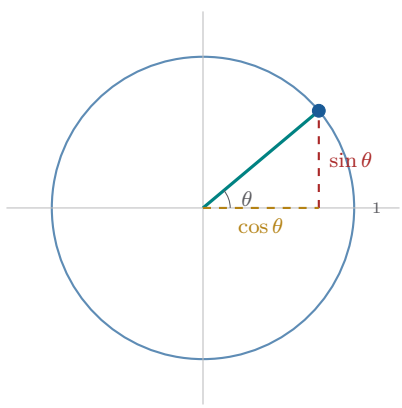
Constraint and Transformation

An Integrated Introduction to

Algebra, Geometry, Trigonometry, and Chemical Stoichiometry



$\sin^2\theta + \cos^2\theta = 1$

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Constraint and Transformation
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Contents

Notation and Symbols **ix**

Preface **xi**

1 Mathematics as Structured Relation **1**
1.1 The Central Thesis . . . 1
1.2 Four Languages, One Pattern . . . 1
1.3 Constraint Networks . . . 2
1.4 The Book's Arc . . . 2
Exercises . . . 3

I Algebra: Relations and Constraint Systems **4**

2 Numbers, Variables, and Representation **5**
Opening Perspective . . . 5
2.1 Numbers as Quantities . . . 7
2.2 The Real Number Line . . . 7
2.3 Proportionality: The Simplest Relation . . . 7
Exercises . . . 8

3 Equations and Balance **9**
Opening Perspective . . . 9
3.1 Equations as Statements of Equality . . . 11
3.1.1 The Balance Model . . . 11
3.2 Solving Linear Equations . . . 11
3.3 Dimensional Analysis . . . 12
Exercises . . . 12

4 Proportions and Scaling **13**
Opening Perspective . . . 13
4.1 Ratios and Proportions . . . 15
4.2 Scaling Laws . . . 15
Exercises . . . 16

5 Systems of Equations **17**
Opening Perspective . . . 17
5.1 Multiple Constraints, Multiple Unknowns . . . 18
5.2 Geometric Interpretation . . . 19
Exercises . . . 19

6 Matrices and Linear Transformations **20**
Opening Perspective . . . 20
6.1 Systems in Matrix Form . . . 21
6.2 Matrices as Transformations of Space . . . 22
6.3 The Stoichiometric Matrix . . . 22
Exercises . . . 22

II Geometry: Spatial Structure and Invariance **23**

7 Points, Lines, and Distance **24**
Opening Perspective . . . 24
7.1 The Coordinate Plane . . . 25
7.2 Angles and Their Measure . . . 26
Exercises . . . 26

8 Triangles and the Pythagorean Relation **27**
Opening Perspective . . . 27
8.1 Right Triangles . . . 29
8.2 Applications . . . 29
Exercises . . . 29

9 Similarity and Scaling in Geometry **30**
Opening Perspective . . . 30
9.1 Similar Figures . . . 32
Exercises . . . 32

10 Transformations of the Plane **33**
Opening Perspective . . . 33
10.1 Isometries: Transformations Preserving Distance . . . 34
Exercises . . . 35

11 Area, Volume, and Spatial Distribution **36**
Opening Perspective . . . 36
11.1 Area Formulas as Algebraic Constraints . . . 38
Exercises . . . 38

III Trigonometry: Cycles and Rotational Systems **39**

12 Angles and Circular Geometry **40**
Opening Perspective . . . 40
12.1 The Circle as Generator of Cycles . . . 41
Exercises . . . 42

13 Sine, Cosine, and the Unit Circle **43**
Opening Perspective . . . 43
13.1 The Fundamental Definitions . . . 44
13.2 Exact Values . . . 45
Exercises . . . 45

14 Trigonometric Functions and Graphs **46**
Opening Perspective . . . 46
14.1 Periodic Functions . . . 47
14.2 Variable Stars: Trigonometric Modelling . . . 48
Exercises . . . 48

15 Modeling Cyclic Phenomena **49**
Opening Perspective . . . 49
15.1 Harmonic Motion . . . 50
Exercises . . . 51

16 Rotations and Harmonic Motion **52**
Opening Perspective . . . 52
16.1 Circular Motion and Projections . . . 53
Exercises . . . 54

IV Stoichiometry: Conservation and Transformation in Chemistry **55**

17 Atoms, Molecules, and Chemical Formulas **56**
Opening Perspective . . . 56
17.1 The Atomic Hypothesis and Conservation . . . 58
Exercises . . . 58

18 Balancing Chemical Equations **59**
Opening Perspective . . . 59
18.1 Conservation as a Linear System . . . 60
Exercises . . . 61

19 Moles and Quantitative Chemistry **62**
Opening Perspective . . . 62
19.1 Stoichiometric Calculations . . . 63
Exercises . . . 64

20 Reaction Ratios and Limiting Reagents **65**
Opening Perspective . . . 65
20.1 The Limiting Reagent . . . 66
20.2 Percent Yield . . . 67
Exercises . . . 67

21 Reaction Networks and Conservation Laws **68**
Opening Perspective . . . 68
21.1 Dynamic Stoichiometry . . . 69

V Integration: Mathematics of Structured Systems **71**

22 Constraint Networks **72**
Opening Perspective . . . 72
22.1 Networks as Mathematical Objects . . . 73

23 Cycles, Flows, and Equilibrium **75**
Opening Perspective . . . 75
23.1 Three Kinds of Balance . . . 76
23.2 Equilibrium as Fixed Point . . . 76

24 Scaling and Multilevel Structure **77**
Opening Perspective . . . 77
24.1 Patterns That Recur Across Scales . . . 78

25 From Elementary Systems to Scientific Models **79**
Opening Perspective . . . 79
25.1 The Architecture of Science . . . 80

A Algebra as Constraint Before Content **82**
A.1 Constraint Systems . . . 82
A.2 Admissibility and Relaxation . . . 82
A.3 Degrees of Freedom . . . 83

B Geometry as Structured Invariance **84**
B.1 Euclidean Space and the Inner Product . . . 84
B.2 Linear Transformations and Invariance . . . 85
B.3 Configuration Spaces . . . 85
B.4 Hilbert Space Structure . . . 85
B.5 Geometric Interpretation . . . 86

C Trigonometry as Oscillation, Rotation, and Recurrence **87**
C.1 The Rotation Group . . . 87
C.2 Harmonic Analysis . . . 88
C.3 Plane Waves and Higher–Dimensional Oscillation . . . 89

D Stoichiometry as Conservation Algebra **90**
D.1 The Stoichiometric Null Space . . . 90
D.2 Dynamic Stoichiometry . . . 91
D.3 Stoichiometric Subspaces . . . 91
D.4 Continuum Limit and Field Conservation . . . 92

E RSVP Field Theory: Foundational Derivation **93**
E.1 The Fundamental Fields . . . 93
E.2 Governing Equations . . . 93
E.3 Lyapunov Functional and Entropic Descent . . . 94
E.4 Linear Perturbation Theory . . . 94

F Entropic Redshift and Non-Expansion Cosmology **96**
F.1 Redshift Without Metric Expansion . . . 96
F.2 Solution of the Frequency Transport Equation . . . 96
F.3 Definition of Redshift . . . 97
F.4 Weak-Field Approximation . . . 97
F.5 Emergence of the Linear Distance–Redshift Law . . . 98

G TARTAN: Recursive Tiling and Annotated Noise **99**
G.1 Multiscale Partition . . . 99
G.2 Annotated Field States . . . 99
G.3 Recursive Field Updates . . . 100
G.4 Noise Decomposition . . . 100
G.5 Trajectory Memory . . . 101
G.6 Multiscale Dynamics . . . 101

H Simulated Agency as Sparse Projection **102**
H.1 State Spaces and the Projection Operator . . . 102
H.2 The Agency Loop . . . 103
H.3 Dynamical Structure of the Agency Loop . . . 103

I Category-Theoretic Semantics of Constraint Systems **104**
I.1 Categories . . . 104
I.2 Functors and Structural Preservation . . . 105
I.3 Symmetric Monoidal Categories . . . 105
I.4 Constraint Systems as Monoidal Structures . . . 106
I.5 Colimits and Merging of Systems . . . 106

J Sheaf-Theoretic Gluing and Local-to-Global Consistency **107**
J.1 Presheaves . . . 107
J.2 Sheaf Condition . . . 107
J.3 Gluing and Consistency . . . 108
J.4 Applications to Constraint Systems . . . 108
J.5 Local-to-Global Reasoning . . . 109

K Variational Principles and Entropic Descent **110**
K.1 Variational Functionals . . . 110
K.2 First Variation and Euler–Lagrange Equation . . . 110
K.3 Gradient Flow Dynamics . . . 111
K.4 Examples Across Domains . . . 112
K.5 Unified Interpretation . . . 112

L Worked Derivations: Bridges Between the Introductory Subjects and the Theory **113**
L.1 From Linear Algebra to Stoichiometry to Field Theory . . . 113
L.2 From Euclidean Distance to the L^2 Metric . . . 115
L.3 From Trigonometric Oscillation to RSVP Eigenmodes . . . 116
L.4 Unified Perspective . . . 116

Glossary **118**

Selected Answers **120**

Abbreviated Periodic Table **121**

Notation and Symbols

The following symbols are used consistently throughout the text. A change of meaning is always announced explicitly.

Symbol	Meaning
$\mathbb{R}$	real number line
$\mathbb{N}$	natural numbers $\{0, 1, 2, \dots\}$
x, y, z	generic real variables
a, b, c	constants or geometric side lengths
f, g, h	function names
θ	angle (radians unless noted)
r	radius of a circle or sphere
n	amount of substance (moles, mol)
M	molar mass (g/mol)
m	mass (grams or kilograms)
L	luminosity of a star (watts)
I	radiation intensity ($\mathrm{W/m^2}$)
d	distance
A	area
V	volume
Φ	scalar field (RSVP plenum density)
$\mathbf{v}$	vector flow field
S	entropy density field
γ	stoichiometric coefficient vector
N	element–species incidence matrix
$\mathcal{J}$	variational functional
$\preceq$	causal priority order on events

Preface

This book is built on one conviction: algebra, geometry, trigonometry, and chemistry are not separate disciplines accidentally packaged together by tradition. They are four dialects of the same language — the language of *constraint and transformation*.

Every equation encountered here is a claim about balance. Algebra encodes symbolic balance among quantities. Geometry encodes spatial balance among dimensions. Trigonometry encodes rotational and cyclic balance. Stoichiometry encodes material balance in chemical reactions. To understand any one of these is to recognize a pattern that recurs across all the others: a system remains coherent because its constitutive relations must simultaneously be satisfied.

This orientation reflects a structural feature of scientific reasoning. Quantitative systems are defined by relations that must remain internally consistent as quantities change. A child discovering that a toy machine follows a hidden rule, an astrophysicist reading the composition of a star from its spectrum, a political theorist analyzing how collective power emerges from coordination, a novelist constructing an ecologically stable imaginary world — each is solving, in different vocabulary, the same problem: *what constraints govern this system, and what transformations preserve them?*

Intellectual influences. The book's perspective has been shaped by several intellectual traditions. The developmental psychology of Gopnik [2] shows how causal inference in children is structurally isomorphic with algebraic constraint satisfaction. The astrophysics of Payne-Gaposchkin [8] demonstrates that geometric and spectroscopic reasoning can reveal chemical composition at stellar distances. The political philosophy of Arendt [1] offers a model in which social systems are organized by relational structure rather than by fixed hierarchies. The complexity science of Juarrero [5] formalizes how constraints propagate through dynamical systems, while the speculative fiction of Guin [3] explores the societal consequences of such constraint structures with rare concreteness. The visual mathematical approach of Needham [6, 7] informs the book's emphasis on diagrammatic reasoning.

How to read this book. Each chapter follows a three-step rhythm: *observation* (a phenomenon in plain language), *formalization* (the mathematical statement), and *extension* (the same idea in a new domain). The main text is self-contained for an introductory reader. Each chapter ends with three tiers of exercises: **Practice** (computation), **Exercise** (conceptual reasoning), and **Challenge** (open-ended exploration). The appendices develop the deeper formal architecture — field theory, category theory, sheaf semantics, variational principles, and recursive simulation — for readers who wish to see how elementary mathematics extends into contemporary research.

Prerequisites. Arithmetic: fractions, decimals, percentages, signed numbers. No prior algebra, geometry, or chemistry is assumed.

Chapter 1

Mathematics as Structured Relation

1.1 The Central Thesis

Scientific descriptions can be viewed as networks of relationships rather than lists of isolated facts. This book proposes that the four subjects it covers — algebra, geometry, trigonometry, and chemistry — are four languages for a single idea: a system is defined not merely by its components but by the *relations* those components must satisfy.

Definition Constraint

A **constraint** is any condition restricting the possible values of one or more quantities. Formally, given variables $x_1, \ldots, x_n$, a constraint is a relation

$$F(x_1, \ldots, x_n) = 0.$$

The **solution set** is $\mathcal{C} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid F(x_1, \ldots, x_n) = 0\}$.

Guiding Principle

Every equation in this book is a claim that two descriptions of the same situation must agree. Solving the equation means finding the values that live inside the solution set $\mathcal{C}$.

1.2 Four Languages, One Pattern

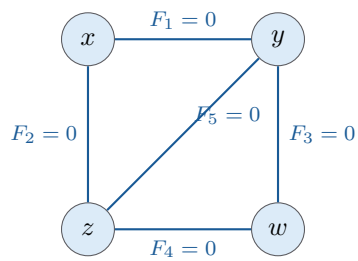
The table below introduces the central motif. Each row names a type of constraint, the domain where it appears, and a representative equation.

Domain	Constraint type	Canonical equation
Algebra	symbolic / numerical	$2x + 3 = 11$
Geometry	spatial / dimensional	$a^2 + b^2 = c^2$
Trigonometry	cyclic / angular	$\sin^2\theta + \cos^2\theta = 1$
Stoichiometry	material / atomic	$2H_2 + O_2 \to 2H_2O$

In every case the equation is a *conservation statement*: it asserts that some quantity — a sum, a squared length, a trigonometric magnitude, or an atom count — is preserved under transformation.

1.3 Constraint Networks

A single equation imposes one constraint. Real systems impose many constraints simultaneously. Representing them as a network reveals their mutual dependencies.



Each node represents a quantity; each edge represents a constraint relating the two adjacent quantities. A consistent system is one in which all constraints can be satisfied simultaneously.

Deeper Structure

The general form of a linear constraint system is $A\mathbf{x} = \mathbf{b}$, where A is an $m \times n$ matrix. Appendix A develops this idea formally, including its connections to the RSVP field framework and Spherepop event-history calculus.

1.4 The Book's Arc

The book progresses from local to global. Part I develops algebraic constraints on individual quantities. Part II extends this to spatial relationships. Part III describes cyclic and rotational constraints. Part IV applies all of this to chemical systems, which are constraint networks over atomic species. Part V synthesizes the four domains into a unified picture of structured systems.

Exercises

Exercise 1.1. Write down three everyday relationships and express each as an equation $F(x, y) = 0$. What does the solution set mean in each case?

Challenge 1.2. Draw a constraint network with five quantities and at least six constraints. Choose a real system (a recipe, a budget, a simple machine) as your model. Explain why some constraints are necessary and others may be redundant.

Part I

Algebra: Relations and Constraint Systems

Chapter 2

Numbers, Variables, and Representation

From Counting to Representation

Human beings began using numbers long before mathematics existed as a formal discipline. Early societies counted livestock, measured land, tracked seasons, and divided resources among households. In each case numbers served a practical role: they were tools for describing the quantitative structure of the world.

At first, numbers were attached to specific objects. Five sheep meant five actual animals standing in a field. Three days meant three rotations of the Earth between sunrise and sunrise. Gradually, however, a profound conceptual shift occurred. People began to treat numbers not merely as labels for physical things but as abstract entities that could be manipulated independently of the objects they described.

This transition—from counting objects to reasoning about numbers themselves—marks the birth of mathematics.

Representation as the foundation of mathematics.

The central idea of this chapter is that numbers allow us to represent quantitative relationships. Representation means that a symbolic structure stands in for a physical or conceptual situation. A number can represent the length of a road, the temperature of the air, the number of molecules in a gas sample, or the brightness of a distant star.

Once a quantity is represented symbolically, we can manipulate the symbols according to consistent rules. Arithmetic operations such as addition and multiplication are not merely computational procedures. They encode structural facts about quantities themselves. If two lengths are combined end to end, their numerical representations add. If an object is scaled by a factor of three, the number describing its length is multiplied by three. Arithmetic therefore reflects the structure of the physical relationships that numbers represent.

In this way mathematics becomes a language for describing patterns of constraint.

Variables and the power of abstraction.

The next conceptual step beyond numbers is the introduction of variables. A variable is a symbol that stands for a quantity whose value may change or may be unknown. Instead

of describing a single specific situation, a variable allows us to describe an entire family of situations at once.

For example, consider the distance travelled by a car moving at constant speed. If the speed is 60 kilometres per hour, then the distance travelled depends on the amount of time that passes. Writing the relationship as

$$d = 60t$$

expresses not just one measurement but infinitely many possible measurements. For every possible value of the time variable t, the distance d is determined by the same rule.

This is the essence of algebra: relationships between quantities are expressed symbolically so that the structure of the relationship becomes visible.

From particular facts to general laws.

Before algebraic notation was developed, mathematicians described relationships in words. Babylonian and Greek texts contain lengthy verbal descriptions of geometric and arithmetic procedures. The introduction of symbolic algebra in the early modern period transformed mathematics because it allowed general laws to be expressed concisely.

A single equation such as

$$y = kx$$

encodes an entire class of proportional relationships. The same equation can describe the distance travelled by a moving object, the amount of chemical produced in a reaction, or the intensity of light spreading through space. What changes from one application to another is the interpretation of the variables.

In other words, algebra separates structure from context.

Numbers as coordinates in a space of possibilities.

One of the most powerful ways to visualize numbers is to place them on a line. The number line transforms arithmetic into geometry: addition becomes movement to the right, subtraction becomes movement to the left, and distance between numbers corresponds to magnitude.

This geometric viewpoint will become increasingly important as the book progresses. Algebraic equations describe sets of points on the number line or in the plane. Systems of equations describe intersections of these sets. Later chapters will extend this idea further, showing how geometric, trigonometric, and chemical relationships all arise from the same underlying structure of constraints among quantities.

Constraint and transformation.

Throughout this book we will repeatedly encounter two complementary ideas.

A *constraint* is a rule that restricts which combinations of quantities are possible. An equation such as $d = 60t$ is a constraint because it specifies the relationship that distance and time must satisfy for an object moving at constant speed.

A *transformation* is a change in the quantities of a system that preserves the constraint. When time increases, distance increases in exactly the way required by the equation. The equation itself remains true throughout the transformation.

Seen from this perspective, mathematics is the study of structures that remain consistent under change. Numbers provide the language in which those structures can be expressed.

The sections that follow introduce the basic elements of this language. We begin with numbers as representations of quantities, then introduce variables that allow relationships to be expressed in general form. The number line provides a geometric picture of these ideas, and the concept of proportionality gives our first example of a mathematical law linking two quantities together.

From these simple ingredients the entire machinery of algebra will grow.

2.1 Numbers as Quantities

A **number** is a symbolic stand-in for a measurable quantity: a count, a length, a temperature. Numbers obey arithmetic: addition, subtraction, multiplication, division.

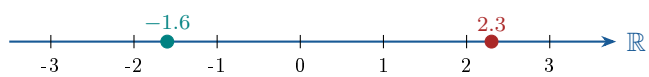
Definition Variable

A **variable** is a letter that stands for an unknown or varying quantity. An **expression** is a combination of numbers and variables using arithmetic operations. An expression does not assert equality; it simply describes a quantity.

The expression $3x + 5$ has a specific numerical value for each specific value of x. It is not an equation and does not claim anything — it is simply a description.

2.2 The Real Number Line

Every real number corresponds to a unique point on an infinite line. Distance from the origin measures magnitude; direction indicates sign.



