \approx 33.0°$.

Chapter 8

1. $\sin(105°) = \frac{\sqrt{6}+\sqrt{2}}{4}$. **3.** $\sin(15°) = \frac{\sqrt{6}-\sqrt{2}}{4}$. **5.** $\sin(A - B) = \sin(A + (-B)) = \sin A\cos(-B) + \cos A\sin(-B) = \sin A\cos B - \cos A\sin B$, using $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$ from Chapter 6. **7.** Applying the rotation formula, $R(40°)R(50°)$ sends $(1, 0)$ to $(\cos 90°, \sin 90°) = (0, 1)$, which is exactly where $R(90°)$ alone sends $(1, 0)$, confirming the two successive rotations agree with a single 90° rotation. **9.** $R(90°) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; applied to $(1, 0)$ this gives $(0, 1)$.

Chapter 9

1. Dividing $\sin^2\theta + \cos^2\theta = 1$ by $\cos^2\theta$ gives $\tan^2\theta + 1 = \sec^2\theta$. **3.** $\frac{\sin\theta}{\cos\theta} = \tan\theta$ directly by the quotient identity's definition. **5.** $\frac{\cos^2\theta}{1-\sin\theta} = \frac{(1-\sin\theta)(1+\sin\theta)}{1-\sin\theta} = 1 + \sin\theta$, using $\cos^2\theta = 1 - \sin^2\theta = (1-\sin\theta)(1+\sin\theta)$. **7.** $\sin(-\theta) + \cos(-\theta) = -\sin\theta + \cos\theta$, using the odd symmetry of sine and the even symmetry of cosine. **9.** Because $\sin\theta$ and $\cos\theta$ can never both exceed 1 in absolute value: the Pythagorean identity forces $\sin^2\theta + \cos^2\theta = 1$, but $2^2 + 3^2 = 13 \neq 1$, so no angle θ can produce that pair.

Chapter 10

1. $\cos(2\theta) = \cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$. **3.** Substituting $\cos^2\theta = 1 - \sin^2\theta$ into $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ gives $\cos(2\theta) = (1 - \sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta$. **5.** With $\cos\theta = -3/5$ in Quadrant II, $\sin\theta = 4/5$, so $\cos(2\theta) = 1 - 2\sin^2\theta = 1 - 2(16/25) = -7/25$. **7.** $\cos(22.5°) = \sqrt{(1 + \cos 45°)/2} = \frac{\sqrt{2+\sqrt{2}}}{2}$. **9.** $\cos A\cos B = \frac{\cos(A+B) + \cos(A-B)}{2}$, obtained by adding the cosine sum and difference formulas and dividing by 2.

Chapter 11

1. $\theta = \dfrac{\pi}{3} + 2\pi k$ or $\theta = \dfrac{2\pi}{3} + 2\pi k$. **3.** $\theta = \dfrac{\pi}{4} + \pi k$. **5.** $\sin\theta = 0$ gives $\theta = \pi k$; $2\cos\theta + 1 = 0$ gives $\cos\theta = -1/2$, so $\theta = \dfrac{2\pi}{3} + 2\pi k$ or $\theta = \dfrac{4\pi}{3} + 2\pi k$. **7.** Using $\cos(2\theta) = 2\cos^2\theta - 1$, the equation becomes $2\cos^2\theta - \cos\theta - 1 = 0$; substituting $u = \cos\theta$ and factoring, $(2u+1)(u-1) = 0$, so $\cos\theta = 1$ (giving $\theta = 2\pi k$) or $\cos\theta = -1/2$ (giving $\theta = 2\pi/3 + 2\pi k$ or $\theta = 4\pi/3 + 2\pi k$). **9.** Because sine is not one-to-one on all of $\mathbb{R}$: $\arcsin(a)$ returns only the one solution lying in $[-\pi/2, \pi/2]$, while infinitely many other angles, differing by the Quadrant II reflection and by multiples of 2π, also satisfy $\sin\theta = a$.