2.3 Proportionality: The Simplest Relation

Theorem 2.1 (Direct Proportionality)**.** *If two quantities x and y satisfy $y = kx$ for a constant $k \neq 0$, we say y is* ***directly proportional*** *to x with proportionality constant k. The graph of this relation is a straight line through the origin.*

Worked Example

A car travels at a constant speed of 60 km/h. Distance d (km) and time t (h) satisfy

$$d = 60\,t.$$

Checking units: km = (km/h) · h. ✓
The slope $k = 60$ is the rate of change: each additional hour yields 60 more kilometres. For $t = 2.5$ h, $d = 150$ km.

Historical Note Alison Gopnik and Causal Inference

In her research on how children understand the world, Alison Gopnik showed that even toddlers spontaneously construct algebraic-style rules from observation [2]. A child who notices that a machine lights up whenever two specific blocks are inserted together is, in effect, discovering a proportional or Boolean rule governing the system. Elementary algebra makes this intuition precise.

Exercises

Practice 2.1. A recipe uses 3 cups of flour per 2 cups of sugar. Write an equation relating flour f to sugar s. How much sugar is needed for 9 cups of flour?

Exercise 2.2. A toy machine activates according to $A = 3B + 2$, where B is the number of blue blocks inserted. If the observed activation value is $A = 14$, determine B and interpret the result.

Challenge 2.3. Collect five proportional relationships from daily life (speed and distance, ingredients in a recipe, etc.) and estimate each proportionality constant. Which relationships are only approximately proportional? What causes the deviation?

Chapter 3

Equations and Balance

Balance as the Logic of Equations

In the previous chapter we introduced numbers as representations of quantities and variables as symbols that allow relationships between quantities to be written in general form. Once variables enter the picture, mathematics acquires a new kind of expressive power. We can describe not only specific measurements but entire classes of situations. The central tool that makes this possible is the equation.

An equation is a statement that two expressions represent the same quantity. At first glance this seems almost trivial. Writing $2 + 3 = 5$ merely confirms something already known. Yet the conceptual importance of the equals sign cannot be overstated. It asserts that two different ways of describing a situation must ultimately agree.

Seen in this light, an equation is not merely a piece of notation but a claim about the structure of a system.

Equality as balance.

One of the most helpful ways to understand an equation is to imagine a balance scale. A balance scale compares two masses placed on opposite sides. When the scale is level, the masses must be equal. If one side is heavier, the scale tilts until the imbalance is corrected.

An equation behaves in exactly the same way. The two sides of the equation correspond to the two sides of the scale. The equality sign indicates that the system is in equilibrium: whatever quantity appears on the left must match the quantity on the right.

If we alter one side of the equation without making the same alteration to the other, the balance is broken. To preserve equality we must perform the same operation on both sides simultaneously. This simple principle lies at the heart of all algebraic manipulation.

Solving equations as restoring equilibrium.

Most equations encountered in algebra contain unknown quantities. The purpose of solving an equation is to determine which value of the variable restores balance between the two sides.

For example, suppose the equation

$$2x + 3 = 11$$

represents a relationship between two quantities. The value of x is not immediately obvious,

but the equation tells us that whatever number x represents must make the left-hand side equal to eleven.

Solving the equation means systematically transforming it until the variable stands alone. Each transformation preserves the balance of the equation while making its structure clearer. Eventually the value of the unknown becomes visible.

In this sense algebraic problem solving is not a matter of guessing numbers but of applying operations that preserve equality.

Equations as constraints.

Within the broader framework of this book, equations play the role of constraints. A constraint is a rule restricting which combinations of quantities are possible. The equation $2x + 3 = 11$ does not permit x to take arbitrary values. Only one value satisfies the condition imposed by the equation.

More complicated systems may contain many equations simultaneously. Each equation contributes its own constraint, and the solution to the system must satisfy all of them at once. Later chapters will show that geometry, trigonometry, and chemistry all rely on exactly this idea. The Pythagorean theorem, the identity $\sin^2\theta + \cos^2\theta = 1$, and the conservation of atoms in a chemical reaction are all equations imposing constraints on the quantities involved.

Thus the humble algebraic equation provides a prototype for the laws of science.

Units and dimensional consistency.

Equations describing real physical systems must obey an additional principle: the units on both sides must match. If one side of an equation is measured in metres, the other side must also represent a length. If one side is measured in joules, the other side must represent energy.

This requirement is known as dimensional consistency. It acts as a powerful error-checking mechanism. If the units do not agree, something in the reasoning must be wrong.

Dimensional analysis therefore complements algebraic manipulation. While algebra preserves equality between numerical expressions, dimensional analysis preserves consistency between the physical quantities those numbers represent.

Equations as the language of science.

Nearly every scientific law can be expressed as an equation. Newton's laws of motion, the ideal gas law, the conservation of energy, and the chemical equations governing reactions are all statements that certain quantities must remain in balance.

The algebra developed in this chapter provides the basic techniques needed to work with such relationships. By learning how to manipulate equations and isolate unknown quantities, we gain the ability to predict how a system will behave under changing conditions.

In the sections that follow we begin with the formal definition of an equation and examine how its structure reflects the idea of balance. From there we develop systematic methods for solving linear equations and apply them to problems drawn from everyday reasoning and scientific measurement.

What begins as a simple principle of equality will ultimately become the foundation for describing the structure of the physical world.

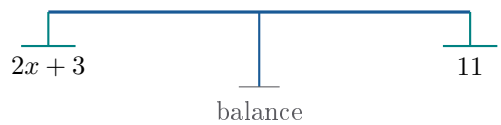
3.1 Equations as Statements of Equality

Definition Equation and Solution

An **equation** is a statement $A = B$ where A and B are expressions. A **solution** is any assignment of values to the variables that makes the statement true.

An equation is therefore a constraint: it asserts that exactly one value (or set of values) satisfies the balance condition.

3.1.1 The Balance Model



Whatever operation is applied to one side of the equation must be applied identically to the other — the scale must remain balanced.

3.2 Solving Linear Equations

Theorem 3.1 (Isolation Principle)**.** *A linear equation $ax + b = c$ (with $a \neq 0$) has the unique solution*

$$x = \frac{c - b}{a}.$$

Proof Sketch. Subtract b from both sides: $ax = c - b$. Divide both sides by $a \neq 0$. □

Worked Example Balancing a simple equation

Solve $2x + 3 = 11$.

$2x + 3 = 11$	original constraint
$2x = 8$	subtract 3
$x = 4$	divide by 2

Verification: $2(4) + 3 = 11$. ✓

Worked Example Gopnik: Hypothesis testing as equation solving

A child hypothesises that a toy machine follows the rule $A = 3G + Y$, where G and Y are the numbers of green and yellow blocks. She observes $A = 8$ and inserts $Y = 2$. Solving for G:

$$3G + 2 = 8 \implies 3G = 6 \implies G = 2.$$

3.3 Dimensional Analysis

Guiding Principle

Every valid equation involving physical quantities must be **dimensionally consistent**: the units on both sides must be identical.

Worked Example Converting units

Convert 72 km/h to m/s:

$$72\ \frac{\text{km}}{\text{h}} \times \frac{1000\ \text{m}}{1\ \text{km}} \times \frac{1\ \text{h}}{3600\ \text{s}} = 20\ \frac{\text{m}}{\text{s}}.$$

Exercises

Practice 3.1. Solve each equation.
(a) $5x - 7 = 18$ (b) $\frac{x}{3} + 4 = 9$ (c) $-2x + 10 = 0$ (d) $3(x - 2) = 15$.

Exercise 3.2. A reaction produces 0.5 mol of gas per minute. Write the constraint equation and determine how many moles are produced in 2.5 hours. Include units at every step.

Challenge 3.3. Arendt [1] argues that political power P emerges from coordination, not mere numbers. Model $P = cn$ where c is coordination strength and n is the number of participants. If $c = 0.2$ initially and the group triples its coordination to $c = 0.6$, by what factor does P change? What does this say about collective action?

Chapter 4

Proportions and Scaling

Patterns That Scale

Many relationships in nature do not depend on the absolute size of a system but on the ratios between its parts. If two ingredients in a recipe must be mixed in a ratio of three to two, the taste of the dish remains the same whether we cook for two people or for two hundred. If a map is drawn at a certain scale, the shapes of mountains and rivers are preserved even though their distances are reduced. When astronomers analyze the light of distant stars, they compare the relative strengths of spectral lines rather than their absolute brightness.

In each case the essential information lies in a proportion.

Ratios as comparative measurements.

A ratio compares two quantities by expressing how large one is relative to the other. When we write a ratio such as $3 : 2$ or $\frac{3}{2}$, we are not describing the absolute size of either quantity individually. We are describing their relationship.

Ratios therefore capture the structure of a system rather than its magnitude. Doubling both quantities in the ratio $3 : 2$ produces $6 : 4$, yet the relationship remains unchanged because the proportion between the quantities is preserved.

This invariance under scaling makes ratios an especially powerful mathematical tool.

Proportions as constraints.

A proportion is a statement that two ratios are equal. Writing

$$\frac{a}{b} = \frac{c}{d}$$

asserts that the relationship between a and b is identical to the relationship between c and d. Such equations appear constantly in science and everyday reasoning. The gears of a machine rotate in fixed ratios determined by their teeth. The concentration of a chemical solution expresses the ratio between solute and solvent. The brightness of a star measured through different spectral lines reveals the proportional abundance of its elements.

Proportions therefore act as constraints linking multiple quantities together.

Scaling and the structure of space.

Ratios become even more powerful when we consider how systems change size. Imagine enlarging a geometric figure such as a triangle. All of its side lengths increase by the same

factor, but the angles remain the same. The figure preserves its shape even though its size changes.

This property is known as similarity, and it arises because the relationships between the lengths of the sides remain proportional. Geometry therefore provides a visual illustration of proportional reasoning: scaling a system preserves its structure when ratios remain constant.

Scaling relationships also govern many physical processes. The area of a square grows with the square of its side length. The volume of a cube grows with the cube of its side length. The brightness of a star observed from Earth decreases with the square of the distance from the observer. In each case a simple change in size produces a predictable change in another quantity.

Understanding these scaling laws is one of the most important goals of mathematics in science.

Proportional reasoning in scientific discovery.

Historically, many major scientific discoveries relied on recognizing proportional relationships. In astronomy, the intensity of spectral lines in starlight reveals the relative abundance of chemical elements. Cecilia Payne-Gaposchkin's groundbreaking work showed that the Sun and most stars consist primarily of hydrogen and helium. Her conclusion followed from proportional comparisons between observed spectral lines and theoretical predictions.

Similar reasoning appears in physics, chemistry, biology, and economics. When scientists compare measurements across systems of different sizes, they often find that the same proportional pattern repeats again and again.

Such patterns hint at underlying laws governing the system.

From ratios to scaling laws.

The concept of proportionality can be extended even further. Sometimes one quantity does not increase in direct proportion to another but instead follows a power relationship such as

$$y = kx^p.$$

When the input quantity x changes by some factor, the output quantity changes by that factor raised to a power. This phenomenon, known as power scaling, appears across an extraordinary range of natural processes.

For example, doubling the radius of a circle quadruples its area. The relationship between radius and area therefore follows a square law. Similarly, the energy emitted by a star depends on the fourth power of its temperature, a relationship known as the Stefan–Boltzmann law.

Power scaling reveals how quantities transform as systems grow or shrink.

Scaling in human systems.

Scaling relationships do not apply only to natural phenomena. They also shape the organization of societies and ecosystems. The carrying capacity of an island, the distribution of resources in a community, and the growth of cities all depend on how space and resources scale with population.

Ursula K. Le Guin's speculative fiction explores such questions in remarkably concrete ways. In her imagined societies, ecological balance often depends on proportional relationships between land area, resource availability, and population size. Mathematical reasoning about scaling therefore becomes a tool for understanding how complex systems remain sustainable.

In this chapter we develop the mathematics needed to work with ratios, proportions, and scaling laws. We begin by examining how ratios express relationships between quantities and how proportions allow unknown quantities to be determined. We then extend these ideas to power laws that describe how systems change when their size changes.

These concepts will reappear throughout the book. Geometry relies on similarity and scale. Trigonometry describes cyclic relationships that repeat proportionally. Chemistry uses stoichiometric ratios to balance reactions. In each domain the same principle applies: understanding the relative relationships between quantities often reveals more about a system than measuring their absolute values.

4.1 Ratios and Proportions

A **ratio** compares two quantities. A **proportion** asserts that two ratios are equal:

$$\frac{a}{b} = \frac{c}{d}.$$

Worked Example Stellar mass ratios

In the hydrogen fusion reaction $4\text{H} \to \text{He}$, the mass ratio of hydrogen consumed to helium produced is approximately $4:1$ (by nucleon count). If a stellar region contains 10^{12} hydrogen nuclei per second, the production rate of helium nuclei satisfies

$$\frac{n_{\text{He}}}{1} = \frac{n_{\text{H}}}{4} \implies n_{\text{He}} = \frac{10^{12}}{4} = 2.5 \times 10^{11} \text{ nuclei/s}.$$

Historical Note Cecilia Payne-Gaposchkin

Cecilia Payne-Gaposchkin (1900–1979) discovered in her 1925 doctoral dissertation that stars are composed primarily of hydrogen and helium — a result initially rejected by the astronomical establishment but later recognised as one of the most important discoveries in astrophysics [8]. Her analysis relied on proportional reasoning connecting spectral line intensities to elemental abundances.

4.2 Scaling Laws

Theorem 4.1 (Power Scaling)**.** *If quantity y scales as the p-th power of quantity x, then $y = kx^p$. When x is multiplied by a factor λ, y is multiplied by λ^p.*

Worked Example Le Guin: island resource capacity

In a circular island settlement of radius $r = 2$ km, the total area is $A = \pi r^2 = 4\pi \approx 12.6$ km^2. If agricultural land occupies half the island and each km^2 supports 500 people:

$$\text{capacity} = 500 \times \frac{4\pi}{2} \times 100 = 500 \times 200\pi \approx 314{,}159 \text{ people}.$$

Doubling the radius to $r = 4$ km quadruples the area and thus quadruples the capacity. Scaling is not linear [3].

Exercises

Practice 4.1. A car uses 8 L per 100 km. How far can it travel on 22 L?

Exercise 4.2. A solution is prepared at 5 g of salt per 200 mL. How much salt is needed for 500 mL? Write the proportion and solve.

Challenge 4.3. Le Guin's utopian novel *The Dispossessed* [3] depicts a society on an arid moon that must balance population against scarce resources. Suppose the moon's habitable area is A km^2, each km^2 supports at most p people, and the society aims to keep population at 80% of capacity. Express the target population as a function of A and p. How does the target change if a drought reduces A by 15%?

Chapter 5

Systems of Equations

Networks of Constraints

In the previous chapters we studied equations involving a single unknown quantity. Such equations impose a constraint on the possible values that the variable may take. Solving the equation means finding the value that satisfies the condition imposed by the relationship.

Many real systems, however, cannot be described by a single equation. They involve several quantities that influence one another simultaneously. The price of goods in a market depends on supply and demand. The motion of planets depends on the gravitational forces of many bodies at once. The composition of a chemical mixture depends on the quantities of all substances present.

In situations like these, the quantities involved form a web of relationships rather than a single isolated rule.

From individual equations to systems.

A system of equations is a collection of constraints that must all be satisfied at the same time. Each equation expresses one relationship between the variables. The full solution must satisfy every equation in the system simultaneously.

This requirement dramatically narrows the set of possible solutions. A single equation in two variables usually describes infinitely many points. But when two independent equations are imposed together, only the points that satisfy both relations remain possible.

In this sense a system of equations acts like a set of intersecting filters, progressively restricting the set of admissible states.

The geometric viewpoint.

One of the most illuminating ways to understand systems of equations is to interpret them geometrically. A linear equation such as

$$ax + by = c$$

represents a straight line in the coordinate plane. Every point on that line satisfies the equation.

If we introduce a second equation involving the same variables, it defines a second line. The point where the two lines intersect is the unique pair of values that satisfies both equations at once.

This geometric picture reveals the logic of the system immediately. If the lines intersect at one point, the system has a unique solution. If the lines are parallel, they never meet, and the system has no solution. If the lines coincide, every point on the line satisfies both equations, producing infinitely many solutions.

Algebra and geometry are therefore describing the same structure in different languages.

Constraint propagation in complex systems.

Systems of equations also provide a simple model of how constraints propagate through larger networks. When one relationship in a system changes, the other relationships must adjust in order to maintain consistency.

Consider a system in which food production depends on labor, and energy production depends on both labor and food supply. A reduction in labor does not merely affect the first equation; it ripples through the entire system. The quantities connected by the equations must reorganize themselves until all constraints are again satisfied.

Philosopher of complexity Alicia Juarrero describes this phenomenon as the propagation of constraints through a network of relations. In a well-structured system, the relationships between components determine how the system responds to change.

Mathematically, systems of equations provide one of the simplest representations of this idea.

From small systems to scientific models.

Although the examples in this chapter involve only two variables and two equations, the same principles apply to much larger systems. Economists model markets with dozens of simultaneous equations. Engineers analyze electrical circuits by solving systems relating currents and voltages. Chemists balance reactions by writing conservation equations for each element involved.

Even modern physical theories often begin as systems of equations relating multiple interacting quantities.

The methods developed in this chapter therefore form the foundation of many areas of scientific reasoning.

In the sections that follow we first define what it means for several equations to form a system. We then explore the geometric interpretation of such systems and learn algebraic techniques for solving them. By viewing equations as interconnected constraints, we begin to see how mathematical relationships organize entire networks of quantities rather than isolated variables.

5.1 Multiple Constraints, Multiple Unknowns

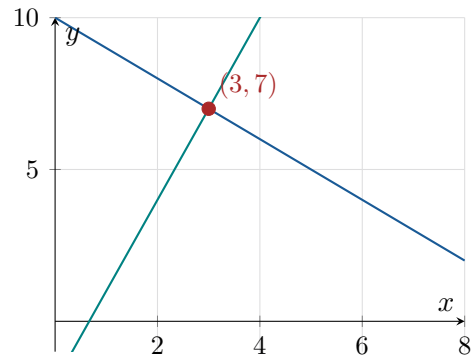
When a situation involves two or more unknowns, a single equation constrains but does not determine. A system of n equations in n unknowns typically yields a unique solution.

Definition System of Equations

A **system of m equations in n unknowns** is a collection $\{F_1 = 0, \ldots, F_m = 0\}$ that must all hold simultaneously. The **solution** is any tuple satisfying every equation at once.

5.2 Geometric Interpretation

Each linear equation $ax+by=c$ defines a **line** in the xy-plane. A system of two such equations corresponds to two lines; the solution is their intersection.



Worked Example Juarrero: constraint propagation through a system

Juarrero [5] emphasises that constraints propagate through systems: a change in one relation alters the whole. Consider a simple model:

$$\begin{cases} F = 2L \\ E = F + L, \end{cases}$$

with $L = 100$. Then $F = 200$, $E = 300$. If a drought reduces L to 80, then $F = 160$ and $E = 240$. The constraint cascade propagates through the entire system.

Worked Example Solving by elimination

Solve $\{x + y = 10,\ 3x - y = 2\}$.
Adding the two equations: $4x = 12$, so $x = 3$. Then $3 + y = 10$, so $y = 7$.
Verification: $3 + 7 = 10$ ✓; $9 - 7 = 2$ ✓.

Exercises

Practice 5.1. Solve the system $\{2x + 3y = 12,\ x - y = 1\}$ by both substitution and elimination.

Exercise 5.2. Interpret geometrically: (a) a system with no solution; (b) a system with infinitely many solutions. What does each case mean for the constraint network?

Challenge 5.3. A chemist mixes Solution A (20% acid) and Solution B (50% acid) to produce 300 mL of 35% acid. Set up and solve the system. Connect this to Juarrero's idea of constraints propagating through a mixture network.

Chapter 6

Matrices and Linear Transformations

Transforming Systems

In the previous chapter we studied systems of equations as collections of constraints that must be satisfied simultaneously. Solving such systems allowed us to determine the values of several unknown quantities at once. Each equation imposed a restriction, and the intersection of those restrictions determined the solution.

Although the algebraic techniques for solving systems are useful in their own right, they also point toward a deeper mathematical idea. Behind every system of linear equations lies a transformation that relates one set of quantities to another.

Matrices provide a compact language for describing these transformations.

From equations to structure.

Consider again a pair of linear equations such as

$$a_1x + b_1y = c_1, \qquad a_2x + b_2y = c_2.$$

At first glance these appear to be two independent statements. Yet both equations involve the same variables, and both share the same coefficients arranged in a particular pattern. Writing these coefficients as a rectangular array reveals the underlying structure that the equations have in common.

This array is called a matrix.

A matrix collects numerical relationships into a single mathematical object. Instead of working with several equations individually, we can manipulate the entire system at once.

Matrices as operators on space.

One of the most remarkable discoveries of nineteenth-century mathematics was that matrices do more than organize equations. They also describe transformations of geometric space.

When a matrix multiplies a vector, it sends that vector to a new location. If we apply the same matrix to every point in the plane, the entire plane is transformed. Lines may rotate, stretch, or shear, but the relationships between points follow precise rules determined by the matrix.

In this way matrices bridge algebra and geometry. A system of equations can be interpreted either as a set of algebraic constraints or as a geometric transformation of space.

Linear transformations.

Transformations represented by matrices have a special property called linearity. A transformation is linear if it preserves two fundamental operations: addition and scalar multiplication.

If two vectors are added together before a linear transformation is applied, the result is the same as transforming each vector first and then adding the results. Similarly, multiplying a vector by a number before the transformation produces the same outcome as transforming the vector and then scaling the result.

These properties ensure that straight lines remain straight lines and that the structure of space is preserved in a predictable way.

Because of this consistency, linear transformations form the foundation of many areas of mathematics and science.

Matrices across scientific disciplines.

Matrices appear wherever systems of relationships must be analyzed simultaneously. In physics they describe rotations, reflections, and the dynamics of coupled systems. In computer graphics they determine how objects are rotated and scaled on a screen. In statistics they encode relationships among many variables at once.

Chemistry provides another striking example. The balancing of chemical reactions can be expressed using a stoichiometric matrix whose rows represent chemical elements and whose columns represent molecular species. Conservation of atoms then becomes the statement that the matrix multiplied by the reaction coefficients equals zero.

This is exactly the same algebraic structure we encountered when solving systems of equations.

From small matrices to large systems.

Although the examples in this chapter involve small matrices, the same principles apply to systems with hundreds or thousands of variables. Modern scientific models often consist of enormous matrices describing networks of interactions among components.

Climate models, neural networks, and economic simulations all rely on linear algebra as a foundational tool. The ability to represent and manipulate systems using matrices makes it possible to analyze complex relationships that would otherwise be impossible to manage.

The sections that follow introduce matrices formally and show how they allow systems of equations to be written in a compact and powerful notation. We then interpret matrices as transformations of geometric space and examine how the same mathematical structure appears in the analysis of chemical reactions.

By the end of the chapter, systems of equations will no longer appear as isolated algebraic puzzles. Instead they will reveal themselves as instances of a broader idea: linear transformations acting on spaces of quantities.

6.1 Systems in Matrix Form

A system $\{a_1x + b_1y = c_1,\ a_2x + b_2y = c_2\}$ may be written as

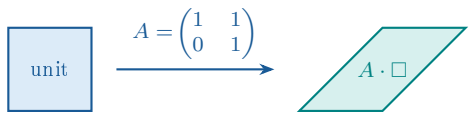
$$A\mathbf{x} = \mathbf{b}, \qquad A = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Definition Matrix

An $m \times n$ **matrix** is a rectangular array of mn numbers arranged in m rows and n columns. It encodes a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ by $T(\mathbf{x}) = A\mathbf{x}$.

6.2 Matrices as Transformations of Space

A 2×2 matrix A transforms every point (x, y) in the plane. The unit square is sent to a parallelogram.



6.3 The Stoichiometric Matrix

Chemical reactions are encoded by a stoichiometric matrix N whose rows index elements and whose columns index species.

Worked Example Water formation as matrix equation

For $2H_2 + O_2 \to 2H_2O$, writing $\gamma = (-2, -1, 2)^\top$ (negative for reactants):

$$N = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix}, \qquad N\gamma = \begin{pmatrix} 2(-2) + 0(-1) + 2(2) \\ 0(-2) + 2(-1) + 1(2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Conservation is encoded by $N\gamma = \mathbf{0}$.

Deeper Structure

Appendix D develops stoichiometry as conservation algebra in full generality, including the reaction rate equation $\dot{c} = Sr(c)$.

Exercises

Practice 6.1. Multiply $\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

Exercise 6.2. Write the system $\{x + 2y = 7,\ 3x - y = 5\}$ as $A\mathbf{x} = \mathbf{b}$ and solve using Cramer's rule or row reduction.

Challenge 6.3. Model Arendt's power equation $P = C\mathbf{n}$ as a matrix transformation, where $\mathbf{n}$ is a vector of participants across three groups and C is a 1×3 coordination matrix. What does the rank of C represent?

Part II

Geometry: Spatial Structure and Invariance

Chapter 7

Points, Lines, and Distance

Space as Structured Relation

Up to this point the relationships we have studied have been expressed primarily through algebraic symbols. Variables represented quantities, equations expressed constraints among those quantities, and systems of equations revealed how multiple constraints interact simultaneously. Although these ideas may seem purely symbolic, they already contain the seeds of geometry.

Geometry arises when quantities are interpreted as positions in space.

From numbers to locations.

A single number can represent a position along a line. When we place numbers on a number line, addition corresponds to movement along the line and subtraction corresponds to movement in the opposite direction. The numerical structure of arithmetic therefore acquires a spatial interpretation.

Extending this idea produces the coordinate plane. Instead of describing position with a single number, we describe it using an ordered pair of numbers. One coordinate measures horizontal position and the other measures vertical position. Together they determine a unique point in the plane.

This simple idea, introduced by René Descartes in the seventeenth century, unified two previously separate branches of mathematics: algebra and geometry.

Geometry as the study of spatial constraints.

Once points in space can be represented by coordinates, geometric relationships can be expressed as equations. A line becomes the set of all points satisfying a linear equation. A circle becomes the set of points satisfying a quadratic relation. More complex curves correspond to more complicated equations.

In this way geometry becomes another language for describing constraints.

Instead of writing an equation and solving for a number, we can visualize the entire set of solutions as a geometric figure. The shape of the figure reflects the structure of the equation.

For example, a linear equation in two variables produces a straight line, while a quadratic equation produces a curved shape such as a circle or parabola. Algebra and geometry are therefore two perspectives on the same underlying mathematical structure.

Distance and the geometry of space.

Among the most fundamental questions in geometry is the concept of distance. Given two points in space, how far apart are they? Measuring distance allows us to compare lengths, describe shapes, and determine the size of geometric figures.

Remarkably, the formula for distance in the coordinate plane follows directly from the Pythagorean theorem. A pair of coordinates defines the legs of a right triangle whose hypotenuse represents the distance between the points. The relationship between algebra and geometry therefore appears again: an equation describing squares of lengths becomes a geometric measurement.

Distance is one of the key invariants of geometry. Certain transformations of space—such as translations and rotations—preserve distance exactly. Understanding these invariants will become central in later chapters.

Angles and rotation.

Another fundamental geometric concept is the angle. While distance measures separation between points, angles measure the amount of rotation between directions.

Angles appear in many contexts: the orientation of a line, the turning of a wheel, the orbit of a planet, or the motion of waves and periodic phenomena. Because of this wide applicability, mathematicians developed a natural unit for measuring angles based on the geometry of circles.

This unit is the radian.

Unlike degrees, which divide a circle into an arbitrary number of parts, the radian measures angles directly in terms of arc length. The angle subtended by an arc equal to the radius of the circle is defined to be one radian. This definition links angles to the geometry of circles in a natural and mathematically convenient way.

Geometry as the bridge between algebra and physics.

The coordinate methods introduced in this chapter form the foundation of analytic geometry, a subject that plays a crucial role in nearly every scientific discipline. Physicists describe motion using coordinates and distances. Astronomers locate stars on celestial coordinate systems. Engineers analyze forces and structures using geometric relationships expressed algebraically.

By translating spatial relationships into equations, analytic geometry allows the methods of algebra to be applied directly to problems involving space.

In the sections that follow we introduce the coordinate plane and show how points, lines, and geometric relationships can be expressed using algebraic formulas. We then examine the concept of angle and introduce the radian as the natural unit for measuring rotation.

These ideas provide the foundation for the geometric viewpoint that will support the rest of the book. Just as equations describe numerical constraints among quantities, geometric figures describe spatial constraints among points. Understanding this connection will allow us to move freely between algebraic reasoning and geometric intuition.

7.1 The Coordinate Plane

René Descartes' insight was that geometric objects can be described algebraically. A point is an ordered pair (x, y); a line is the solution set of a linear equation; a circle is the solution set of a quadratic relation.

Theorem 7.1 (Distance Formula)**.** *The distance between points* (x_1, y_1) *and* (x_2, y_2) *in the plane is*

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This formula is a consequence of the Pythagorean theorem, which we derive in the next chapter.

7.2 Angles and Their Measure

Definition Radian

One **radian** is the angle subtended at the centre of a circle by an arc equal in length to the radius. Since the full circumference is $2\pi r$, a full rotation equals 2π radians.

Conversion: $\theta_{\text{rad}} = \frac{\pi}{180}\theta_{\text{deg}}$.

Exercises

Practice 7.1. Convert: (a) 60 to radians; (b) $\frac{3\pi}{4}$ to degrees.

Exercise 7.2. Show algebraically that the midpoint of the segment from (x_1, y_1) to (x_2, y_2) is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Chapter 8

Triangles and the Pythagorean Relation

Geometry of Orthogonality

In the previous chapter we introduced the coordinate plane and showed how geometric objects can be described algebraically. Points became ordered pairs of numbers, lines became the solution sets of linear equations, and circles emerged from quadratic relations. Geometry and algebra were revealed as two different descriptions of the same mathematical structure.

The next step is to understand how distances in the plane are related to one another.

Among all geometric figures, the triangle plays a uniquely fundamental role. Any polygon can be decomposed into triangles, and the geometry of curved surfaces can often be approximated by triangulating them into many small triangular regions. Because of this universality, triangles serve as the basic building blocks of much of geometry.

Within the family of triangles, right triangles occupy a special position.

Orthogonality and right angles.

A right triangle contains an angle of exactly ninety degrees. The two sides forming that angle are called the legs of the triangle, and the side opposite the right angle is called the hypotenuse.

This configuration introduces a fundamental geometric relationship: the directions of the two legs are perpendicular, or orthogonal, to one another.

Orthogonality appears in many areas of mathematics and science. The horizontal and vertical axes of the coordinate plane are orthogonal. The directions north and east used in navigation are orthogonal. In physics, independent components of motion are often resolved along orthogonal directions.

Right triangles therefore provide the simplest geometric model of independent directions in space.

The Pythagorean relation.

The central result governing right triangles is the Pythagorean theorem, one of the oldest and most celebrated results in mathematics. It states that the square of the hypotenuse equals the sum of the squares of the two legs.

$$a^2 + b^2 = c^2.$$

Although the theorem is ancient, its significance extends far beyond the geometry of triangles. The equation expresses a deeper relationship between orthogonal components of a system.

If two quantities represent independent directions, the magnitude of their combined effect is determined by adding the squares of their components. This idea reappears throughout mathematics and physics.

In analytic geometry it becomes the formula for the distance between two points in the plane. In physics it appears in vector addition and in the energy relations of mechanical systems. In statistics it emerges in the definition of variance and Euclidean distance between data points.

The simple geometric picture of a right triangle therefore encodes a general rule for combining independent contributions.

Squares and geometric measurement.

The presence of squares in the Pythagorean theorem is not accidental. Area plays a central role in the geometric interpretation of the relationship.

The classical proof, illustrated in the figure that follows, shows that the areas of the squares constructed on the legs of a right triangle combine exactly to equal the area of the square constructed on the hypotenuse. In other words, the theorem is a statement about how areas transform when geometric figures are rearranged.

Many different proofs of the Pythagorean theorem exist, ranging from purely geometric arguments to algebraic demonstrations. Each proof reveals a different perspective on the same invariant relationship.

From triangles to spatial laws.

Although the Pythagorean theorem arises from elementary geometry, its influence extends into many branches of science. One important example is the inverse-square law governing the spread of radiation, sound, and gravitational influence.

When energy radiates outward from a source, it spreads over the surface of an expanding sphere. The surface area of that sphere grows in proportion to the square of its radius. As a result, the intensity of the radiation decreases with the square of the distance from the source.

This geometric principle explains phenomena ranging from the brightness of stars to the strength of gravitational attraction.

Astronomers such as Cecilia Payne-Gaposchkin relied on these geometric relations to interpret measurements of stellar radiation and to infer the physical properties of distant stars.

In the sections that follow we examine right triangles in detail and develop the Pythagorean theorem as a fundamental geometric relation. From this theorem we derive practical methods for computing distances and analyzing spatial relationships.

Although the theorem concerns a simple triangular figure, it expresses a deep principle about the structure of space: independent directions combine according to a precise geometric rule. Understanding this rule will allow us to analyze a wide variety of problems involving distance, measurement, and the geometry of physical systems.

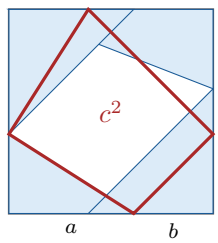
8.1 Right Triangles

Theorem 8.1 (Pythagorean Theorem)**.** *In any right triangle with legs a, b and hypotenuse c,*

$$a^2 + b^2 = c^2.$$

The length of the hypotenuse is completely determined by the two legs.

Proof Sketch. Place four congruent right triangles inside a square of side $a+b$. The uncovered central region is a square of side c.



Area of outer square: $(a+b)^2 = a^2 + 2ab + b^2$. Four triangles: $4 \cdot \frac{1}{2}ab = 2ab$. Inner square: $c^2 = (a+b)^2 - 2ab = a^2 + b^2$. □

8.2 Applications

Worked Example Payne-Gaposchkin: radiation and distance

Light from a star of luminosity L (W) spreads uniformly over a sphere of radius r. Surface area $A = 4\pi r^2$ (m^2), so intensity is

$$I = \frac{L}{4\pi r^2} \text{ W/m}^2.$$

For the Sun: $L = 3.8 \times 10^{26}$ W at $r = 1.5 \times 10^{11}$ m:

$$4\pi r^2 \approx 2.83 \times 10^{23} \text{ m}^2, \qquad I \approx 1340 \text{ W/m}^2.$$

The inverse-square law is a constraint on spatial geometry.

Exercises

Practice 8.1. Find the diagonal of a 7×24 cm rectangle.

Exercise 8.2. Two ships leave the same port. One travels 40 km due north; the other 30 km due east. How far apart are they? Draw and label the constraint diagram.

Challenge 8.3. Star A has luminosity $2L$ and is at distance $3d$. Star B has luminosity L and is at distance d. Which appears brighter to an observer, and by what factor? Generalise: express the apparent brightness ratio as a function of the luminosity and distance ratios.

Chapter 9

Similarity and Scaling in Geometry

Shape, Scale, and Invariance

In the previous chapter we examined right triangles and the Pythagorean relation, discovering how distances combine when directions are orthogonal. The geometry of space revealed a precise rule connecting the lengths of sides in a triangle. Yet another question naturally arises once distance is understood: how do geometric figures behave when they change size?

The study of similarity answers this question.

Similarity concerns the preservation of shape under changes of scale. When two figures are similar, one can be obtained from the other by enlarging or shrinking it while preserving the relative relationships among its parts. Angles remain the same, and the ratios between corresponding lengths remain constant.

This property reveals one of the most important ideas in mathematics: structure can remain invariant even when scale changes.

Shape versus size.

Two geometric figures may differ greatly in size yet still share exactly the same shape. A small triangle drawn on paper and a large triangular roof on a building may have identical angles and proportional sides. Although their absolute dimensions differ, the pattern of relationships between their sides is preserved.

Similarity therefore separates the concept of form from the concept of magnitude. Geometry is not only concerned with measuring objects but also with understanding the structural patterns that remain unchanged when objects are enlarged or reduced.

This distinction between shape and size is central to many scientific disciplines.

Scaling and dimensionality.

When a figure is scaled by a factor k, every linear dimension becomes k times larger. Yet quantities that depend on two or three dimensions change more dramatically.

Areas scale with the square of the scale factor, and volumes scale with the cube of the scale factor. A square whose sides double in length has four times the area. A cube whose edges double in length has eight times the volume.

These relationships arise because area measures two-dimensional extent and volume measures three-dimensional extent. Each independent spatial dimension contributes an additional factor of the scale change.

The simple act of enlarging a figure therefore produces predictable changes in geometric measurements.

Scaling laws in nature.

The same principles appear throughout the natural world. The surface area of a sphere grows with the square of its radius, while its volume grows with the cube. These geometric relationships influence many physical processes.

The brightness of a star decreases with the square of the distance from the observer because the star's radiation spreads across the surface of an expanding sphere. The strength of gravitational attraction follows a similar inverse-square relationship.

Even biological systems are shaped by scaling laws. The surface area of an organism determines how it exchanges heat with its environment, while its volume determines its mass and metabolic needs. As organisms increase in size, these quantities scale differently, placing constraints on how large biological structures can become.

Geometry therefore provides a mathematical language for describing how structure changes with size.

Similarity as an invariant transformation.

From a geometric viewpoint, similarity transformations belong to a family of operations that preserve certain properties of figures while altering others. Rotations and reflections preserve both shape and size. Translations move figures without altering their geometry at all. Scaling transformations preserve shape but change size.

Understanding which quantities remain invariant under each type of transformation allows mathematicians to classify geometric structures according to their fundamental properties.

Similarity transformations preserve angles and proportional relationships among lengths. These invariants make it possible to study entire families of figures at once rather than analyzing each individual instance separately.

From geometric similarity to mathematical models.

The idea of similarity also plays a crucial role in modeling physical systems. Engineers use scaled models of buildings, bridges, and aircraft to study their behavior before constructing the full structures. Astronomers interpret observations of distant objects by comparing them to known physical models scaled to cosmic dimensions.

In each case the assumption of similarity allows small systems to reveal the behavior of much larger ones.

Mathematically, similarity provides the bridge between geometry and the scaling laws introduced earlier in the study of proportions. What began as a simple ratio between numbers now becomes a transformation of entire geometric figures.

In the sections that follow we formalize the concept of similarity and examine how scaling transformations affect geometric measurements such as length, area, and volume. These ideas will allow us to analyze how shapes change under enlargement and reduction while preserving their essential structure.

By understanding similarity, we gain a powerful perspective on how patterns persist across scales—from small geometric constructions to the vast structures of the physical universe.

9.1 Similar Figures

Definition Similarity

Two figures are **similar** (written $\sim$) if one can be obtained from the other by scaling, rotation, or reflection. Corresponding angles are equal; corresponding lengths are proportional with scale factor k.

Theorem 9.1 (Scaling Laws)**.** *If two similar figures have linear scale factor k, then areas scale as k^2 and volumes scale as k^3.*

Proof Sketch. Area involves two length dimensions; volume involves three. Each scales as a power of k equal to the number of dimensions. □

Exercises

Practice 9.1. Two similar triangles have scale factor $3:5$. If the smaller has area $27\ \text{cm}^2$, find the area of the larger.

Exercise 9.2. A spherical star with radius r has surface area $4\pi r^2$. If the star expands to radius $2r$, by what factor does its surface area increase? Its volume?

Chapter 10

Transformations of the Plane

Motion and Invariance in the Plane

Up to this point our study of geometry has focused on the relationships between static figures. Points, lines, triangles, and circles were described by equations and measurements that determined their shape and size. Yet geometry also concerns the ways figures may move within space while preserving their fundamental structure.

A geometric transformation describes such motion.

Instead of asking what a figure is, we ask how it changes when every point of the plane is mapped to a new position according to a rule. This viewpoint shifts attention from individual shapes to the operations that act upon them.

Geometry as transformation.

In classical geometry figures were often studied as fixed objects. A triangle had certain side lengths and angles, and the goal was to discover relationships among those measurements. Modern geometry, however, emphasizes the transformations that carry one figure to another.

A transformation is a function that assigns a new location in the plane to every point of the original figure. When the same rule is applied to all points simultaneously, the entire geometric configuration moves in a coherent way.