Chapter 12

1. (a) $b \approx 20.8$; (b) $a \approx 12.1$ (with $B = 65°$); (c) $c \approx 17.4$ (with $A = 67°$). **3.** $A = 75°$, $a \approx 18.5$, $b \approx 16.2$. **5.** (a) $h \approx 8.60 > a = 8$: no triangle. (b) $a = 12 \geq b = 10$: one triangle. (c) $h \approx 7.61 < a = 9 < b = 18$: two triangles. **7.** $B \approx 35.1°$, $C \approx 94.9°$, $c \approx 20.8$ (only one triangle, since $a \geq b$). **9.** (a) $c \approx 8.48$; (b) $c \approx 19.2$; (c) $c \approx 9.19$. **11.** The travelers are approximately 26.8 km apart. **13.** The triangle is right ($6^2 + 8^2 = 10^2$); the acute angles are approximately 36.9° and 53.1°. **15.** Three side lengths satisfying the triangle inequality fix every angle uniquely through the Law of Cosines, since arccos returns exactly one angle in $[0, \pi]$ for each computed ratio, unlike the SSA case, where arcsin leaves a supplementary angle to check. **17.** Area $\approx 2060\ \text{m}^2$. **19.** (a) $c \approx 8.48$, $A \approx 46.3°$, $B \approx 83.7°$; (b) $C = 70°$, $b \approx 16.6$, $c \approx 16.9$; (c) $A \approx 39.4°$, $B \approx 54.7°$, $C \approx 85.9°$. **21.** The ship is approximately 200 km from its starting point (using an included angle of 145° at the turning point). **23.** The Law of Sines is preferable whenever a side-angle pair is already known (AAS, ASA, or SSA); the Law of Cosines is preferable when no such pair is available (SAS or SSS). **25.** The correction term measures how far the triangle departs from a right angle at C: it vanishes when $C = 90°$, shortens the opposite side when C is acute, and lengthens it when C is obtuse. **27.** The Law of Sines propagates a known side-angle ratio to the rest of the triangle; the Law of Cosines reconstructs an unknown side or angle directly from the other two sides, without requiring any angle to be known in advance. **29.** Area $= 84$ square units (using $\cos C = 140/364$, giving $C \approx 67.4°$). **31.** Dropping an altitude from the vertex at angle C splits the opposite side into two segments; expressing each resulting right triangle's legs in terms of a, b, and C and applying the Pythagorean theorem to the full triangle reproduces $c^2 = a^2 + b^2 - 2ab\cos C$ without any coordinate system.

Chapter 13

1. (a) 11; (b) 33; (c) -26. **3.** -6. **5.** $\mathbf{u}\cdot\mathbf{v} = (6)(10)\cos 120° = -30$. **7.** (a) $6 - 6 = 0$: orthogonal. (b) $8 - 8 = 0$: orthogonal. (c) $5 - 6 = -1 \neq 0$: not orthogonal. **9.** $\langle 8, 3\rangle$ (or any scalar multiple), since $\langle -3, 8\rangle \cdot \langle 8, 3\rangle = -24 + 24 = 0$. **11.** $\theta = \arccos(6/\sqrt{85}) \approx 49.4°$. **13.** By the geometric formula, $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$; for nonzero vectors this product is zero exactly when $\cos\theta = 0$, which occurs precisely at $\theta = 90°$. **15.** Scalar projection $= 4$; vector projection $= \langle 4, 0\rangle$. **17.** The scalar projection $|\mathbf{u}|\cos\theta$ measures the length of the parallel component, and multiplying by the unit vector $\mathbf{v}/|\mathbf{v}|$ turns that length into the parallel component vector itself; subtracting this vector projection from $\mathbf{u}$ leaves the perpendicular component. **19.** $W = \mathbf{F}\cdot\mathbf{d} = 36 + 20 = 56$. **21.** A current pushing due north is perpendicular to eastward motion, so the angle between them is 90° and $\mathbf{F}\cdot\mathbf{d} = |\mathbf{F}||\mathbf{d}|\cos 90° = 0$. **23.** Yes: $\langle 4, 1\rangle \cdot \langle 1, -4\rangle = 4 - 4 = 0$, so the two vectors are perpendicular. **25.** Setting $\mathbf{v} = \mathbf{u}$ in the geometric formula gives