This perspective reveals an important idea: many geometric properties remain unchanged under certain transformations. These unchanged quantities are called invariants.

Distance as a geometric invariant.

Among the most fundamental invariants of Euclidean geometry is distance. If the distance between two points remains unchanged after a transformation, then the transformation preserves the shape and size of all figures in the plane.

Transformations with this property are called isometries.

When an isometry is applied to a geometric figure, the figure may move to a new position or orientation, but its internal structure remains exactly the same. Angles remain equal, lengths remain unchanged, and the overall shape is preserved.

Isometries therefore describe motions of the plane that do not distort geometry.

The basic motions of the plane.

There are three fundamental ways a figure can move while preserving distance. A figure may slide from one location to another without changing its orientation. This motion is called

a translation. A figure may rotate about a point, turning through some angle while preserving its shape. Finally, a figure may be reflected across a line, producing a mirror image.

Every distance-preserving transformation of the plane can be constructed from combinations of these three basic motions.

These operations form the foundation of the symmetry structure of Euclidean space.

Transformations and matrices.

The study of transformations connects geometry back to the algebraic methods introduced earlier in the book. Many geometric transformations can be represented by matrices acting on coordinate vectors. When a matrix multiplies a coordinate pair (x, y), it produces a new pair representing the transformed point.

Rotations provide a particularly elegant example. A rotation through an angle θ can be expressed using a matrix involving the trigonometric functions $\cos\theta$ and $\sin\theta$. This representation allows geometric motion to be analyzed using the tools of linear algebra.

In this way algebra and geometry once again converge.

Symmetry and invariance.

The concept of invariance under transformation extends far beyond geometry. In physics, symmetries of space and time determine the laws governing motion and conservation. In chemistry, molecular structures are classified according to their symmetry groups. In art and architecture, patterns and designs rely on repeating transformations of basic shapes.

Mathematically, the collection of transformations that preserve a given set of properties forms a group, a structure that captures the internal symmetry of a system.

The isometries of the plane therefore provide one of the simplest examples of symmetry in mathematics.

In the sections that follow we define transformations of the plane and study the special class of isometries that preserve distance. We examine translations, rotations, and reflections and explore how these transformations can be combined to produce more complex motions.

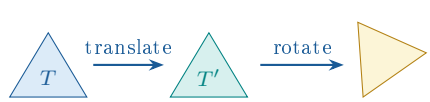
Through this study we will see that geometry is not only the study of shapes but also the study of the transformations that leave those shapes unchanged. Understanding these invariances will prepare us for the next stage of the book, where rotational motion leads naturally to the periodic structures studied in trigonometry.

10.1 Isometries: Transformations Preserving Distance

A **transformation** is a function mapping the plane to itself. An **isometry** preserves distances: $d(T(P), T(Q)) = d(P, Q)$ for all points P, Q.

The three basic isometries are:

- **Translation** by vector $\mathbf{v}$: $T(P) = P + \mathbf{v}$.
- **Rotation** by angle θ around origin: encoded by $R(\theta)$.
- **Reflection** across a line ℓ.



Guiding Principle

Every isometry preserves shape and size. It is a constraint-preserving transformation: the distance relation $d(P, Q) = c$ becomes $d(T(P), T(Q)) = c$.

Historical Note Hannah Arendt and Invariance

Arendt's analysis of political action distinguishes between acts that alter the social space and those that maintain it. An isometry is the spatial analogue of a norm-preserving act: the shape of relations is maintained even as positions change. The rotation matrix $R(\theta)$ is the canonical mathematical example.

Exercises

Exercise 10.1. Show that the composition of two rotations around the origin is another rotation. What is the resulting angle?

Challenge 10.2. A rotation by θ is represented by the matrix $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Compute $R(\theta)^\top R(\theta)$ and explain why this confirms that $R(\theta)$ is an isometry.

Chapter 11

Area, Volume, and Spatial Distribution

Measuring Space

Up to this point we have examined the structure of geometric figures and the transformations that preserve them. Points were located by coordinates, triangles revealed relations among lengths, and transformations showed how shapes could move through space without changing their essential structure. Yet geometry also concerns a more practical question: how much space does a figure occupy?

The measurement of space gives rise to the concepts of area and volume.

Area measures the extent of a figure in two dimensions, while volume measures the extent of a solid body in three dimensions. These quantities allow us to compare regions of space, determine capacities, and analyze the physical constraints imposed by spatial structure.

From shape to magnitude.

Earlier chapters emphasized relationships among lengths and angles. A triangle could be classified by its angles or by the proportional relationships among its sides. But when we ask how large a triangular region is, we must introduce a new type of measurement.

Area quantifies the size of a two–dimensional region.

The formulas used to compute area often reflect the geometric structure of the figure itself. A rectangle has area equal to the product of its length and width because its interior can be tiled by a grid of equal squares. A triangle has half the area of a rectangle with the same base and height because two congruent triangles fill that rectangle exactly.

Thus area formulas arise from simple geometric decompositions.

Curved shapes and geometric constants.

Not all geometric figures are composed of straight edges. Circles and spheres introduce curvature, and their measurement reveals one of the most famous constants in mathematics: π.

The area of a circle depends on the square of its radius, while the surface area of a sphere depends on the square of its radius multiplied by a constant factor. These relationships illustrate again how scaling laws operate in geometry. Doubling the radius of a circle quadruples its area, and doubling the radius of a sphere quadruples its surface area while increasing its volume by a factor of eight.

Such scaling relationships are direct consequences of the dimensional structure of space.

Volume and the geometry of solids.

While area measures two–dimensional extent, volume measures the capacity of three–dimensional objects. Many volume formulas can be understood as extensions of area formulas. For example, the volume of a cylinder is obtained by multiplying the area of its circular base by its height. The same principle applies to prisms and many other solids: the volume is the cross–sectional area multiplied by the length of the object.

This method reflects an important idea in geometry: complex spatial quantities can often be understood by analyzing simpler lower–dimensional structures.

Spatial constraints in natural and social systems.

The measurement of area and volume is not merely an abstract exercise. These quantities determine the capacity of physical systems and the limits within which many processes must operate.

Agricultural productivity depends on the available land area. The amount of water a reservoir can hold depends on its volume. The intensity of radiation emitted by a source decreases as it spreads over larger and larger surfaces.

In ecological and social systems, spatial distribution often imposes fundamental constraints. A settlement must balance its population against the productive capacity of the land that supports it. Urban planning, resource management, and environmental sustainability all require careful reasoning about how area and volume translate into available resources.

The speculative societies described in the work of Ursula K. Le Guin often illustrate such constraints vividly. Communities organized around limited ecological resources must constantly consider how population, land area, and productivity interact.

Geometry as the mathematics of spatial limits.

Area and volume formulas therefore serve as algebraic expressions of spatial constraints. They translate geometric structure into numerical relationships that can be used to analyze real systems.

When we write $A = \pi r^2$ or $V = \pi r^2 h$, we are not merely describing shapes; we are expressing the rules governing how space is distributed within those shapes.

In the sections that follow we examine several common geometric figures and the formulas used to measure their area and volume. By interpreting these formulas as constraints relating geometric quantities, we gain a deeper understanding of how spatial structure determines the limits of physical and ecological systems.

11.1 Area Formulas as Algebraic Constraints

Shape	Formula	Constraint expressed
Rectangle	$A = lw$	product of perpendicular dimensions
Triangle	$A = \frac{1}{2}bh$	half the enclosing rectangle
Circle	$A = \pi r^2$	area-radius relation
Sphere	$A = 4\pi r^2$	surface of constant curvature
Cylinder	$V = \pi r^2 h$	circular cross-section times height

Worked Example Le Guin: sustainable population

A circular settlement of radius $r = 3$ km devotes 60% of its area to crops. Each km^2 supports 400 people.

$$A = \pi(3)^2 = 9\pi \text{ km}^2, \qquad A_c = 0.6 \times 9\pi = 5.4\pi \text{ km}^2.$$

Population capacity: $400 \times 5.4\pi \times 100 \approx 679{,}000$. (Factor of 100 converts km^2 to hectares.) Le Guin's ecological societies are governed by exactly such spatial constraints [3].

Exercises

Practice 11.1. A circle has area 50π cm^2. Find its circumference.

Exercise 11.2. A cylindrical grain silo has radius 2 m and height 6 m. Find its volume and total surface area. Interpret each quantity in the context of grain storage.

Part III

Trigonometry: Cycles and Rotational Systems

Chapter 12

Angles and Circular Geometry

Rotation and Periodicity

The preceding chapters explored geometry as the study of spatial structure. We examined points and distances, the relationships within triangles, the invariance of shapes under transformations, and the ways in which area and volume quantify the extent of regions in space. Throughout these topics one geometric figure appeared repeatedly: the circle.

The circle occupies a unique place in mathematics because it naturally connects geometry with motion.

Rotation as a geometric process.

A point moving along a straight line changes position without changing direction. A point moving along a circle, however, continually changes its direction while maintaining a constant distance from a central point. This motion is called rotation.

Rotation introduces a new type of measurement: the angle. Angles measure how far a point has rotated around a center. Unlike distance, which measures separation between points, angles measure change in direction.

Understanding rotation therefore requires a mathematical language that describes how positions change as a point moves around a circle.

The circle as a generator of periodic motion.

When a point travels around a circle at constant speed, its coordinates do not change randomly. Instead they follow a repeating pattern.

The horizontal coordinate increases and decreases smoothly as the point moves from the right side of the circle to the left and back again. The vertical coordinate follows a similar oscillation as the point moves from the top of the circle to the bottom and back.

These oscillations repeat every full revolution.

Trigonometry arises from the mathematical description of this periodic behavior. The trigonometric functions $\sin\theta$ and $\cos\theta$ record the vertical and horizontal coordinates of a point rotating around the unit circle. As the angle θ increases, these functions trace smooth waves that repeat indefinitely.

Thus trigonometry translates circular motion into algebraic functions.

Angles and the radian measure.

Earlier chapters introduced the radian as the natural unit for measuring angles. The radian arises directly from the geometry of circles: it is defined so that the length of an arc on a circle equals the radius of the circle when the subtended angle is one radian.

This definition connects angular measurement with linear distance. Because of this connection, many formulas in trigonometry and calculus take their simplest form when angles are measured in radians rather than degrees.

For example, the arc length of a circular segment is given simply by $s = r\theta$ when θ is measured in radians.

Cycles in the natural world.

The periodic relationships described by trigonometry appear throughout nature. Planetary orbits, the oscillation of waves, the vibration of strings, and the rotation of mechanical systems all exhibit cyclic behavior. Even biological and ecological processes often follow rhythms that repeat over time.

Astronomers have long relied on such periodic patterns to understand the structure of the universe. The regular oscillations of certain stars, known as Cepheid variables, provide one of the most important tools for measuring cosmic distances. By observing the period of a star's brightness variation, astronomers can infer its intrinsic luminosity and therefore estimate how far away it must be.

Cecilia Payne-Gaposchkin contributed significantly to the study of such stellar phenomena, helping to establish the role of periodic processes in astrophysical measurement.

Trigonometry as the mathematics of cycles.

Trigonometry therefore extends geometry into the study of rotational and periodic systems. Where earlier chapters analyzed static shapes, trigonometry analyzes continuous motion along circular paths.

The circle becomes a generator of oscillations, and angles become the parameter describing position within a cycle. By understanding the relationships between angles, arc lengths, and coordinates on the circle, we gain the mathematical tools needed to model waves, rotations, and repeating phenomena.

In the sections that follow we examine the geometry of the circle and derive the basic relationships connecting angles, arc length, and coordinates. These ideas will lead naturally to the trigonometric functions, which describe the oscillatory patterns generated by circular motion.

Through this development the circle will emerge not merely as a geometric figure but as a fundamental source of periodic structure in mathematics and science.

12.1 The Circle as Generator of Cycles

A circle of radius r centred at the origin is the solution set of $x^2 + y^2 = r^2$. As a point moves around the circle, its x- and y-coordinates each trace out a periodic oscillation. Trigonometry is the mathematics of this periodicity.

Historical Note Cyclic patterns in astrophysics

Cecilia Payne-Gaposchkin also studied Cepheid variable stars, whose brightness oscillates with remarkable regularity. The period of oscillation encodes the star's intrinsic luminosity — a fact used to measure cosmic distances. Trigonometric modelling of such cycles was central to her work [8].

Exercises

Practice 12.1. State the arc length formula $s = r\theta$ (where θ is in radians). Find the arc length subtended by an angle of $\pi/3$ on a circle of radius 5 cm.

Chapter 13

Sine, Cosine, and the Unit Circle

Coordinates of Rotation

In the previous chapter we saw that circular motion naturally produces periodic patterns. A point moving around a circle generates oscillations in its horizontal and vertical coordinates. These repeating patterns form the basis of trigonometry.

To describe this motion precisely, mathematicians introduced two fundamental functions: sine and cosine.

These functions record the position of a rotating point on a circle.

The unit circle as a geometric reference system.

Although circles may have any radius, trigonometry becomes especially simple when the radius is chosen to be one. Such a circle is called the unit circle.

When a point moves around the unit circle, its distance from the origin remains exactly one. The coordinates of the point therefore describe the direction of the radius rather than its length.

This geometric situation leads directly to the definitions of sine and cosine. The horizontal coordinate of the rotating point is called the cosine of the angle, and the vertical coordinate is called the sine.

Thus the trigonometric functions arise naturally from the geometry of rotation.

From triangles to circular motion.

Historically, trigonometric ratios were first defined using right triangles. In a right triangle with angle θ, the ratio between the length of the opposite side and the hypotenuse defines $\sin\theta$, while the ratio between the adjacent side and the hypotenuse defines $\cos\theta$.

The unit circle interpretation extends these definitions beyond the limitations of triangles. Instead of restricting θ to acute angles, the circular definition allows angles to take any value, including angles larger than 90° and even negative angles.

In this way trigonometry becomes a theory of rotational position rather than merely a set of triangle ratios.

Coordinates and vectors.

The pair $(\cos\theta, \sin\theta)$ can also be interpreted as a vector of length one pointing in direction θ. Because the vector lies on the unit circle, its components satisfy a special relationship: the square of the horizontal component plus the square of the vertical component equals one.

This relationship is the trigonometric form of the Pythagorean theorem.

$$\sin^2\theta + \cos^2\theta = 1.$$

The identity expresses the geometric fact that the coordinates of any point on the unit circle must satisfy the equation of that circle.

Thus the most fundamental trigonometric identity emerges directly from geometry.

Angles, symmetry, and periodic structure.

As the angle θ increases, the point on the unit circle moves continuously around the circle, and the values of $\sin\theta$ and $\cos\theta$ vary smoothly between -1 and 1. After a full rotation of 2π radians, the point returns to its starting position and the cycle repeats.

This periodic behavior makes trigonometric functions ideal tools for describing oscillatory systems. Waves, vibrations, rotating machines, alternating electrical currents, and many astronomical motions all follow patterns that repeat with regular periods.

Because sine and cosine describe the geometry of rotation, they provide a natural mathematical language for analyzing such phenomena.

Trigonometry and linear transformations.

Trigonometric functions also appear naturally in the study of geometric transformations. A rotation of the plane through an angle θ can be represented by a matrix whose entries involve $\sin\theta$ and $\cos\theta$.

This matrix rotates every vector while preserving its length, a property closely related to the Pythagorean identity. In this way trigonometry connects circular geometry with the linear algebra of transformations studied earlier.

In the sections that follow we formalize the definitions of sine and cosine using the unit circle and examine the fundamental identities that govern their behavior. We also compute the exact values of these functions for several special angles, which serve as important reference points for trigonometric reasoning.

Through these ideas the unit circle becomes a geometric framework for understanding rotation, periodicity, and the oscillatory patterns that appear throughout mathematics and science.

13.1 The Fundamental Definitions

Definition Sine and Cosine

For a point on the unit circle at angle θ from the positive x-axis,

$$\cos\theta = x\text{-coordinate}, \qquad \sin\theta = y\text{-coordinate}.$$

Equivalently, in a right triangle with adjacent side a, opposite side o, and hypotenuse h:

$$\cos\theta = \frac{a}{h}, \qquad \sin\theta = \frac{o}{h}, \qquad \tan\theta = \frac{o}{a}.$$

Theorem 13.1 (Pythagorean Identity)**.** *For all θ,*

$$\sin^2\theta + \cos^2\theta = 1.$$

The squared components of a unit vector always sum to one.

Proof Sketch. A point $(\cos\theta, \sin\theta)$ lies on the unit circle by definition. The circle's equation $x^2 + y^2 = 1$ gives the identity directly. $\square$

13.2 Exact Values

θ	radians	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	0	1	0
30	$\pi/6$	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$
45	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
90	$\pi/2$	1	0	undef

Exercises

Practice 13.1. Without a calculator, compute $\sin(120)$ and $\cos(135)$.

Exercise 13.2. Prove that $\sin(90 - \theta) = \cos\theta$ using the unit circle definitions.

Challenge 13.3. The rotation matrix $R(\theta)$ preserves the norm of every vector. Express this statement as an identity involving $\sin\theta$ and $\cos\theta$, and relate it to the Pythagorean identity.

Chapter 14

Trigonometric Functions and Graphs

Waves, Oscillations, and Repeating Patterns

In the previous chapter we introduced the sine and cosine functions as coordinates of a point moving around the unit circle. As the point rotates, its horizontal and vertical coordinates vary smoothly between -1 and 1, tracing a repeating pattern. This geometric description reveals the essential nature of trigonometric functions: they encode periodic motion.

A periodic function is one whose values repeat after a fixed interval. Such functions are fundamental in mathematics because they provide a natural way to describe cycles.

From circular motion to waves.

When a point travels around a circle at constant angular speed, its coordinates oscillate in a regular rhythm. If we plot the vertical coordinate of that point as a function of time, the resulting curve is a wave.

This wave is the graph of the sine function.

Similarly, the horizontal coordinate produces the cosine function. Although these graphs appear different at first glance, they represent the same underlying motion viewed from different starting positions along the circle.

In this way circular motion and wave motion become two perspectives on the same mathematical phenomenon.

Parameters of oscillation.

Real oscillatory systems rarely follow the simple form $\sin t$ or $\cos t$ exactly. Instead they may oscillate with larger or smaller amplitudes, repeat their cycles more quickly or more slowly, or begin their motion at different points in the cycle.

To capture these variations, trigonometric functions are often written in a general form:

$$f(t) = A\sin(\omega t + \phi) + D.$$

Each parameter has a clear physical interpretation. The amplitude A determines how large the oscillation is. The angular frequency ω determines how rapidly the system repeats its cycle. The phase ϕ shifts the entire pattern forward or backward in time, and the constant D raises or lowers the equilibrium level around which the oscillation occurs.

Together these parameters allow trigonometric functions to model a wide range of periodic phenomena.

Oscillations in natural systems.

Periodic behavior appears throughout the natural world. Ocean tides rise and fall with predictable rhythms. Sound waves propagate through air as vibrations of pressure. Mechanical systems such as springs and pendulums oscillate around equilibrium positions. Even biological systems often exhibit regular cycles, including daily circadian rhythms and seasonal patterns of growth.

Astronomy provides especially striking examples of periodic behavior. Certain stars, known as variable stars, change their brightness in regular cycles. By analyzing these oscillations, astronomers can infer properties of the stars and even estimate cosmic distances.

The work of Cecilia Payne-Gaposchkin contributed significantly to the study of such variable stars. Her research helped establish the connection between stellar structure and observable brightness oscillations, demonstrating how periodic patterns reveal information about distant astrophysical systems.

Graphs as visual representations of cycles.

Graphing trigonometric functions provides a powerful way to visualize periodic behavior. When sine or cosine is plotted against time or angle, the resulting curve displays the repeating structure of the oscillation. Peaks correspond to maximum values, troughs correspond to minimum values, and the midpoint represents the equilibrium level of the system.

Because these graphs repeat indefinitely, they offer a concise visual representation of cycles that continue without bound.

In the sections that follow we formalize the concept of periodic functions and examine the graphs of sine and cosine. We then extend these ideas to general sinusoidal functions that include parameters for amplitude, frequency, phase, and vertical shift.