$\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}||\mathbf{u}| \cos 0 = |\mathbf{u}|^2$, so $|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$. **27.** Because $\cos\theta$ itself is negative for $90° < \theta < 180°$, and the geometric formula multiplies this negative cosine by the two positive magnitudes. **29.** It decreases monotonically from $|\mathbf{u}||\mathbf{v}|$ at $\theta = 0°$ to $-|\mathbf{u}||\mathbf{v}|$ at $\theta = 180°$, passing through zero at $\theta = 90°$. **31.** Expanding $|\mathbf{u} + \mathbf{v}|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})$ using coordinates and comparing to the Law of Cosines applied to the triangle formed by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} + \mathbf{v}$ (with interior angle $\pi - \theta$ at their join) yields the same identity $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$ as the $|\mathbf{u} - \mathbf{v}|^2$ derivation. **33.** All vectors of the form $t\langle 4, -3\rangle$ for real t, since $\langle 3, 4\rangle \cdot \langle 4, -3\rangle = 12 - 12 = 0$.

Chapter 14

1. (a) $(2\sqrt{3}, 2)$; (b) $(-3\sqrt{2}, 3\sqrt{2})$; (c) $(-5, 0)$. **3.** $r = 4$, $\theta = -\pi/3$ (equivalently $5\pi/3$); polar form $(4, -\pi/3)$. **5.** (a) Quadrant II; (b) Quadrant III; (c) Quadrant IV. **7.** $(1, 1)$ and $(-1, -1)$ both satisfy $y/x = 1$, but the first lies in Quadrant I and the second in Quadrant III; the ratio y/x alone cannot distinguish them; only checking the signs of x and y separately identifies the correct quadrant and hence the correct polar angle. **9.** Converting both to rectangular coordinates: $(5, \pi/6)$ gives $(5\cos 30°, 5\sin 30°) = (5\sqrt{3}/2, 5/2)$, and $(-5, 7\pi/6)$ gives $(-5\cos 210°, -5\sin 210°) = (5\sqrt{3}/2, 5/2)$ as well, confirming the same point. **11.** Because sine and cosine are periodic with period 2π (Chapter 6), adding any multiple of 2π to θ leaves $\cos\theta$ and $\sin\theta$, and hence the point $(r\cos\theta, r\sin\theta)$, unchanged; the negative-radius identity $(r, \theta) = (-r, \theta + \pi)$ supplies a second infinite family for every first one. **13.** $r = 2$ and $r = 5$ are concentric circles of radius 2 and 5 centered at the pole; $\theta = \pi/4$ is the line through the origin inclined at 45°. **15.** Every point satisfying $\theta = c$ shares the same direction from the origin while r is free to take any value, positive or negative, which traces out a full line through the origin rather than just a ray. **17.** Increasing a enlarges the rose proportionally, stretching each petal's maximum distance from the pole to a, without changing the number of petals, which is governed by n. **19.** $\theta = 0 \Rightarrow r = 0$; $\theta = \pi/2 \Rightarrow r = \pi/2 \approx 1.57$; $\theta = \pi \Rightarrow r = \pi \approx 3.14$; $\theta = 3\pi/2 \Rightarrow r = 3\pi/2 \approx 4.71$; $\theta = 2\pi \Rightarrow r = 2\pi \approx 6.28$; the point spirals steadily outward as it completes each quarter turn. **21.** $(120, 35°)$. **23.** Circular motion keeps r constant while only θ changes, which polar coordinates record directly with a single varying quantity, whereas rectangular coordinates require both $x(t)$ and $y(t)$ to vary together in a coupled way. **25.** Rectangular coordinates $(3, 4)$ emphasize horizontal and vertical displacement from the axes; polar coordinates $(5, \arctan(4/3))$ emphasize distance from the origin and direction, the same point described through two different structural lenses. **27.** A circle centered at the origin consists of every point at a fixed distance $r = a$ from the pole regardless of angle, which polar coordinates state directly as $r = a$, while rectangular coordinates must instead express the same set through the implicit equation $x^2 + y^2 = a^2$. **29.** Rectangular coordinates assign a unique horizontal and vertical displacement to each point, but polar coordinates measure angle, which is periodic; every representation $(r, \theta + 2\pi k)$ and $(-r, \theta + \pi + 2\pi k)$ names the same point, so infinitely many coordinate pairs are available where rectangular coordinates offer only one. **31.** Since $r_1 = r_2$ (both being the positive distance to the same point), the conditions $r_1 \cos\theta_1 = r_2 \cos\theta_2$ and $r_1 \sin\theta_1 = r_2 \sin\theta_2$ reduce to $\cos\theta_1 = \cos\theta_2$ and $\sin\theta_1 = \sin\theta_2$; by the coterminal-angle theorem of Chapter 4, this forces $\theta_2 - \theta_1$ to be an integer multiple of 2π. **33.** Polar coordinates already separate a point's description into a magnitude and an angle acted on independently, which is exactly the structure Chapter 15's amplitwist $A = sR(\theta)$ formalizes as a single operator with two independent parameters.

Chapter 15

1. $A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\sqrt{2}\\\sqrt{2}\end{pmatrix}$; the length doubles from 1 to 2, and the direction rotates 45° counterclockwise. **3.** $\begin{pmatrix}2\sqrt{3} & -2\\ 2 & 2\sqrt{3}\end{pmatrix}$. **5.** $(4, 4\sqrt{3})$ (magnitude 8 at angle 60°). **7.** $(2.5, 90°)$. **9.** $(1, 0°)$, the identity transformation. **11.** $A^{-1} = \frac{1}{5}R(-40°)$. **13.** $AA^{-1} = 2R(60°) \cdot \frac{1}{2}R(-60°) = 1 \cdot R(0°) = I$. **15.** Composing a scaling by s and a rotation by θ with its inverse must return every vector to itself exactly, so the inverse must undo both effects: scaling by $1/s$ cancels the amplitude change, and rotating by $-\theta$ cancels the rotation, matching $A^{-1} = \frac{1}{s}R(-\theta)$. **17.** $(4, 90°)$. **19.** $(16, 180°)$, equivalent to scaling by 16 and reversing direction, i.e. the map $\mathbf{v} \mapsto -16\mathbf{v}$. **21.** $A = (1.2, 30°)$, $A^2 = (1.44, 60°)$, $A^3 = (1.728, 90°)$, $A^4 = (2.0736, 120°)$; the points spiral outward, moving farther from the origin while rotating by 30° at each step. **23.** $A^5 = (1, 360°) = (1, 0°)$, the identity: five successive 72° rotations return exactly to the starting orientation, since $5 \times 72° = 360°$. **25.** For $s > 1$, repeated amplitwists spiral outward; for $0 < s < 1$, they spiral inward toward the origin; for $s = 1$, the points remain on a fixed circle, tracing pure rotation with no radial change. **27.** Composing (s_1, θ_1) and (s_2, θ_2) produces $(s_1 s_2, \theta_1 + \theta_2)$, which is itself of the form (s, θ) with $s = s_1 s_2 > 0$, hence again an amplitwist; the set is therefore closed under composition. **29.** If B and B' both satisfy $AB = BA = I$ and $AB' = B'A = I$, then $B = BI = B(AB') = (BA)B' = IB' = B'$, so the inverse is unique. **31.** $A^2 = I$ requires $s^2R(2\theta) = 1 \cdot R(0°)$, so $s^2 = 1$ (giving $s = 1$, since $s > 0$) and 2θ is a multiple of 360°, giving $\theta = 0°$ or $\theta = 180°$: the only solutions are $A = R(0°) = I$ and $A = R(180°)$. **33.** $AB = sR(\theta)\,tR(\phi) = stR(\theta + \phi)$ and $BA = tR(\phi)\,sR(\theta) = tsR(\phi + \theta)$; since ordinary multiplication of s and t commutes and $\theta + \phi = \phi + \theta$, $AB = BA$. **35.** An orientation-preserving similarity fixing the origin preserves all angles and scales every length by the same constant factor $s > 0$ without reflecting the plane; the only linear transformations with this combination of properties are exactly the maps $sR(\theta)$, since the image of the two basis vectors $(1, 0)$ and $(0, 1)$ under such a map must remain perpendicular, of equal length s, and correctly oriented, which forces the matrix into the form of Section 15.4's amplitwist for a unique pair (s, θ).