By studying these graphs we gain the mathematical tools needed to analyze oscillatory systems, from the motion of waves to the rhythmic patterns observed throughout nature and science.

14.1 Periodic Functions

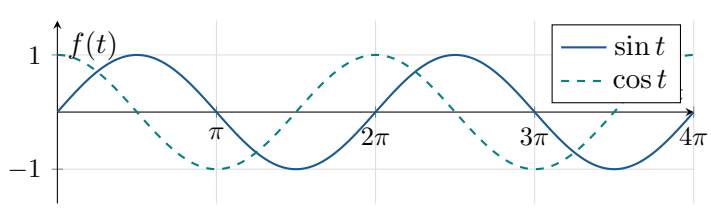
Definition Period

A function f is **periodic** with period $T > 0$ if $f(t+T) = f(t)$ for all t. The sine and cosine functions have period 2π.

The general sinusoidal function

$$f(t) = A\sin(\omega t + \phi) + D$$

has **amplitude** A, **angular frequency** ω, **phase** ϕ, and **vertical shift** D. The period is $T = 2\pi/\omega$.



14.2 Variable Stars: Trigonometric Modelling

Worked Example Payne-Gaposchkin: stellar brightness cycles

A Cepheid variable star's brightness is modelled by

$$B(t) = 10 + 2\sin\left(\frac{2\pi}{5}t\right),$$

with amplitude 2 and equilibrium brightness = 10.
At $t = 2$ days:

$$B(2) = 10 + 2\sin\left(\frac{4\pi}{5}\right) = 10 + 2\sin(144) \approx 10 + 2(0.588) = 11.18.$$

This model was used by Payne-Gaposchkin to link oscillation period to distance measurement.

Exercises

Practice 14.1. Identify the amplitude, period, and phase of $f(t) = 3\sin(2t - \pi/4)$.

Exercise 14.2. Gopnik [2] notes that children alternate between exploration and exploitation in a roughly periodic pattern. Model the exploration intensity as $E(t) = 0.5 + 0.5\sin(\omega t)$. When is exploration at maximum? When is it at minimum? What is the period if the cycle is one week?

Chapter 15

Modeling Cyclic Phenomena

Oscillation as a Universal Pattern

In the previous chapter we examined trigonometric functions as the mathematical description of periodic patterns. The sine and cosine functions arise naturally from the geometry of circular motion, and their graphs reveal repeating waves that extend indefinitely in time. Such functions provide a compact language for describing cycles.

Many physical and biological systems exhibit precisely this type of repeating behavior.

A vibrating string moves back and forth around an equilibrium position. A pendulum swings periodically under the influence of gravity. Electrical currents in alternating circuits oscillate with predictable frequency. Even astronomical systems often display rhythmic patterns, from the rotation of planets to the pulsation of certain stars.

The mathematics that describes these phenomena is known as harmonic motion.

Restoring forces and equilibrium.

Harmonic motion arises when a system is displaced from an equilibrium position and experiences a restoring influence that pulls it back toward that equilibrium. The simplest example occurs in a mass attached to a spring. If the mass is pulled away from its resting position, the spring exerts a force proportional to the displacement, pulling the mass back toward the center.

Because the restoring force depends linearly on displacement, the resulting motion follows a remarkably regular pattern: the system oscillates back and forth in a repeating cycle.

Mathematically, this behavior leads to a differential equation whose solutions are sine and cosine functions.

Trigonometric functions as solutions of motion.

The equation governing simple harmonic motion relates the acceleration of a system to its displacement. When the restoring force is proportional to the displacement, the acceleration becomes proportional to the negative of the position. This relationship produces an equation of the form

$$\frac{d^2x}{dt^2} + \omega^2 x = 0.$$

Although this expression may appear abstract, its solutions describe the same oscillatory patterns already encountered in trigonometry. The functions $\sin(\omega t)$ and $\cos(\omega t)$ provide the

fundamental building blocks of these solutions.

Thus trigonometry, which began as the study of circular geometry, becomes the mathematical language of oscillatory motion.

Superposition and interference.

Many real systems involve not just a single oscillation but several oscillations interacting simultaneously. When two waves overlap, their effects combine according to the principle of superposition. The resulting motion is simply the sum of the individual waves.

Depending on their relative phases, waves may reinforce or cancel each other. When their peaks align, the amplitudes add and produce a larger wave. When a peak meets a trough, the waves partially or completely cancel.

This phenomenon, known as interference, plays a central role in the behavior of sound, light, and other wave-like systems.

Cycles within cycles.

Oscillatory behavior often appears in nested or interacting layers. Planetary systems exhibit orbital cycles within larger galactic rotations. Biological organisms maintain rhythms within rhythms, including heartbeats, breathing cycles, and circadian patterns.

Philosopher Alicia Juarrero describes such systems as networks of constraints in which cycles interact to produce coherent behavior. The mathematical study of oscillators offers a simplified model of this idea. By analyzing how periodic processes interact, we gain insight into how complex systems maintain stability through recurring patterns.

From geometry to dynamics.

The development of trigonometry in this part of the book has gradually shifted attention from static geometric structures to dynamic processes. Where geometry focused on shapes and spatial relationships, trigonometry reveals how those relationships evolve through time.

The circle, originally studied as a geometric figure, now becomes the source of oscillations that describe motion, vibration, and repeating phenomena.

In the sections that follow we introduce the mathematical equation governing simple harmonic motion and explore how combinations of trigonometric functions model interacting waves. These ideas will illustrate how the geometry of the circle provides a powerful framework for understanding cyclic processes across many areas of science.

15.1 Harmonic Motion

Simple harmonic motion arises whenever a restoring force is proportional to displacement. The governing equation is

$$\frac{d^2x}{dt^2} + \omega^2 x = 0,$$

with general solution $x(t) = A\cos(\omega t) + B\sin(\omega t)$.

Worked Example Wave interference

Two waves $f_1 = A\sin(\omega t)$ and $f_2 = A\sin(\omega t + \delta)$ combine to

$$f_1 + f_2 = 2A\cos\left(\frac{\delta}{2}\right)\sin\left(\omega t + \frac{\delta}{2}\right).$$

When $\delta = \pi$, amplitude vanishes (destructive interference). The constraint $\delta = \pi$ is exactly the condition for cancellation.

Exercises

Exercise 15.1. Two sides of a triangle are 7 and 9; the included angle is 40. Use the Law of Cosines to find the third side. Show all unit checks.

Challenge 15.2. Juarrero [5] argues that organisms maintain coherent behaviour through nested cycles of constraint. Model two interacting oscillators $x(t) = \sin(t)$, $y(t) = \sin(t + \phi)$. For what value of ϕ do they cancel? For what value do they reinforce? Discuss the biological analogy.

Chapter 16

Rotations and Harmonic Motion

Circular Motion as Hidden Oscillation

In the previous chapter we studied harmonic motion as a model of oscillation. A system displaced from equilibrium and subjected to a restoring force was shown to move in a regular, repeating cycle, and the solutions to its governing equation were expressed in terms of sine and cosine. Those functions described the motion algebraically, but a deeper geometric question remains: why do trigonometric functions appear so naturally in the study of oscillations?

The answer lies in circular motion.

A rotating point and its projections.

Imagine a point moving around a circle at constant angular speed. Its distance from the center never changes, yet its horizontal and vertical coordinates continually vary. If we observe only the horizontal motion of the point, ignoring its vertical position, the point appears to move back and forth in a smooth oscillation. The same is true for the vertical component.

In other words, uniform circular motion contains within it two hidden oscillations: one along the x-axis and one along the y-axis.

This fact reveals one of the deepest connections in elementary mathematics. Circular motion and harmonic motion are not separate phenomena described by coincidentally similar formulas. They are two different views of the same underlying geometric process.

The geometry behind sine and cosine.

When a point rotates around a circle of radius R, its position can be described in terms of angle. The horizontal coordinate is $R\cos(\omega t)$ and the vertical coordinate is $R\sin(\omega t)$. These expressions are not arbitrary formulas imposed from outside. They arise directly from the geometry of the circle and the definitions of sine and cosine.

Thus trigonometric functions become the natural coordinate language of rotation.

If we project the rotating point onto one axis, we obtain a sinusoidal motion. If we project onto the other axis, we obtain another sinusoidal motion shifted in phase by one quarter of a cycle. Harmonic motion is therefore the shadow of circular motion cast onto a line.

Velocity, change, and invariance.

Circular motion also provides a natural setting for understanding how position and velocity are related. As a point moves around the circle, its velocity changes direction continuously

even if its speed remains constant. The motion is therefore governed by both variation and invariance: the direction changes at every instant, but the distance from the center and the magnitude of the speed remain fixed.

This combination of change and preservation is central to the broader theme of the book. A system may transform continuously while still maintaining invariant structure. In the case of circular motion, the invariants are radius and speed; the changing quantities are the coordinates and velocity components.

From geometric motion to physical systems.

The relation between rotation and oscillation is not merely a geometric curiosity. It appears throughout science. Alternating current in an electrical circuit can be represented by rotating vectors in the complex plane. Wave motion in physics is often analyzed as the projection of rotational or phase-based structure. Planetary motion, mechanical vibration, and signal analysis all depend on the same trigonometric framework.

By recognizing sinusoidal motion as the projection of circular motion, we gain a unified picture of periodic systems.

A bridge between geometry and dynamics.

This chapter serves as a culmination of the trigonometric part of the book. Earlier chapters introduced angles, the unit circle, and the graphs of sine and cosine. Then we studied harmonic motion as a dynamic system described by those functions. Here the two strands come together: geometry and dynamics are revealed as different expressions of a single rotational structure.

The circle is not only a static figure and not only the source of trigonometric ratios. It is also the hidden geometric engine of oscillation.

In the sections that follow we examine a point moving around a circle at constant angular velocity and derive the sinusoidal forms of its coordinate projections. We then analyze the corresponding velocities and use the Pythagorean identity to show that the speed remains constant.

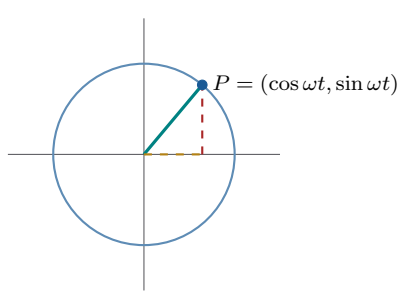
Through this analysis, circular motion and harmonic motion will appear not as separate topics but as two mathematically equivalent ways of describing periodic change.

16.1 Circular Motion and Projections

A point moving at constant angular velocity ω on a circle of radius R has position

$$(x, y) = (R\cos(\omega t), R\sin(\omega t)).$$

The x- and y-projections are independent sinusoidal oscillations, which is why circular motion and wave motion share the same mathematics.



Exercises

Exercise 16.1. Show that if a point traces the unit circle at constant speed, its x- and y-velocities are $-\sin(\omega t)$ and $\cos(\omega t)$ respectively. Verify using the Pythagorean identity that the speed is constant.

Part IV

Stoichiometry: Conservation and Transformation in Chemistry

Chapter 17

Atoms, Molecules, and Chemical Formulas

Matter as Conserved Structure

The earlier parts of this book developed a sequence of mathematical ideas that at first may have seemed abstract. Algebra introduced the language of variables and constraints. Geometry described the spatial relations among points and figures. Trigonometry revealed the cyclic patterns arising from rotation and oscillation. Despite their different forms, all of these topics shared a common theme: systems are governed by relationships that remain invariant under transformation.

Chemistry provides a striking physical realization of this principle.

Transformation without loss.

Chemical reactions often appear dramatic. Substances change color, gases form, heat is released, and entirely new materials emerge. Yet beneath these visible transformations lies a simple rule: the atoms themselves are not destroyed. They are merely rearranged.

This principle is known as the law of conservation of mass.

If a chemical reaction begins with a certain number of atoms of each element, the same number must appear in the products. The atoms change partners, forming new molecules, but the total count of each element remains constant.

In this sense a chemical reaction resembles the balance equations studied earlier in algebra. The quantities involved may change form, but the underlying totals must remain consistent.

Atoms as discrete units of matter.

The atomic hypothesis provides the conceptual framework for this conservation principle. According to this idea, all matter is composed of discrete particles called atoms. Atoms of different elements combine in specific numerical ratios to form molecules.

For example, a molecule of water contains two hydrogen atoms and one oxygen atom. Carbon dioxide contains one carbon atom and two oxygen atoms. These numerical patterns define the chemical identity of each substance.

Chemical formulas therefore function as symbolic descriptions of atomic structure.

Counting atoms with the mole.

Because atoms are extraordinarily small, chemists rarely count them individually. Instead they measure quantities of substances using the mole, a unit that represents a fixed number of particles. One mole contains approximately 6.022×10^{23} entities, a quantity known as Avogadro's number.

The mole serves as a bridge between the microscopic world of atoms and the macroscopic quantities measured in the laboratory. By relating mass to the number of particles present, chemists can determine how much of a substance participates in a reaction.

This relationship is expressed through the concept of molar mass.

Chemical formulas as algebraic relations.

Just as equations in algebra describe relationships between variables, chemical formulas describe relationships between atoms within a molecule. The formula H_2O states that two hydrogen atoms are combined with one oxygen atom. The formula CO_2 expresses a ratio of one carbon atom to two oxygen atoms.

When reactions occur, these ratios must be preserved across the entire system.

Balancing a chemical equation therefore resembles solving a system of algebraic constraints. Each element imposes its own conservation rule, and the final reaction must satisfy all of them simultaneously.

Chemical composition in the cosmos.

The importance of stoichiometric reasoning extends far beyond laboratory chemistry. Astronomers use similar principles to interpret the spectra of stars. By analyzing the patterns of light absorbed and emitted by stellar atmospheres, scientists can determine the relative abundances of elements present.

In her pioneering work, Cecilia Payne-Gaposchkin demonstrated that the Sun and other stars consist primarily of hydrogen and helium. Her analysis relied on comparing observed spectral intensities with theoretical predictions, effectively performing a kind of cosmic stoichiometry.

The proportions of elements in a star determine its structure, temperature, and energy production.

Conservation as a unifying theme.

The transition from mathematics to chemistry in this part of the book is therefore not abrupt but natural. The balance principles introduced in algebra, the invariants studied in geometry, and the periodic structures described by trigonometry all reflect the broader idea that certain quantities remain constant even as systems transform.

Chemical reactions provide a concrete example of this principle. Matter changes form, molecules rearrange, and energy flows through the system, but the total number of atoms of each element remains unchanged.

Understanding this conservation principle is the first step in learning how to analyze chemical reactions quantitatively.

In the sections that follow we examine the atomic hypothesis in more detail and introduce the mole as the standard unit for measuring the amount of a substance. We then learn how chemical formulas encode stoichiometric relationships among atoms and how these relationships allow us to compute the quantities involved in chemical processes.

Through this approach, chemistry becomes another domain in which mathematical reasoning reveals the hidden structure of transformation.

17.1 The Atomic Hypothesis and Conservation

In ordinary chemical reactions, atoms are neither created nor destroyed — they are rearranged. This is the **law of conservation of mass**, the chemical analogue of every balance principle studied so far.

Definition Mole and Molar Mass

The **mole** (mol) is the SI unit of chemical amount: one mole contains $N_A = 6.022 \times 10^{23}$ particles (Avogadro's number). The **molar mass** M (g/mol) is the mass of one mole of a substance. The relationship

$$n = \frac{m}{M}$$

converts between mass and amount of substance.

Historical Note Payne-Gaposchkin: hydrogen and helium in stars

Payne-Gaposchkin's 1925 discovery that stars are overwhelmingly hydrogen and helium was resisted because it contradicted the then-prevailing view. Her argument was essentially stoichiometric: matching observed spectral absorption strengths to theoretical abundances. The molar proportions of H to He in a star constrain its entire energy budget [8].

Worked Example Moles of water

How many moles are present in 36 g of H_2O?

$$M(H_2O) = 2(1.008) + 16.00 = 18.016\,\text{g/mol}. \qquad n = \frac{36}{18.016} \approx 2.0\,\text{mol}.$$

Exercises

Practice 17.1. Find the molar mass of (a) NaCl; (b) CO_2; (c) $C_6H_{12}O_6$.

Exercise 17.2. How many molecules are in 44 g of CO_2? Express in scientific notation.

Challenge 17.3. Payne-Gaposchkin computed that the Sun's atmosphere is approximately 92% hydrogen by number of atoms. If the Sun's atmosphere contains N total atoms, write an expression for the number of hydrogen atoms and the number of helium atoms (assuming the remaining 8% is helium). What mole ratio does this represent?

Chapter 18

Balancing Chemical Equations

Chemical Reactions as Constraint Systems

The previous chapter introduced the atomic hypothesis and the mole as the fundamental unit for measuring chemical quantities. Molecules were described as combinations of atoms arranged in fixed numerical ratios, and chemical formulas were shown to encode these structural patterns.

When a chemical reaction occurs, those molecular structures change. Atoms break old bonds and form new ones, producing different substances. Yet the atoms themselves remain present throughout the process. No atom disappears, and none is created out of nothing.

This requirement leads directly to one of the central quantitative tasks in chemistry: balancing chemical equations.

Reactions as rearrangements of atoms.

A chemical equation represents a transformation of matter. The substances written on the left-hand side of the equation are called reactants, and those on the right-hand side are called products. The equation asserts that the reactants rearrange into the products through a specific pattern of molecular changes.

However, the equation must also satisfy a conservation constraint.

Because atoms are conserved during ordinary chemical reactions, the total number of atoms of each element must be the same before and after the reaction. If a reaction begins with two carbon atoms, six hydrogen atoms, and several oxygen atoms, the products must contain exactly the same numbers of those atoms.

Balancing a chemical equation therefore ensures that these conservation conditions are satisfied.

From counting atoms to solving equations.

At first glance balancing reactions might appear to be a matter of trial and error: adjusting coefficients until the numbers of atoms match on both sides. While this approach can work for simple reactions, it quickly becomes cumbersome when many elements or molecules are involved.

A more systematic approach emerges when the problem is interpreted mathematically.

Each chemical element imposes a conservation rule. The number of atoms of that element on the reactant side must equal the number on the product side. If a reaction involves several elements, each one contributes an equation expressing its conservation.

The coefficients multiplying each chemical species then become unknown variables in a system of equations.

Balancing the reaction is equivalent to solving that system.

Stoichiometric matrices and linear algebra.

Earlier chapters introduced matrices as tools for representing systems of linear relationships. The same idea can be applied to chemical reactions. A matrix can be constructed whose rows correspond to elements and whose columns correspond to chemical species. The entries of the matrix record how many atoms of each element appear in each molecule.

When this matrix multiplies the vector of stoichiometric coefficients, the result must be zero if the reaction is balanced. This condition expresses the conservation of every element simultaneously.

Thus the balancing of chemical equations becomes a problem in linear algebra.

The mathematical techniques developed earlier in the book—systems of equations, matrices, and constraint networks—now find a direct application in chemical reasoning.

Stoichiometry as quantitative chemistry.

The systematic study of these numerical relationships is called stoichiometry. Once the coefficients of a reaction are determined, they reveal how much of each substance participates in the reaction. The balanced equation tells us how many molecules or moles of reactants are required to produce a given quantity of products.

In industrial chemistry and laboratory practice, these ratios determine how reactions are carried out efficiently. They guide the calculation of reactant quantities, product yields, and energy requirements.

A bridge between mathematics and chemistry.

Balancing chemical equations therefore illustrates a deeper theme of this book: the same mathematical structures appear across seemingly different domains. The conservation rules of chemistry correspond to constraint systems in algebra. The stoichiometric matrix mirrors the matrix representations studied earlier in linear transformations. The solutions of these systems describe how matter transforms while preserving its fundamental components.

Chemistry, in this sense, becomes another arena in which mathematical structure governs physical processes.

In the sections that follow we formalize the conservation principles underlying chemical reactions and express them using systems of linear equations. We then develop practical methods for balancing reactions and interpreting the resulting coefficients.

Through this approach the balancing of chemical equations emerges not as a mechanical procedure but as a clear example of conservation and constraint operating within a physical system.