Chapter 16

1. $i^7 = -i$. **3.** $7 - 3i$. **5.** 5. **7.** The point $(-4, 1)$, four units left and one unit up from the origin. **9.** The complex plane associates $a + bi$ with the point (a, b), using exactly the same horizontal/vertical structure as the Cartesian plane of Chapter 2, now with the horizontal axis interpreted as real parts and the vertical axis as imaginary parts. **11.** $(3 + i)i = -1 + 3i$. **13.** Multiplication by -1 rotates every point 180° about the origin, reversing its direction while leaving its distance from the origin unchanged. **15.** $1 + i = \sqrt{2}(\cos 45° + i \sin 45°)$. **17.** $1 + \sqrt{3}\,i$. **19.** Polar form separates a complex number into a modulus and an angle, and Chapter 16's multiplication theorem shows that multiplying two complex numbers simply multiplies their moduli and adds their angles, avoiding the more cumbersome expansion required in rectangular form. **21.** $e^{i\pi/2} = i$. **23.** $e^{i3\pi/2} = -i$. **25.** $\cos 60° + i \sin 60° = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. **27.** $\cos 180° + i \sin 180° = -1$. **29.** $81(\cos 80° + i \sin 80°) \approx 14.1 + 79.8i$. **31.** Each additional factor of $re^{i\theta}$ in a product adds one more copy of θ to the total angle, by the polar multiplication theorem, so raising to the n-th power accumulates n copies of θ. **33.** $z = 1$, $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$. **35.** $z = e^{i(60k)°}$ for $k = 0, \ldots, 5$: 1, $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, -1, $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$, $\frac{1}{2} - \frac{\sqrt{3}}{2}i$. **37.** Six points spaced 60° apart around the unit circle, starting at $(1, 0)$.

39. The fourth roots of unity are $1, i, -1, -i$, each at distance 1 from the origin and spaced exactly 90° apart, so consecutive roots are connected by equal side lengths and right-angle turns, the defining properties of a square. **41.** All n-th roots of unity lie on the unit circle, equally spaced by $2\pi/n$ radians, which is precisely the condition for a set of points to form the vertices of a regular n-gon. **43.** $-\frac{3}{2} + \frac{3\sqrt{3}}{2}i \approx -1.5 + 2.60i$. **45.** $|e^{i\theta}z| = |e^{i\theta}||z| = 1 \cdot |z| = |z|$, since $|e^{i\theta}| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$ by the Pythagorean identity. **47.** Both statements say that composing two scale-and-rotate operations multiplies the two amplitudes and adds the two angles; the amplitwist law expresses this with matrices, and Euler's formula expresses the identical rule with complex exponentials. **49.** Modulus $\sqrt{2}$, angle 45° ($\pi/4$). **51.** Chapter 15 built the operator $A = sR(\theta)$ and derived its composition law purely geometrically, using matrices, before any complex number had been introduced; that same law reappears in this chapter as ordinary complex multiplication, so Chapter 15 supplies the geometric picture that this chapter's algebra formalizes. **53.** If $z_1, z_2 \neq 0$ then $|z_1|, |z_2| > 0$, so $|z_1 z_2| = |z_1||z_2| > 0$, which means $z_1 z_2 \neq 0$. **55.** Multiplication by i sends $(a, b) \mapsto (-b, a)$, which matches the action of the rotation matrix $R(90°)$ exactly, as shown in Section 16.5. **57.** $e^{i\theta_1}e^{i\theta_2} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$, which expands and simplifies via the angle-addition identities of Chapter 8 to $\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = e^{i(\theta_1+\theta_2)}$. **59.** Writing $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, the multiplication theorem gives $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$, whose modulus is $r_1 r_2 = |z_1||z_2|$. **61.** A complex number's polar form already separates magnitude from angle, so multiplying two complex numbers, which multiplies their real and imaginary parts according to the distributive law, cannot help but multiply the two magnitudes and add the two angles simultaneously, exactly the amplitwist composition law. **63.** $z = e^{i(45k)^\circ}$ for $k = 0, \ldots, 7$, at $0°, 45°, 90°, 135°, 180°, 225°, 270°, 315°$. **65.** Since $z \neq 1$, the finite geometric series formula of Chapter 18 gives $1 + z + \cdots + z^{n-1} = (1 - z^n)/(1 - z)$; because z is an n-th root of unity, $z^n = 1$, so the numerator is 0 and the entire sum is 0. **67.** By Exercise 45's result, multiplying by a fixed complex number of modulus 1 never changes a point's modulus, so every point generated by repeated multiplication remains at the same fixed distance from the origin, tracing a circle. **69.** $z = 1$ (polar $1e^{i0^\circ}$, rectangular 1); $z = e^{i120^\circ}$ (rectangular $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$); $z = e^{i240^\circ}$ (rectangular $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$). **71.** Every nonzero complex number $z = re^{i\theta}$ corresponds to exactly one amplitwist $A = rR(\theta)$ with amplitude r and angle θ; since Chapter 15 showed every such amplitwist is a unique orientation-preserving similarity fixing the origin, every nonzero complex number likewise determines one such transformation uniquely, and every such transformation corresponds to exactly one nonzero complex number.

Chapter 17

1. $\theta = 0.5$: ≈ 0.9589; $\theta = 0.1$: ≈ 0.9983; $\theta = 0.01$: ≈ 0.99998. **3.** As θ shrinks toward 0, the ratio $\sin\theta/\theta$ moves steadily closer to 1, matching the numerical pattern of Exercise 1 and confirming that $\sin\theta$ and θ themselves grow closer together in value as $\theta \to 0$. **5.** See the proof of the theorem in Section 17.4: dividing the sandwich inequality $\sin\theta < \theta < \tan\theta$ by $\sin\theta$ gives $\cos\theta < \sin\theta/\theta < 1$, and since $\cos\theta \to 1$ as $\theta \to 0$, the Squeeze Theorem forces $\sin\theta/\theta \to 1$ as well. **7.** $f'(x) = -\sin x$. **9.** $f'(x) = 2\cos(2x)$. **11.** $f'(x) = 5\cos(5x) - 2\sin(2x)$. **13.** $\mathbf{v}(t) = \langle -\sin t, \cos t\rangle$. **15.** $\mathbf{r}(t) \cdot \mathbf{v}(t) = \cos t(-\sin t) + \sin t(\cos t) = 0$, confirming the velocity is always perpendicular to, and hence tangent to, the unit circle. **17.** $\mathbf{v}(\pi) = \langle -\sin\pi, \cos\pi\rangle = \langle 0, -1\rangle$. **19.** $\frac{d^2}{dx^2}\sin x = \frac{d}{dx}\cos x = -\sin x$. **21.** Since $\frac{d^2}{dx^2}\sin x = -\sin x$ and $\frac{d^2}{dx^2}\cos x = -\cos x$, both functions satisfy $y'' = -y$ directly by substitution, so any combination $y = A\sin x + B\cos x$ solves the equation as well. **23.** Differentiating the pair $(\cos x, \sin x)$

produces $(-\sin x, \cos x)$, exactly the 90°-rotated pair; multiplying $\cos x + i \sin x$ by i in Chapter 16 produces $-\sin x + i \cos x$, the same rotation expressed as complex multiplication. Differentiation and multiplication by i therefore perform the identical quarter-turn operation, one written as a vector-valued derivative and the other as complex arithmetic.