18.1 Conservation as a Linear System

Theorem 18.1 (Stoichiometric Conservation)**.** *A chemical equation is balanced if and only if, for each element, the total atom count on the left side equals the total atom count on the right side. Equivalently, the stoichiometric coefficient vector γ satisfies $N\gamma = \mathbf{0}$, where N is the element–species incidence matrix.*

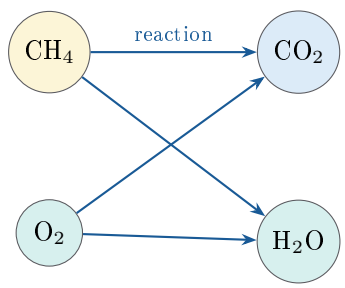
Worked Example Combustion of ethane

Balance $C_2H_6 + O_2 \to CO_2 + H_2O$.
Assign coefficients a, b, c, d and write conservation equations:

$$\begin{aligned} \mathrm{C}: &\quad 2a = c \\ \mathrm{H}: &\quad 6a = 2d \\ \mathrm{O}: &\quad 2b = 2c + d \end{aligned}$$

Set $a = 1$: $c = 2$, $d = 3$, $b = 7/2$. Multiply by 2:

$$2C_2H_6 + 7O_2 \to 4CO_2 + 6H_2O.$$



$CH_4 + 2O_2 \to CO_2 + 2H_2O$

Exercises

Practice 18.1. Balance (a) $Fe + O_2 \to Fe_2O_3$; (b) $C_3H_8 + O_2 \to CO_2 + H_2O$.

Exercise 18.2. Write the stoichiometric matrix N for the reaction $CH_4 + 2O_2 \to CO_2 + 2H_2O$ and verify $N\gamma = \mathbf{0}$.

Challenge 18.3. Balance $KMnO_4 + HCl \to KCl + MnCl_2 + H_2O + Cl_2$ by writing and solving the linear system. This is a non-trivial redox reaction with 7 unknowns.

Chapter 19

Moles and Quantitative Chemistry

Counting Matter in Chemical Reactions

The previous chapter showed that balancing a chemical equation is equivalent to satisfying conservation constraints for every element involved in a reaction. Once a reaction is balanced, the coefficients in the equation express the ratios in which molecules participate in the transformation. These coefficients therefore encode the structural logic of the reaction.

To apply this information in practice, however, we must connect the symbolic language of chemical equations with measurable quantities of matter.

This connection is provided by the mole.

From molecules to measurable quantities.

Chemical equations describe reactions in terms of molecules or atoms. For example, the equation

$$2H_2 + O_2 \rightarrow 2H_2O$$

states that two molecules of hydrogen combine with one molecule of oxygen to produce two molecules of water. While this description is precise at the microscopic level, real chemical experiments involve enormous numbers of molecules. Even a tiny droplet of water contains trillions upon trillions of particles.

Because such numbers are impractical to count individually, chemists use the mole as a standard counting unit.

The mole as a bridge between scales.

One mole represents a fixed number of particles, known as Avogadro's number:

$$N_A = 6.022 \times 10^{23}.$$

This constant plays a role similar to the idea of a dozen in everyday counting. Just as a dozen always means twelve objects, one mole always means 6.022×10^{23} entities, whether those entities are atoms, molecules, or ions.

The mole therefore links the microscopic world of particles with the macroscopic measurements of mass used in the laboratory.

Molar mass provides the conversion between these two scales. Because each substance has a characteristic molar mass, dividing the measured mass of a sample by its molar mass reveals the number of moles present.

Stoichiometric ratios as quantitative rules.

Once quantities are expressed in moles, the coefficients of a balanced chemical equation can be interpreted directly as ratios between amounts of substances. In the reaction

$$2H_2 + O_2 \rightarrow 2H_2O,$$

two moles of hydrogen react with one mole of oxygen to produce two moles of water. If four moles of hydrogen react, two moles of oxygen are required and four moles of water will be formed.

The balanced equation therefore functions as a quantitative rule governing the transformation of matter.

Stoichiometric calculations simply apply these ratios in combination with the mole–mass conversion.

Conservation revisited.

The ability to compute quantities in chemical reactions reflects the same conservation principles that have appeared throughout this book. Earlier chapters described balance conditions in algebra, invariants under geometric transformations, and conserved relationships in cyclic systems. In chemistry the invariant is the number of atoms of each element.

Stoichiometry translates this conservation law into numerical relationships that allow chemists to predict the outcomes of reactions.

Quantitative reasoning in chemistry.

These calculations are not merely classroom exercises. In laboratory practice and industrial chemistry, stoichiometric reasoning determines how much reactant must be supplied to produce a desired quantity of product. It also reveals how efficiently a reaction proceeds and how much material might remain unused.

Whether producing pharmaceuticals, refining fuels, or analyzing environmental samples, chemists rely on these quantitative relationships to control chemical processes.

In the sections that follow we develop systematic methods for converting between mass, moles, and molecular quantities. Using balanced chemical equations, we then compute the amounts of substances produced or consumed in reactions.

Through these calculations the abstract conservation principles of chemistry become practical tools for measuring and predicting the transformation of matter.

19.1 Stoichiometric Calculations

Worked Example Mass of water produced

From $2H_2 + O_2 \rightarrow 2H_2O$, how many grams of water form from 4 g of H_2 (excess O_2)?

$$n(H_2) = \frac{4\ \text{g}}{2.016\,\text{g/mol}} \approx 1.984\,\text{mol}$$
$$n(H_2O) = 1.984\,\text{mol} \times \frac{2\ \text{mol}\ H_2O}{2\ \text{mol}\ H_2} = 1.984\,\text{mol}$$
$$m(H_2O) = 1.984 \times 18.016 \approx 35.7\ \text{g}$$

Exercises

Practice 19.1. From $C_3H_8 + 5O_2 \rightarrow 3CO_2 + 4H_2O$, how many grams of CO_2 are produced by burning 11 g of propane ($M = 44.1\,\text{g/mol}$)?

Chapter 20

Reaction Ratios and Limiting Reagents

Scarcity and Constraint in Chemical Systems

In the previous chapter we learned how balanced chemical equations translate the conservation of atoms into quantitative relationships between substances. By converting masses to moles and applying the stoichiometric ratios encoded in a reaction, we can determine how much product a reaction could theoretically produce.

However, real chemical reactions rarely proceed under perfectly balanced conditions. In practice, the reactants present at the beginning of a reaction are not always available in the exact ratios required by the balanced equation.

This imbalance introduces a new constraint.

Reactions as resource transformations.

A chemical reaction may be viewed as a transformation of resources. Each reactant provides a supply of atoms that can participate in the formation of products. For the reaction to proceed, the required atoms must be available in the proportions specified by the balanced equation.

If one reactant runs out before the others, the reaction can no longer continue, even if other substances remain present.

The reactant that is consumed first therefore determines how far the reaction can proceed.

The limiting reagent principle.

The reactant that restricts the extent of the reaction is called the limiting reagent. Because it is used up first, it sets the maximum possible quantity of product that can form. All other reactants are present in excess and remain partially unused.

This idea is analogous to many everyday situations involving limited resources. If a baker has flour, sugar, and eggs but only enough eggs to make ten cakes, then the eggs determine how many cakes can be baked, regardless of how much flour or sugar remains.

In chemistry the same logic applies. The scarcest required component propagates its limitation through the entire reaction.

Constraint propagation in reaction networks.

When a reaction involves multiple substances, each reactant imposes a potential constraint on the system. The limiting reagent is the one that becomes active first, restricting the overall outcome.

Philosopher Alicia Juarrero has described similar processes in complex systems as cascades of constraints. When one constraint becomes active, it influences the behavior of the entire system, shaping what outcomes are possible.

Chemical reactions provide a clear quantitative example of this idea. The availability of each reactant determines how far the reaction can proceed and how much product can be formed.

Theoretical yield and real outcomes.

Once the limiting reagent is identified, it determines the theoretical yield of the reaction: the maximum amount of product that could form if the reaction proceeded perfectly according to stoichiometric ratios.

In real laboratory conditions, however, reactions rarely achieve this ideal. Side reactions, incomplete conversion, and practical losses during purification often reduce the amount of product actually obtained.

To measure this difference, chemists compare the actual yield obtained in an experiment with the theoretical yield predicted by stoichiometric calculations.

The ratio between these two quantities, expressed as a percentage, is called the percent yield.

Stoichiometry as system analysis.

The concepts of limiting reagents and percent yield reveal how chemical reactions can be analyzed as constrained systems. Each reactant contributes a resource, and the reaction proceeds only as far as the most restrictive resource allows.

By identifying which component limits the system, chemists can predict the maximum output of a reaction and design experiments that use materials efficiently.

In the sections that follow we formalize the definition of the limiting reagent and develop methods for identifying it in chemical reactions. We then introduce the concept of percent yield and show how it compares theoretical predictions with experimental outcomes.

Through these ideas the stoichiometric analysis of reactions becomes a study of how conservation laws and resource constraints determine the possible transformations of matter.

20.1 The Limiting Reagent

Definition Limiting Reagent

The **limiting reagent** is the reactant consumed first, setting the maximum possible yield. All other reactants are **excess reagents**.

Worked Example Juarrero: constraint propagation in a reaction

Consider $N_2 + 3H_2 \rightarrow 2NH_3$. Begin with 2 mol N_2 and 5 mol H_2.
N_2 alone could produce 4 mol NH_3; H_2 alone could produce $5 \times \frac{2}{3} \approx 3.33$ mol NH_3.
H_2 is limiting. Yield $= 3.33$ mol NH_3.
Juarrero [5] calls this *constraint cascade*: the scarcest resource propagates its limitation through the entire network.

20.2 Percent Yield

$$\%\text{yield} = \frac{\text{actual yield}}{\text{theoretical yield}} \times 100\%.$$

Exercises

Exercise 20.1. How many moles of excess N_2 remain after the reaction in the example above?

Challenge 20.2. Model a three-step reaction cascade $A + B \to C$, $C + D \to E$, $E + F \to G$. Starting with 4 mol each of A, B, D, F and none of the intermediates, determine the theoretical yield of G. Identify which reagent is limiting at each stage.

Chapter 21

Reaction Networks and Conservation Laws

Networks of Transformation

The previous chapters introduced stoichiometry as the quantitative analysis of chemical reactions. Individual reactions were balanced by enforcing conservation of atoms, and stoichiometric ratios allowed us to compute how much product could be formed from given reactants. These methods treated reactions largely as isolated events.

In reality, however, most chemical processes do not occur in isolation. They form networks of interacting transformations.

From single reactions to reaction networks.

In many natural and industrial systems, the products of one reaction become the reactants of another. A sequence of reactions may convert raw materials into intermediate compounds, which are then transformed into final products. Biological metabolism, atmospheric chemistry, and industrial synthesis all involve such chains of reactions.

When many reactions occur simultaneously, the system becomes a reaction network. Each reaction alters the concentrations of multiple chemical species, and these changes propagate through the network as the system evolves over time.

The analysis of such systems requires a mathematical framework capable of describing how many interacting transformations occur at once.

Stoichiometric structure of networks.

The key mathematical object in the study of reaction networks is the stoichiometric matrix. Each column of this matrix corresponds to a reaction, and each row corresponds to a chemical species. The entries of the matrix record how many molecules of each species are consumed or produced by each reaction.

In this representation, the entire network of chemical transformations is encoded within a single matrix.

When this matrix acts on a vector of reaction rates, it determines how the concentrations of all species change over time. The resulting equations describe the dynamic evolution of the chemical system.

Conservation laws and invariant quantities.

Even within complex reaction networks, certain quantities remain constant. Because atoms are conserved, combinations of species that contain the same atoms must satisfy conservation relations. These relations correspond mathematically to vectors in the kernel of the stoichiometric matrix.

If a vector lies in this kernel, it represents a quantity that does not change as reactions proceed.

Thus conservation laws appear as invariants within the mathematical structure of the reaction network.

Flows of matter and mathematical continuity.

The equations governing reaction networks resemble other conservation equations encountered in physics. For example, fluid dynamics describes the motion of matter using continuity equations that express the principle that matter cannot disappear or appear spontaneously. The rate of change of density within a region must equal the net flow of material into or out of that region.

Although the mathematical forms differ, the underlying principle is the same. Whether we are studying molecules reacting in a chemical system or mass flowing through a fluid, the equations ensure that conserved quantities remain balanced.

From discrete reactions to continuous fields.

Reaction networks therefore provide a bridge between discrete chemical processes and continuous physical systems. The same conservation principles appear in algebraic stoichiometry, in differential equations describing reaction kinetics, and in field theories describing the flow of matter or energy through space.

Seen from this perspective, chemistry becomes one example of a broader mathematical pattern: systems evolve through transformations that preserve certain underlying quantities.

In the sections that follow we formalize the mathematical description of reaction networks using the stoichiometric matrix and reaction rate functions. These tools allow us to describe how concentrations change over time and how conservation laws emerge naturally from the structure of the network.

Through this framework we will see that the same balance principles encountered throughout this book—algebraic constraints, geometric invariants, and periodic structures—also govern the dynamic evolution of chemical systems.

21.1 Dynamic Stoichiometry

In a reaction network, concentrations $\mathbf{c}$ evolve by

$$\frac{d\mathbf{c}}{dt} = S\,\mathbf{r}(\mathbf{c}),$$

where S is the stoichiometric matrix and $\mathbf{r}(\mathbf{c})$ is the vector of reaction rates. Conservation is encoded in $\ker(S)$.

Worked Example Continuity in disguise

The equation above is the discrete analogue of the field-theoretic continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$

describing conservation of density ρ in a flowing medium with velocity $\mathbf{v}$. Algebra, stoichiometry, and field theory all encode the same structural idea: *what flows in must flow out.*

Deeper Structure

Appendix E derives the RSVP field equations as the continuum limit of this conservation structure.

Part V

Integration: Mathematics of Structured Systems

Chapter 22

Constraint Networks

Systems of Relations

The preceding parts of this book developed several distinct mathematical languages. Algebra described relationships among quantities through equations. Geometry revealed invariants under spatial transformation. Trigonometry analyzed cyclic behavior generated by rotation. Chemistry introduced conservation laws governing the transformation of matter. Although these topics appear to belong to different domains, they share a common structural theme.

Each system can be understood as a collection of interconnected constraints.

From isolated equations to systems of relations.

In early chapters we studied individual equations and small systems of equations. Each equation imposed a restriction on the possible values of variables. When multiple equations were combined, the set of allowable solutions became progressively narrower until a consistent solution emerged.

This logic extends naturally to larger systems.

In many scientific and social contexts, relationships do not occur in pairs but within complex webs of interaction. Variables influence one another through chains of dependencies. A change in one component propagates through the entire structure of relations.

To represent such systems clearly, mathematics introduces the concept of a network.

Nodes, edges, and structure.

A network, or graph, consists of nodes connected by edges. The nodes represent entities within a system, while the edges represent relationships or constraints linking those entities.

Depending on the context, nodes might represent physical quantities, chemical species, individuals in a social system, or locations in an ecological landscape. Edges may represent equations, flows of material, communication links, or causal influences.

By representing systems as networks, we focus attention not only on the individual components but on the structure of their interactions.

Propagation of constraints.

One of the most important features of a constraint network is that changes propagate through the structure. When the value of one node changes, the constraints connecting it to other nodes require those nodes to adjust as well. This chain reaction continues until the system reaches a new consistent configuration.

Earlier chapters provided several examples of this phenomenon. In a system of linear equations, changing one variable affects the values of others. In a chemical reaction network, altering the concentration of one species influences the entire set of reaction rates. In ecological models, changing the availability of a resource affects every process that depends on it.

Constraint propagation therefore provides a unified perspective on many types of systems.

Networks across disciplines.

The language of networks appears in many areas of science and social analysis. Physicists describe interacting particles as networks of forces. Biologists model metabolic pathways as networks of chemical reactions. Computer scientists analyze information flow through networks of processors or communication channels.

Philosopher Hannah Arendt described political life as a web of relationships among individuals whose actions and communications create a shared public space. Similarly, speculative social systems imagined in the work of Ursula K. Le Guin often revolve around carefully balanced networks of ecological and social constraints.

Although the contexts differ, the underlying mathematical idea remains the same: complex systems are defined not only by their components but by the relationships connecting those components.

Networks as mathematical objects.

Modern mathematics studies networks using the theory of graphs and related structures. These tools allow us to analyze connectivity, stability, and flow within complex systems. Techniques from linear algebra, probability, and dynamical systems all contribute to the study of networks.

By representing a system as a network of nodes and edges, we obtain a framework capable of describing interactions that would be difficult to analyze in isolation.

In the sections that follow we formalize the concept of a network and explore how systems of equations, ecological relationships, and social interactions can all be represented as constraint networks. Through this approach we see that many of the mathematical ideas developed throughout this book are different manifestations of a single underlying principle: structure emerges from the relationships that link the parts of a system together.

22.1 Networks as Mathematical Objects

A **network** (or graph) consists of **nodes** (vertices) and **edges** (links). When nodes represent quantities and edges represent constraints, the network encodes the structure of a system.

Worked Example Social constraint network after Arendt

Arendt's analysis of political space can be formalised as a constraint network in which nodes are actors and directed edges are communicative acts [1]. The system is stable when the network satisfies certain consistency conditions — analogous to the solvability of a linear system.

Worked Example Ecological resource balance after Le Guin

The allocation system $F = 2L$, $E = F + L$ from Chapter 4 is a two-node constraint network. Le Guin's utopian societies must maintain exactly such networks at equilibrium [3].

Chapter 23

Cycles, Flows, and Equilibrium

Balance in Mathematical Systems

Throughout this book we have encountered many different mathematical structures: algebraic equations, geometric relations, trigonometric identities, chemical reaction balances, and differential equations describing physical flows. At first glance these subjects appear to belong to separate domains.

Yet each of them expresses a common structural idea: balance.

In algebra, balance appears as the equality of two expressions. In geometry it appears as the invariant relationship between the sides of a right triangle. In trigonometry it appears as the identity linking sine and cosine on the unit circle. In chemistry it appears as the conservation of atoms in a reaction. In physics it appears as the continuity equation governing the flow of matter.

Although the mathematical symbols differ, the underlying principle is the same. A conserved quantity constrains how a system can change.

Balance as a structural law.

A balanced relation states that certain quantities cannot change independently. If one part of the relation varies, another part must vary in a compensating way.

For example, if the sides of a right triangle change, the relationship $a^2 + b^2 = c^2$ must still hold. If the concentrations of species in a chemical reaction evolve, the total number of atoms of each element remains constant. If mass flows through a region of space, the change in density must equal the net flow entering or leaving that region.

Such constraints define the allowable states of the system.

Dynamic systems and equilibrium.

Many systems evolve over time. Chemical concentrations change, physical fields move through space, and populations fluctuate as resources are consumed and replenished. The evolution of these systems is described by rules that transform one state into another.

Sometimes these dynamics reach a configuration in which further change ceases. The system has reached equilibrium.

Mathematically, equilibrium corresponds to a fixed point of the system's transformation rule. Applying the system dynamics to that state produces no further change.

Unifying patterns.

The purpose of this chapter is to place the different mathematical ideas studied throughout the book within a single conceptual framework. We will see that algebraic equations, geometric identities, trigonometric relations, chemical balances, and physical continuity equations all express the same fundamental idea: systems evolve under constraints that preserve certain quantities.

Recognizing these shared structures allows us to understand seemingly different phenomena as manifestations of the same underlying principles of balance, flow, and equilibrium.

23.1 Three Kinds of Balance

All the subjects in this book converge on three structural ideas:

Type	Domain	Equation
Algebraic balance	variable equations	$ax = b$
Spatial balance	Pythagorean geometry	$a^2 + b^2 = c^2$
Rotational balance	unit circle	$\sin^2 + \cos^2 = 1$
Material balance	stoichiometry	$N\gamma = \mathbf{0}$
Dynamic balance	field continuity	$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$

23.2 Equilibrium as Fixed Point

Definition Equilibrium

A system state $\mathbf{x}^*$ is an **equilibrium** if applying the system's dynamics to $\mathbf{x}^*$ returns $\mathbf{x}^*$: $T(\mathbf{x}^*) = \mathbf{x}^*$.

Chapter 24

Scaling and Multilevel Structure

Structure Across Scales

One of the most striking features of natural systems is that similar patterns often appear at very different scales. Structures that govern microscopic processes frequently reappear in macroscopic phenomena. Mathematics provides the language that allows us to recognize these recurring patterns.