Chapter 18

1. $a_{10} = 4 + 9(3) = 31$. **3.** Writing S_n forward and backward and adding, as in the proof of Section 18.3, gives $2S_n = n(a_1 + a_n)$ regardless of the specific sequence, so any independent derivation should arrive at the same identity $S_n = \frac{n}{2}(a_1 + a_n)$ by this pairing argument. **5.** $a_8 = 3 \cdot 2^7 = 384$. **7.** $\sum_{k=0}^{\infty} (1/3)^k = \dfrac{1}{1 - 1/3} = \dfrac{3}{2}$. **9.** The ball travels a total of 40 meters (10 meters for the initial drop, plus $2 \times 15 = 30$ meters for the infinite sequence of up-and-down bounces, using $\sum_{k=1}^{\infty} 10(0.6)^k = 15$). **11.** These are the three cube roots of unity from Chapter 16, and by the geometric-series identity, $1 + z + z^2 = (1 - z^3)/(1 - z) = (1 - 1)/(1 - z) = 0$ since $z^3 = 1$. **13.** $1 + i + i^2 + i^3 = 1 + i - 1 - i = 0$; the partial sums are 1, $1 + i$, i, and 0, corresponding to the points $(1, 0)$, $(1, 1)$, $(0, 1)$, and the origin, tracing a closed square path in the complex plane. **15.** Balance $= 2000(1.04)^8 \approx \$2737.14$. **17.** In $A^n = s^n R(n\theta)$, consecutive scaling factors s^n and s^{n+1} have constant ratio $s^{n+1}/s^n = s$, satisfying the definition of a geometric sequence with common ratio s. **19.** The modulus r^n changes by a constant multiplicative factor r with each increase of n, a geometric progression, while the angle $n\theta$ changes by a constant additive amount θ with each increase of n, an arithmetic progression; both patterns coexist in the same expression because amplitude and angle combine multiplicatively and additively, respectively, under repeated multiplication. **21.** If $|r| \geq 1$, then $|r^n| = |r|^n$ either stays fixed at 1 (when $|r| = 1$) or grows without bound (when $|r| > 1$) as $n \to \infty$, so r^n never approaches 0; consequently $S_n = a(1 - r^n)/(1 - r)$ has no finite limit, and the infinite series cannot converge.

Chapter 19

1. Frequencies 1 and 4. **3.** $\sin(7x)$ oscillates more rapidly. **5.** Frequency 7, period $2\pi/7$. **7.** $\cos(2x)$ (or $\sin(2x)$), since it completes twice as many cycles as $\cos x$ in the same interval. **9.** A wave resembling $\sin x$ with a smaller secondary bump added near its peak and trough, since the second harmonic completes two full cycles for every one cycle of the first. **11.** Superposition adds two oscillations pointwise, preserving both original frequencies as distinct components of the sum; multiplication instead produces amplitude modulation, in which the frequencies combine into new sum and difference frequencies rather than appearing separately. **13.** The fundamental ($\cos x$) and its fourth harmonic ($\cos 4x$). **15.** $\sin x$ and $\sin(2x)$ complete their cycles at different rates, so their sum does not repeat with the simple period of either term alone; the combined waveform develops additional peaks and asymmetries that neither original term possesses by itself. **17.** $e^{i3x} = \cos(3x) + i \sin(3x)$. **19.** By Euler's formula, $e^{-ix} = \cos(-x) + i \sin(-x)$; since cosine is even and sine is odd (Chapter 6), this equals $\cos x - i \sin x$. **21.** $e^{ix} - e^{-ix} = (\cos x + i \sin x) - (\cos x - i \sin x) = 2i \sin x$. **23.** $\cos x + \sin x$ is already a finite Fourier sum with $a_0 = 0$, $a_1 = 1$, $b_1 = 1$, and every other coefficient zero. **25.** Constant term 3; harmonic terms $2 \cos x$ (first cosine harmonic) and $-\sin(3x)$ (third sine harmonic). **27.** Each individual term $\cos(nx)$ or $\sin(nx)$ is periodic with period $2\pi/n$, a divisor of 2π; a finite sum of such terms therefore repeats once every term has completed a whole number of cycles, which is guaranteed after an interval of 2π. **29.** Treating them as distinct directions lets the amplitude associated with each frequency be read off independently, in the same way a vector's components along perpendicular axes can be measured one at a time without interference from the others. **31.** A signal's Fourier coefficients measure how much of each harmonic direction is present, exactly as a vector's coordinates measure how much of each basis direction is present when projected onto the coordinate axes of Chapter 2. **33.** All three: 220 Hz is the fundamental, 440 Hz is its second harmonic, and 660 Hz is its third harmonic. **35.** Noise-cancelling headphones generate a sound wave that is approximately the negative of the ambient noise; superposing the two waveforms produces a sum close to zero, cancelling the perceived sound. **37.** Astronomers use Fourier methods to extract the period of a variable star's brightness fluctuations or a transiting exoplanet's orbit from a periodic light curve. **39.** Chapter 8's rotation composition and angle-addition identities are exactly what let e^{inx} be manipulated algebraically, supplying the underlying machinery each harmonic term in a Fourier sum depends on. **41.** As a point on the unit circle, $e^{i\theta}$ is an ordinary complex number with modulus 1; as an operator, multiplying by $e^{i\theta}$ rotates any complex number it is applied to by angle θ, so the same object is simultaneously a number and a transformation. **43.** Rotation (Chapter 8) establishes that angles add under composition; the amplitwist (Chapter 15) combines rotation with scaling into a single operator; complex multiplication (Chapter 16) realizes that same operator algebraically; Fourier analysis (this chapter) combines many such rotating, scaled terms at different frequencies to build an arbitrary periodic function. **45.** Similarity (Chapter 1) identifies scale-invariant ratios; trigonometry (Chapters 3–4) names those ratios as functions of angle; rotation (Chapter 8) reveals how angles combine under composition; complex numbers (Chapter 16) unify scaling and rotation into a single algebraic object; Fourier analysis (this chapter) combines many such objects, at different frequencies, into a representation of an arbitrary periodic function. **47.** Chapter 15 supplies the geometric operator (simultaneous scaling and rotation) underlying each harmonic term; Chapter 16 supplies the algebraic notation e^{inx} for that operator; Chapter 18 supplies the finite-sum machinery for combining many such terms into a single superposition. **49.** Subtracting $e^{-ix} = \cos x - i \sin x$ from $e^{ix} = \cos x + i \sin x$ gives $e^{ix} - e^{-ix} = 2i \sin x$ (Exercise 21); dividing both sides by $2i$ gives $\sin x = (e^{ix} - e^{-ix})/(2i)$. **51.** Applying Euler's formula to each term $c_k e^{ikx}$ expands it into $\cos(kx)$ and $\sin(kx)$ parts; collecting these real and imaginary contributions across all terms of the sum, as in Section 19.6, converts the entire expression into a sum of sines and cosines alone. **53.** The Fourier coefficients a_n, b_n describe how a function is expressed relative to a chosen basis of harmonics, a statement about representation, not a set of instructions for computing the function's value at a particular input, which is what a computational procedure would provide. **55.** Open-ended; a strong essay should trace the book's own chain from Chapter 1's similarity through rotation, amplitwists, complex numbers, and finally to this chapter's superposition of harmonics, using the specific results derived at each stage rather than restating the chapter titles alone.

Chapter 20

1. The displacement $\mathbf{v} = \langle 1, 1 \rangle$ in the standard basis becomes $(\sqrt{2}, 0)$ in a basis rotated 45°; the underlying displacement itself does not change, only the numbers used to record it relative to the chosen frame, exactly as Section 20.1 argues for a general coordinate representation. **3.** A short window captures fast fluctuations in the event rate that a long window averages away entirely, so the three window lengths compute genuinely different statistical summaries of the same underlying process, none of which is more intrinsically "the" rate than the others without further justification. **5.** $\ker(P) = \{(0, 0, z) : z \in \mathbb{R}\}$, the z-axis. Recovering a label from $P(x) = (x_1, x_2)$ shows only that the label correlates with the first two coordinates of x; it says nothing about whatever value of z actually occurred, so two systems differing only in z would be indistinguishable to this projection even though they might differ in ways the projection cannot detect. **7.** A square wave can be built either from a Fourier sine series or from a sum of localized rectangular pulses; both bases reconstruct the same signal, but neither the Fourier coefficients nor the pulse amplitudes should be regarded as physically separate components stored inside the signal itself, since the signal is a single object and the decomposition is a property of the chosen basis, not of the signal in isolation. **9.** A decoder only demonstrates that an observer, using some fitted mapping, can recover a target quantity from measured data; this shows the projection and the target are correlated, but says nothing about whether the system being measured performs that same mapping internally, which is a separate and stronger claim requiring its own evidence. **11.** Let $P(x, y) = x$ on $\mathbb{R}^2$. Then $\ker(P) = \{(0, y) : y \in \mathbb{R}\}$, which contains the nonzero element $(0, 1)$; directly, $P(1, 2) = 1$ and $P(1, 2) + (0, 1)) = P(1, 3) = 1$ as well, so the two states $(1, 2)$ and $(1, 3)$, which differ by an element of the kernel, are indistinguishable to this projection, exactly as the Representation Theorem predicts.

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