In earlier chapters we studied scaling relationships such as $y = kx^p$. These relations describe how one quantity changes when another is enlarged or reduced. The exponent p determines how rapidly the dependent quantity grows relative to the independent one.

Many important physical laws are scaling laws. The surface area of a sphere grows with the square of its radius, while its volume grows with the cube of the radius. The intensity of light from a star decreases with the square of the distance from the source. Such relationships arise because geometric dimensions change in predictable ways when objects are enlarged.

Self-similar patterns.

When the same mathematical relationship appears at multiple scales, the system is said to exhibit scale invariance or self-similarity. A pattern that governs a small system also governs larger versions of the system, provided that the relevant variables are scaled appropriately.

This property occurs in many scientific contexts. Physical laws remain valid across enormous ranges of size, from laboratory experiments to astronomical systems. Ecological networks exhibit similar structural principles whether they involve microorganisms in a soil community or entire planetary biospheres.

The recurrence of such patterns suggests that certain structural relationships are fundamental features of the systems being studied.

Multilevel systems.

Many complex systems consist of multiple interacting levels. Chemical interactions determine the properties of minerals. Geological processes determine the distribution of those minerals within planetary crusts. Biological activity can in turn alter mineral diversity by introducing new chemical environments.

Each level of organization operates according to its own mechanisms, yet the same constraint logic often governs the emergence of structure.

Scaling as a unifying idea.

Understanding how relationships behave under scaling allows us to connect phenomena that occur at very different magnitudes. The same mathematical principles that describe laboratory reactions, planetary processes, or stellar radiation fields may be applied across many orders of magnitude.

In this chapter we examine how patterns repeat across scales and how mathematical scaling laws provide a framework for understanding multilevel structure in complex systems.

24.1 Patterns That Recur Across Scales

Theorem 24.1 (Self-Similarity Under Scaling)**.** *If a relationship $y = kx^p$ holds at one scale, and the same exponent p governs the relationship at larger or smaller scales, the pattern is* ***scale-invariant****.*

The inverse-square law $I = L/(4\pi r^2)$ is scale-invariant: it holds whether r is one metre or one light-year.

Worked Example Hazen: mineral evolution

Hazen et al. [4] trace how the diversity of mineral species on Earth grew in proportion to the complexity of planetary processes. At each scale — atomic bonding, crystal structure, tectonic activity, biological metabolism — the same constraint logic applies: new mineral species form precisely when new constraints (temperature, pressure, chemical environment) are satisfied.

Chapter 25

From Elementary Systems to Scientific Models

From Simple Relations to Scientific Models

The mathematical ideas introduced in this book may appear elementary when considered individually. Ratios, triangles, trigonometric identities, and balanced chemical equations are often presented as separate topics within secondary education. Yet these concepts form the essential building blocks of nearly every quantitative scientific model.

Scientific modeling begins with the identification of relationships between quantities. These relationships may describe spatial structure, temporal variation, material transformation, or flows of energy and matter. Once such relationships are identified, mathematics provides the language needed to express them precisely.

From relations to systems.

Early chapters focused on simple equations and geometric relations. A single equation constrains the possible values of variables, while a geometric relation restricts the shape of a figure. When multiple relations are combined, they form a system whose behavior is determined by the interaction of its constraints.

More advanced scientific models follow exactly the same pattern. Physical laws are expressed as systems of equations whose solutions describe the evolution of a system over time or the configuration of a system in space. Whether one studies planetary motion, ecological dynamics, or chemical reactions, the mathematical structure always consists of interacting constraints.

Transformation and invariance.

Two complementary ideas organize scientific reasoning. The first is transformation: the rules that describe how a system changes. The second is invariance: the quantities that remain constant despite those changes.

For example, geometric transformations may rotate or translate a shape while preserving distances. Chemical reactions rearrange atoms while preserving elemental counts. Physical fields may flow through space while conserving mass or energy.

Recognizing invariant quantities often reveals the fundamental laws governing a system.

Layered mathematical structures.

The diagram in this chapter illustrates how the topics studied in this book connect to one another. Algebra provides the symbolic language used to express relations. Geometry describes spatial structure. Trigonometry captures periodic behavior arising from rotation. Stoichiometry expresses conservation in chemical transformations.

These ideas can be extended to more advanced frameworks. Field theories describe how quantities vary continuously through space and time, while modern mathematical structures such as category theory provide abstract ways of representing relationships between entire systems.

Scientific models as structured networks.

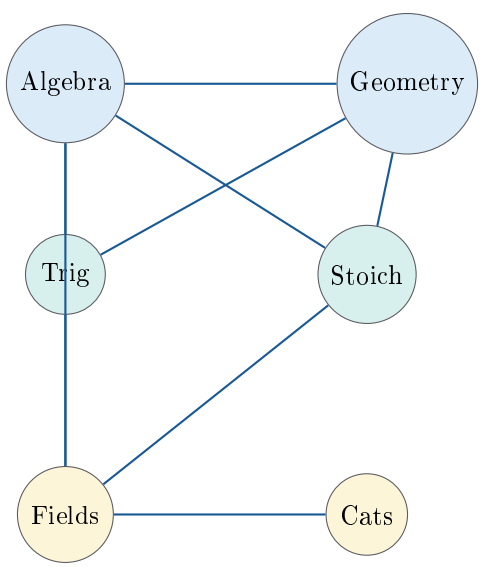
At its core, a scientific model is a structured network of mathematical relations. Each component of the model represents a quantity or process, and the equations linking those components encode the constraints that govern their interactions.

Although the models used in advanced science may appear complex, their architecture rests on the same fundamental ideas encountered throughout this book: equations, transformations, invariants, and conservation laws.

The goal of this final chapter is to place the mathematical tools we have studied into a broader scientific perspective. By recognizing the common structure underlying algebra, geometry, trigonometry, and stoichiometry, we can see how these elementary systems grow into the sophisticated models used to understand the natural world.

25.1 The Architecture of Science

The tools introduced in this book form the foundation of all quantitative science. Every advanced model is an elaboration of the same constraint structure encountered in Chapter 1.



Guiding Principle

Scientific understanding grows by identifying patterns of constraint and transformation across domains. Transformation reveals structure; invariance reveals law.

Appendix A

Algebra as Constraint Before Content

A.1 Constraint Systems

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ denote a vector of system variables. A **constraint system** is a collection of equations

$$F_\alpha(x) = 0, \qquad \alpha = 1, \ldots, m,$$

where each function $F_\alpha : \mathbb{R}^n \to \mathbb{R}$ specifies a restriction on the admissible configurations of the variables.

The set of all points satisfying every constraint is the **solution set** (or constraint locus)

$$\mathcal{C} = \{x \in \mathbb{R}^n \mid F_\alpha(x) = 0 \text{ for all } \alpha\}.$$

Geometrically, $\mathcal{C}$ is the intersection of the level sets $F_\alpha^{-1}(0)$. Each additional constraint restricts the admissible region of the state space.

In the special case where each F_α is linear,

$$F_\alpha(x) = a_{\alpha 1}x_1 + \cdots + a_{\alpha n}x_n - b_\alpha,$$

the system may be written compactly in matrix form as

$$A\mathbf{x} = \mathbf{b},$$

where $A \in \mathbb{R}^{m \times n}$ is the coefficient matrix, $\mathbf{x} \in \mathbb{R}^n$ is the vector of unknowns, and $\mathbf{b} \in \mathbb{R}^m$ is the vector of constants.

The solution set of a consistent linear system is an affine subspace of $\mathbb{R}^n$.

A.2 Admissibility and Relaxation

A constraint system may fail to admit an exact solution. If the number of constraints exceeds the number of variables ($m > n$), the system is **overdetermined**. In this case the equations may be mutually incompatible.

To measure the degree to which a configuration violates the constraints, define the **residual functional**

$$\mathcal{R}(\mathbf{x}) = \sum_{\alpha=1}^{m} \left|F_\alpha(\mathbf{x})\right|^2.$$

Configurations that minimise $\mathcal{R}$ provide the closest possible approximate solution. For linear systems this leads to the classical least–squares formulation.

Proposition A.1 (Least–Squares Solution)**.** *For the linear system $A\mathbf{x} = \mathbf{b}$, the minimiser of*

$$\mathcal{R}(\mathbf{x}) = \|A\mathbf{x} - \mathbf{b}\|^2$$

*satisfies the **normal equations***

$$A^\top A\mathbf{x} = A^\top \mathbf{b}.$$

Proof. Let

$$\mathcal{R}(\mathbf{x}) = (A\mathbf{x} - \mathbf{b})^\top (A\mathbf{x} - \mathbf{b}).$$

Expanding gives

$$\mathcal{R}(\mathbf{x}) = \mathbf{x}^\top A^\top A\mathbf{x} - 2\mathbf{b}^\top A\mathbf{x} + \mathbf{b}^\top \mathbf{b}.$$

Differentiating with respect to $\mathbf{x}$ and setting the gradient to zero yields

$$\nabla\mathcal{R} = 2A^\top A\mathbf{x} - 2A^\top \mathbf{b} = 0,$$

which simplifies to

$$A^\top A\mathbf{x} = A^\top \mathbf{b}.$$

□

The least–squares solution therefore represents the configuration that minimises the total squared violation of the constraints.

A.3 Degrees of Freedom

When the number of variables exceeds the number of independent constraints ($m < n$), the system is **underdetermined**. In this case multiple solutions exist.

Let $r = \operatorname{rank}(A)$. The nullspace

$$\ker(A) = \{\mathbf{v} \in \mathbb{R}^n \mid A\mathbf{v} = 0\}$$

has dimension $n - r$ by the rank–nullity theorem.

If $\mathbf{x}_0$ is a particular solution of $A\mathbf{x} = \mathbf{b}$, then the complete set of solutions is

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{v}, \qquad \mathbf{v} \in \ker(A).$$

Thus the solution set forms an affine space of dimension

$$n - \operatorname{rank}(A).$$

These remaining dimensions represent the **degrees of freedom** of the system: directions in which the variables may vary while preserving all constraints.

In many scientific contexts these degrees of freedom correspond to the internal flexibility of a system. They determine how the system can reconfigure itself while remaining consistent with the governing relations.

Appendix B

Geometry as Structured Invariance

B.1 Euclidean Space and the Inner Product

Let $\mathbb{R}^n$ denote n–dimensional Euclidean space. A point $x \in \mathbb{R}^n$ is represented by a coordinate vector $x = (x_1, \ldots, x_n)$.

The fundamental geometric structure on $\mathbb{R}^n$ is the **inner product**

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i,$$

which assigns to each pair of vectors a real number measuring their mutual alignment.

From the inner product we obtain the **norm**

$$\|x\| = \sqrt{\langle x, x \rangle},$$

which represents the Euclidean length of a vector.

Distance.

The distance between two points $x, y \in \mathbb{R}^n$ is defined by

$$d(x, y) = \|x - y\|.$$

Expanding the square of the norm gives

$$\|x - y\|^2 = \langle x - y, x - y \rangle = \langle x, x \rangle - 2\langle x, y \rangle + \langle y, y \rangle.$$

In two dimensions this identity yields the familiar

$$(x_1 - y_1)^2 + (x_2 - y_2)^2,$$

which is the algebraic form of the Pythagorean theorem.

Angles.

The inner product also determines the angle between vectors. For nonzero vectors x and y, define

$$\cos\theta = \frac{\langle x, y \rangle}{\|x\|\|y\|}.$$

Orthogonality corresponds to the condition

$$\langle x, y \rangle = 0.$$

Thus Euclidean geometry can be understood as the study of invariants arising from the inner product structure.

B.2 Linear Transformations and Invariance

A linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ preserves Euclidean geometry if it preserves the inner product:

$$\langle Tx, Ty \rangle = \langle x, y \rangle \quad \text{for all } x, y.$$

Such transformations form the **orthogonal group**

$$O(n) = \{A \in \mathbb{R}^{n \times n} \mid A^\top A = I\}.$$

Matrices in $O(n)$ preserve lengths and angles:

$$\|Ax\| = \|x\|.$$

Rotations and reflections therefore represent the fundamental symmetry operations of Euclidean space.

B.3 Configuration Spaces

Many physical and mathematical systems cannot be described by a finite number of coordinates. Instead, their states are represented by functions defined over a spatial region Ω.

In such cases the state space becomes infinite–dimensional.

For the RSVP framework, the system state consists of three fields: a scalar density Φ, a vector flow field $\mathbf{v}$, and an entropy field S. The configuration space is therefore

$$\mathcal{M} = \left\{ (\Phi, \mathbf{v}, S) \;\middle|\; \Phi : \Omega \to \mathbb{R},\ \mathbf{v} : \Omega \to \mathbb{R}^3,\ S : \Omega \to \mathbb{R} \right\}.$$

Each element of $\mathcal{M}$ represents a complete spatial configuration of the system.

B.4 Hilbert Space Structure

To extend Euclidean geometry to this setting, we equip $\mathcal{M}$ with an L^2 inner product.

Let

$$X = (\Phi, \mathbf{v}, S), \qquad Y = (\Phi', \mathbf{v}', S').$$

Define

$$\langle X, Y \rangle = \int_\Omega \left(\Phi\, \Phi' + \mathbf{v} \cdot \mathbf{v}' + S\, S' \right) \, dx.$$

The corresponding norm is

$$\|X\| = \sqrt{\langle X, X \rangle}.$$

With this structure, $\mathcal{M}$ becomes a Hilbert space.

Distances between configurations.
The distance between two field configurations X and Y is

$$d(X,Y) = \|X - Y\| = \left(\int_\Omega \big((\Phi - \Phi')^2 + \|\mathbf{v} - \mathbf{v}'\|^2 + (S - S')^2 \big) dx \right)^{1/2}.$$

Thus two configurations are close if their scalar, vector, and entropy fields differ only slightly across the domain.

B.5 Geometric Interpretation

The transition from $\mathbb{R}^n$ to the configuration space $\mathcal{M}$ represents a shift from finite–dimensional geometry to the geometry of function spaces. In finite dimensions, a vector is an ordered tuple of numbers describing a point or direction in space. In the configuration space $\mathcal{M}$, the role of vectors is played by fields defined over the spatial domain Ω. Thus a system state is no longer specified by finitely many coordinates but by functions that assign values to every point in the domain.

The algebraic operations familiar from Euclidean geometry extend naturally to this setting. The dot product between vectors becomes a spatial integral of pointwise products of fields. The length of a vector becomes the norm of a field, obtained by integrating the square of the field magnitude over the domain. Distances between points in $\mathbb{R}^n$ generalize to distances between configurations, measured by the integral of squared differences between the corresponding fields.

In this way the structure of Euclidean geometry is preserved while the objects under consideration become functions rather than finite coordinate vectors. The resulting space retains the essential geometric notions of length, distance, orthogonality, and projection, but these notions now describe relationships between entire field configurations. Such infinite–dimensional geometric spaces provide the natural mathematical setting for continuum physics, variational principles, and field theories.

The study of such configuration spaces forms the mathematical foundation of modern field theories, variational principles, and continuum models. Within this framework, physical dynamics can be interpreted as flows through an infinite–dimensional geometric space whose structure is determined by the underlying invariants of the system.

Appendix C

Trigonometry as Oscillation, Rotation, and Recurrence

C.1 The Rotation Group

Consider the linear transformation of the plane defined by the matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \theta \in \mathbb{R}.$$

For any vector $x = (x_1, x_2) \in \mathbb{R}^2$, the action of $R(\theta)$ rotates the vector counterclockwise through an angle θ about the origin. Explicitly,

$$R(\theta)x = \begin{pmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{pmatrix}.$$

The matrices $R(\theta)$ form a one–parameter subgroup of the general linear group $\mathrm{GL}(2, \mathbb{R})$. Under matrix multiplication they satisfy the composition rule

$$R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2),$$

which follows directly from the trigonometric addition formulas

$$\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2, \qquad \sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2.$$

The identity element corresponds to $\theta = 0$,

$$R(0) = I,$$

and the inverse of a rotation is given by

$$R(\theta)^{-1} = R(-\theta).$$

The set

$$\mathrm{SO}(2) = \{R(\theta) \mid \theta \in \mathbb{R}\}$$

is therefore a Lie group, known as the *special orthogonal group* in two dimensions.

A defining property of SO(2) is the preservation of the Euclidean inner product. For any vectors $x, y \in \mathbb{R}^2$,

$$\langle R(\theta)x, R(\theta)y \rangle = \langle x, y \rangle.$$

Equivalently,

$$R(\theta)^\top R(\theta) = I.$$

This orthogonality condition ensures that rotations preserve both lengths and angles. In particular,

$$\|R(\theta)x\| = \|x\|.$$

The determinant of the rotation matrix is

$$\det R(\theta) = \cos^2\theta + \sin^2\theta.$$

Since $\det R(\theta) = 1$ for all θ, the trigonometric identity

$$\sin^2\theta + \cos^2\theta = 1$$

can be interpreted as the algebraic condition ensuring that rotations preserve oriented area.

Thus the fundamental identity of trigonometry expresses the invariance property defining the rotation group.

C.2 Harmonic Analysis

Trigonometric functions also arise as solutions of fundamental differential equations describing oscillatory motion.

Consider the second–order linear ordinary differential equation

$$\ddot{x} + \omega^2 x = 0, \qquad \omega > 0.$$

This equation governs the dynamics of simple harmonic oscillators such as springs, pendula (for small amplitudes), and many wave phenomena.

To solve the equation, assume a solution of exponential form

$$x(t) = e^{\lambda t}.$$

Substitution yields the characteristic equation

$$\lambda^2 + \omega^2 = 0,$$

whose roots are

$$\lambda = \pm i\omega.$$

Therefore the general complex solution is

$$x(t) = Ae^{i\omega t} + Be^{-i\omega t}.$$

Using Euler's formula

$$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t),$$

the real solutions can be written in trigonometric form

$$x(t) = C\cos(\omega t) + D\sin(\omega t).$$

These functions describe periodic oscillations with angular frequency ω and period

$$T = \frac{2\pi}{\omega}.$$

The connection between rotations and oscillations becomes evident when the system is expressed as a first–order dynamical system. Define the state vector

$$\mathbf{z}(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}.$$

Then the harmonic oscillator equation can be written as

$$\dot{\mathbf{z}} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \mathbf{z}.$$

The solution of this linear system is

$$\mathbf{z}(t) = e^{tA}\mathbf{z}(0), \qquad A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}.$$

The matrix exponential e^{tA} generates a continuous rotation in the phase plane $(x, \dot{x})$, showing that harmonic motion corresponds geometrically to uniform circular motion in phase space.

C.3 Plane Waves and Higher–Dimensional Oscillation

In spatially extended systems the same oscillatory structure appears in the form of traveling waves.

Let $\mathbf{x} \in \mathbb{R}^3$ denote spatial position and t denote time. A plane wave solution takes the form

$$\psi(\mathbf{x}, t) = e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)},$$

where $\mathbf{k} \in \mathbb{R}^3$ is the wavevector and ω is the angular frequency.

Taking real parts yields the familiar trigonometric representation

$$\psi(\mathbf{x}, t) = \cos(\mathbf{k}\cdot\mathbf{x} - \omega t).$$

These solutions arise naturally when linearizing many physical systems around a uniform equilibrium state. Small perturbations decompose into superpositions of plane waves, each characterized by a wavevector $\mathbf{k}$ and frequency ω.

Within the RSVP framework, linearization of the field equations around a uniform background state produces perturbative modes of precisely this form,

$$e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}.$$

The trigonometric recurrence underlying elementary oscillations thus extends naturally to higher–dimensional field configurations, where the geometry of rotations and periodic motion governs the propagation of waves through space.

Appendix D

Stoichiometry as Conservation Algebra

D.1 The Stoichiometric Null Space

Consider a chemical reaction network involving s chemical species and r reactions. Let

$$\mathbf{c}(t) = (c_1(t), \ldots, c_s(t))^\top \in \mathbb{R}^s$$

denote the vector of species concentrations.

The structure of the reaction network is encoded in the **stoichiometric matrix**

$$S \in \mathbb{Z}^{s \times r}.$$

The entry S_{ij} represents the net number of molecules of species i produced by reaction j. Negative entries correspond to reactants and positive entries correspond to products.

Let $\boldsymbol{\gamma} \in \mathbb{R}^r$ denote a vector of reaction coefficients. The condition for a balanced reaction is

$$S\boldsymbol{\gamma} = \mathbf{0}.$$

Thus admissible reaction coefficients lie in the **null space** (kernel) of the stoichiometric matrix:

$$\ker(S) = \{\boldsymbol{\gamma} \in \mathbb{R}^r \mid S\boldsymbol{\gamma} = 0\}.$$

Vectors in $\ker(S)$ correspond to combinations of reactions that preserve the total count of each conserved element.

If the matrix S has rank k, then by the rank–nullity theorem

$$\dim(\ker S) = r - k.$$

This dimension represents the number of independent conservation relations within the reaction network.

Another important structure is the **left null space**

$$\ker(S^\top) = \{\mathbf{w} \in \mathbb{R}^s \mid \mathbf{w}^\top S = 0\}.$$

Vectors $\mathbf{w}$ in $\ker(S^\top)$ correspond to conserved quantities. If $\mathbf{w}^\top S = 0$, then

$$\mathbf{w}^\top \dot{\mathbf{c}}(t) = \mathbf{w}^\top S\, \mathbf{r}(\mathbf{c}) = 0.$$

Thus the quantity $\mathbf{w}^\top \mathbf{c}(t)$ remains constant over time. These conserved quantities correspond to elemental conservation laws or other invariant totals.

D.2 Dynamic Stoichiometry

The time evolution of the system is governed by the reaction rate vector

$$\mathbf{r}(\mathbf{c}) = (r_1(\mathbf{c}), \ldots, r_r(\mathbf{c}))^\top.$$

Each component $r_j(\mathbf{c})$ specifies the rate at which reaction j occurs. Under deterministic mass–action kinetics,

$$r_j(\mathbf{c}) = k_j \prod_{i=1}^{s} c_i^{\nu_{ij}},$$

where ν_{ij} denotes the stoichiometric coefficient of species i in the reactant complex of reaction j, and k_j is the reaction rate constant.

The evolution of the concentration vector is therefore given by

$$\dot{\mathbf{c}} = S\,\mathbf{r}(\mathbf{c}).$$

This system of differential equations describes how the concentrations change as reactions proceed.

Each reaction contributes a vector $S_{\cdot j}$ in concentration space, and the dynamics of the system are obtained by summing these contributions weighted by their reaction rates.

Thus the stoichiometric matrix determines the directions in which the system may evolve, while the reaction rate functions determine the speed of motion along those directions.

D.3 Stoichiometric Subspaces

The trajectories of the dynamical system

$$\dot{\mathbf{c}} = S\,\mathbf{r}(\mathbf{c})$$

are constrained to remain within affine subspaces known as **stoichiometric compatibility classes**.

Let $\mathbf{c}_0$ be an initial concentration vector. The associated stoichiometric compatibility class is

$$\mathbf{c}_0 + \mathrm{im}(S) = \{\mathbf{c}_0 + S\mathbf{y} \mid \mathbf{y} \in \mathbb{R}^r\}.$$

All solutions starting from $\mathbf{c}_0$ remain within this affine subspace.

This follows from the conservation relations. If $\mathbf{w}^\top S = 0$, then

$$\mathbf{w}^\top \mathbf{c}(t) = \mathbf{w}^\top \mathbf{c}_0.$$

Thus conserved quantities restrict the system to a lower–dimensional region of concentration space.

D.4 Continuum Limit and Field Conservation

The algebraic structure of stoichiometry extends naturally to continuum systems.

Let $\rho(x,t)$ denote the density of a conserved quantity within a spatial domain $\Omega \subset \mathbb{R}^3$. The conservation principle states that the rate of change of density within a region equals the net flux entering or leaving that region.

This principle leads to the **continuity equation**

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0,$$

where $\mathbf{v}(x,t)$ is the velocity field describing transport.

The relationship between stoichiometric dynamics and field conservation can be understood by interpreting the reaction terms as discrete sources and sinks of matter.

In the discrete case, conservation is expressed by the algebraic constraint

$$S\boldsymbol{\gamma} = 0.$$

In the continuum case, conservation becomes a differential constraint on the density field.

Thus the same structural principle appears in two different mathematical forms: finite-dimensional linear algebra in reaction networks and partial differential equations in field theory.

Appendix E

RSVP Field Theory: Foundational Derivation

E.1 The Fundamental Fields

Let $\Omega \subset \mathbb{R}^3$ be a spatial domain with coordinates $x = (x_1, x_2, x_3)$ and time variable $t \in \mathbb{R}$. The state of the system at each spacetime point (x, t) is described by three coupled fields

$$\Phi(x,t) \in \mathbb{R}, \qquad \mathbf{v}(x,t) \in \mathbb{R}^3, \qquad S(x,t) \in \mathbb{R}.$$

The scalar field Φ represents the density of the plenum. The vector field $\mathbf{v}$ represents the local flow velocity of the plenum medium. The scalar field S represents the entropy density, which governs dissipative relaxation processes within the system.

Mathematically the configuration space of the theory is therefore

$$\mathcal{M} = \left\{ (\Phi, \mathbf{v}, S) \;\middle|\; \Phi : \Omega \times \mathbb{R} \to \mathbb{R},\ \mathbf{v} : \Omega \times \mathbb{R} \to \mathbb{R}^3,\ S : \Omega \times \mathbb{R} \to \mathbb{R} \right\}.$$

The evolution of the system is governed by coupled partial differential equations expressing conservation, transport, and thermodynamic relaxation.

E.2 Governing Equations

The dynamics of the fields are described by the system

$$\partial_t \Phi + \nabla \cdot (\Phi \mathbf{v}) = D_\Phi \Delta \Phi + F_\Phi(\Phi, \mathbf{v}, S), \tag{A}$$

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P(\Phi, S) + D_v \Delta \mathbf{v} + F_v(\Phi, \mathbf{v}, S), \tag{B}$$

$$\partial_t S + \nabla \cdot (S\mathbf{v}) = D_S \Delta S + \sigma(\Phi, \mathbf{v}, S). \tag{C}$$

The left–hand sides of these equations contain the transport operators governing advection of the fields by the velocity $\mathbf{v}$. The operator

$$\partial_t + \mathbf{v} \cdot \nabla$$

is the material derivative describing the rate of change observed along flow trajectories.

The diffusion terms

$$D_\Phi \Delta\Phi, \qquad D_v \Delta\mathbf{v}, \qquad D_S \Delta S$$

represent smoothing processes that dissipate spatial gradients.

The nonlinear functions

$$F_\Phi(\Phi, \mathbf{v}, S), \qquad F_v(\Phi, \mathbf{v}, S), \qquad \sigma(\Phi, \mathbf{v}, S)$$

encode the internal coupling between density, flow, and entropy.

Equation (A) expresses conservation of plenum density with diffusive and source corrections. Equation (B) governs the momentum balance of the flow field and generalizes the Euler–Navier–Stokes structure. Equation (C) describes entropy transport and production.

E.3 Lyapunov Functional and Entropic Descent

The dissipative structure of the theory can be formulated using a Lyapunov functional defined on the configuration space.

Let

$$\mathcal{F}[\Phi, \mathbf{v}, S] = \int_\Omega \left[\frac{1}{2}|\mathbf{v}|^2 + U(\Phi, S) + \frac{\kappa_\Phi}{2}|\nabla\Phi|^2 + \frac{\kappa_S}{2}|\nabla S|^2 \right] dx.$$

The first term represents kinetic energy associated with the flow field. The potential function $U(\Phi, S)$ encodes the thermodynamic coupling between density and entropy. The gradient terms penalize spatial inhomogeneities and represent interfacial or coherence energy.

A dissipative dynamics is obtained by defining the evolution of each field as a gradient descent in the functional $\mathcal{F}$:

$$\partial_t X = -\frac{\delta\mathcal{F}}{\delta X}, \qquad X \in \{\Phi, \mathbf{v}, S\}.$$

Under this evolution the functional decreases monotonically:

$$\frac{d}{dt}\mathcal{F} = \int_\Omega \frac{\delta\mathcal{F}}{\delta X}\, \partial_t X \, dx = -\int_\Omega \left| \frac{\delta\mathcal{F}}{\delta X} \right|^2 dx \leq 0.$$

Thus $\mathcal{F}$ acts as a Lyapunov functional for the dynamics. Configurations evolve toward states that minimize the free–energy functional.

Within the conceptual framework of RSVP cosmology this descent represents the tendency of the plenum to reorganize toward smoother field configurations of lower free energy.

E.4 Linear Perturbation Theory

To study stability, consider perturbations around a homogeneous background state

$$\Phi(x,t) = \Phi_0 + \delta\Phi(x,t), \qquad \mathbf{v}(x,t) = \delta\mathbf{v}(x,t), \qquad S(x,t) = S_0 + \delta S(x,t).$$

Substituting these expressions into the governing equations and retaining only first–order terms yields a linearized system for the perturbations.

Solutions of the linearized equations can be expressed as Fourier modes

$$\delta X(x,t) = \hat{X}\, e^{i(\mathbf{k}\cdot x - \omega t)},$$

where $\mathbf{k} \in \mathbb{R}^3$ is the wavevector and ω is the angular frequency.

Substituting this ansatz into the linearized equations produces a dispersion relation

$$\omega = \omega(\mathbf{k})$$

that determines the growth, decay, or propagation of perturbations.

If $\mathrm{Im}(\omega) > 0$ the perturbation grows exponentially, indicating instability of the background state. If $\mathrm{Im}(\omega) < 0$ the perturbation decays, indicating that the state is linearly stable.

Thus the dispersion relation characterizes the propagation of waves and the emergence of structure within the RSVP field system.

Appendix F

Entropic Redshift and Non-Expansion Cosmology

F.1 Redshift Without Metric Expansion

In the standard cosmological model, observed redshift is interpreted as a consequence of the expansion of the spacetime metric. Within the RSVP framework, an alternative mechanism is proposed in which photon energy attenuates gradually as radiation propagates through a structured entropic medium.

Let $\mathcal{E}(x,t)$ denote an **entropic attenuation field** defined over spacetime. This field represents the cumulative effect of microscopic interactions between propagating radiation and the thermodynamic structure of the plenum.

Consider a photon trajectory parameterized by an affine parameter λ. Let $x(\lambda)$ denote the spacetime path of the photon. The evolution of the photon frequency $\nu(\lambda)$ is assumed to obey the differential equation

$$\frac{d\nu}{d\lambda} = -\alpha\,\mathcal{E}(x(\lambda),t)\,\nu,$$

where α is a coupling constant describing the strength of the interaction between radiation and the entropic field.

This equation expresses the assumption that frequency attenuation is proportional both to the instantaneous frequency and to the local strength of the entropic field.

F.2 Solution of the Frequency Transport Equation

The differential equation

$$\frac{d\nu}{d\lambda} = -\alpha\,\mathcal{E}(x(\lambda),t)\,\nu$$

is a first-order linear ordinary differential equation.

Separating variables yields

$$\frac{d\nu}{\nu} = -\alpha\,\mathcal{E}(x(\lambda),t)\,d\lambda.$$

Integrating along the photon path from $\lambda = 0$ to λ gives

$$\int_{\nu_0}^{\nu(\lambda)} \frac{d\nu'}{\nu'} = -\alpha \int_0^{\lambda} \mathcal{E}(x(\lambda'), t)\, d\lambda'.$$

Evaluating the integral on the left side yields

$$\ln\left(\frac{\nu(\lambda)}{\nu_0}\right) = -\alpha \int_0^{\lambda} \mathcal{E}(x(\lambda'), t)\, d\lambda'.$$

Exponentiating both sides gives the solution

$$\nu(\lambda) = \nu_0 \exp\left(-\alpha \int_0^{\lambda} \mathcal{E}(x(\lambda'), t)\, d\lambda'\right).$$

Thus the photon frequency decays exponentially according to the integrated entropic attenuation encountered along the path.

F.3 Definition of Redshift

Observed cosmological redshift is defined by the ratio of emitted and observed frequencies:

$$1 + z = \frac{\nu_0}{\nu(\lambda)}.$$

Substituting the solution for $\nu(\lambda)$ yields

$$1 + z = \exp\left(\alpha \int_0^{\lambda} \mathcal{E}(x(\lambda'), t)\, d\lambda'\right).$$

Thus the redshift is determined by the accumulated entropic attenuation along the photon trajectory.

F.4 Weak-Field Approximation

If the attenuation field is approximately homogeneous and weak along the propagation path, one may approximate

$$\mathcal{E}(x(\lambda), t) \approx \mathcal{E}_0.$$

The integral then simplifies to

$$\int_0^{\lambda} \mathcal{E}_0\, d\lambda' = \mathcal{E}_0\, \lambda.$$

The redshift relation becomes

$$1 + z = \exp(\alpha \mathcal{E}_0 \lambda).$$

For small values of $\alpha \mathcal{E}_0 \lambda$, the exponential may be expanded to first order:

$$1 + z \approx 1 + \alpha \mathcal{E}_0 \lambda.$$

Therefore

$$z \approx \alpha \mathcal{E}_0 \lambda.$$

F.5 Emergence of the Linear Distance–Redshift Law

If the affine parameter λ is proportional to physical distance d, the previous relation implies

$$z \propto d.$$

This reproduces the empirically observed linear distance–redshift relation traditionally attributed to cosmological expansion.

In the RSVP framework the same phenomenology emerges without requiring a global expansion of the spacetime metric. Instead, the redshift arises from cumulative entropic attenuation experienced by photons as they propagate through the structured plenum.

Thus the observed large-scale redshift–distance relation can be interpreted as a transport phenomenon governed by the interaction between radiation and the entropy field of the cosmological medium.

Appendix G

TARTAN: Recursive Tiling and Annotated Noise

G.1 Multiscale Partition

Let $\Omega \subset \mathbb{R}^d$ be a spatial domain. The TARTAN framework represents the system using a hierarchy of spatial partitions that resolve the domain at progressively finer scales.

Formally, let

$$\mathcal{T}_0 \supseteq \mathcal{T}_1 \supseteq \mathcal{T}_2 \supseteq \cdots$$

be a nested sequence of partitions of Ω. Each $\mathcal{T}_k$ is a tiling of Ω into a finite or countable set of tiles

$$\mathcal{T}_k = \{T_{k,i}\}_{i \in I_k}, \qquad \Omega = \bigcup_{i \in I_k} T_{k,i}, \qquad T_{k,i} \cap T_{k,j} = \varnothing \text{ for } i \neq j.$$

The refinement structure is encoded by projection maps

$$\pi_k : \mathcal{T}_{k+1} \to \mathcal{T}_k,$$

which associate each fine-scale tile with the unique coarser tile that contains it. Thus if $T' \in \mathcal{T}_{k+1}$ then

$$T' \subseteq \pi_k(T').$$

This hierarchical structure defines a multiscale representation of the domain in which spatial resolution increases with the level index k.

The collection

$$(\mathcal{T}_k, \pi_k)$$

therefore forms an inverse system of partitions. In the limit $k \to \infty$, the tiles converge toward infinitesimal regions of the underlying continuum.

G.2 Annotated Field States

Each tile $T \in \mathcal{T}_k$ carries a local state vector

$$X_k(T) = (\Phi_k(T), \mathbf{v}_k(T), S_k(T), \eta_k(T)).$$

The first three components correspond to discretized values of the RSVP fields: plenum density Φ, vector flow $\mathbf{v}$, and entropy density S. The additional variable $\eta_k(T)$ represents a semantic annotation attached to the tile.

The annotation variable records contextual information associated with the state of the tile. Examples include trajectory provenance, classification labels, or metadata describing the origin of the local configuration. This component enables the framework to propagate not only physical state variables but also higher-level descriptive information through the simulation hierarchy.

G.3 Recursive Field Updates

State evolution occurs through recursive updates across scales.

Let $T' \in \mathcal{T}_{k+1}$ be a tile at level $k+1$. Its parent tile at the coarser level is $\pi_k(T') \in \mathcal{T}_k$. The state of T' is computed by applying an update operator

$$F_k : \mathcal{X}_k \times \Xi_k \to \mathcal{X}_{k+1},$$

where $\mathcal{X}_k$ denotes the state space at level k and Ξ_k denotes the space of stochastic perturbations.

The recursive update rule is therefore

$$X_{k+1}(T') = F_k\big(X_k(\pi_k(T')),\, \xi_k(T')\big)\,.$$

Thus the state of a fine-scale tile depends on the state of its parent tile together with a stochastic perturbation term.

G.4 Noise Decomposition

The perturbation term

$$\xi_k(T')$$

represents stochastic fluctuations introduced during the refinement process. This noise is decomposed into two components,

$$\xi = \xi_{\text{white}} + \xi_{\text{structured}}.$$

The white component represents uncorrelated random fluctuations, typically modeled as Gaussian noise.

The structured component represents correlated perturbations that encode semantic or dynamical information. In the TARTAN framework these perturbations carry metadata describing the origin or interpretation of the noise source. Consequently the stochastic forcing term may contain information about previously observed patterns or constraints within the evolving system.

G.5 Trajectory Memory

In addition to spatial hierarchy, the system retains a record of previous states through a trajectory buffer.

For each tile T and time t, define the history buffer

$$\mathcal{B}(T,t) = \{X(T,t-\tau_j)\}_{j=1}^{L},$$

where $\tau_1, \ldots, \tau_L$ denote discrete time offsets.

The evolution of the tile state is then given by the update operator

$$X(T,t+\Delta t) = G(X(T,t), \mathcal{B}(T,t), \xi(T,t)),$$

where G represents the local dynamical rule governing the system.

This formulation introduces explicit history dependence into the update process. The future state of the system therefore depends not only on its present configuration but also on a finite window of past states.

G.6 Multiscale Dynamics

The combination of recursive spatial refinement and trajectory memory produces a dynamical system operating simultaneously across multiple scales.

At coarse levels the model captures large-scale field structure, whereas finer levels resolve local variations and noise-driven fluctuations. The refinement maps π_k ensure consistency between levels, while the update operators F_k propagate information from coarser to finer resolutions.

This architecture allows the TARTAN framework to represent complex systems in which large-scale patterns emerge from locally stochastic processes while preserving traceable provenance information throughout the hierarchy.

Appendix H

Simulated Agency as Sparse Projection

H.1 State Spaces and the Projection Operator

Let $\mathcal{W}$ denote the space of possible world states and $\mathcal{M}$ the space of internal model states available to an agent. Elements $w \in \mathcal{W}$ represent configurations of the external environment, whereas elements $m \in \mathcal{M}$ represent internal representations constructed by the agent.

In general $\mathcal{W}$ is extremely high–dimensional and may describe a complex physical or informational environment. The internal model space $\mathcal{M}$ is typically of much lower dimension, reflecting the limited representational capacity of the agent.

The process of internal representation is therefore described by a projection map

$$P : \mathcal{W} \to \mathcal{M}, \qquad m = P(w).$$

This mapping compresses the full world state into a reduced internal representation that retains only task-relevant structure.

To formalize the notion of optimal representation, consider an objective functional balancing two competing criteria: fidelity to the task and economy of representation.

Let $L_{\text{task}}(m; w)$ denote a loss function measuring how well the internal model m captures the aspects of the world state w relevant to the agent's task. Let $C(m)$ denote a complexity measure associated with the representation itself. The parameter $\beta > 0$ controls the trade–off between these two terms.

The optimal internal representation is therefore defined by

$$m^* = \arg\min_{m \in \mathcal{M}} \big[L_{\text{task}}(m; w) + \beta\, C(m) \big].$$

This formulation expresses the principle of sparse projection: the agent seeks the simplest internal model capable of performing the required task with acceptable accuracy.

The complexity term $C(m)$ may take several forms depending on the representation scheme. Examples include the dimensionality of the representation, the number of active features, or an information–theoretic measure such as description length.

H.2 The Agency Loop

The interaction between an agent and its environment forms a dynamical feedback loop linking world states, internal representations, and actions.

Let $F : \mathcal{W} \times \mathcal{A} \to \mathcal{W}$ denote the world transition function, where $\mathcal{A}$ is the action space. The environment evolves according to

$$w_{t+1} = F(w_t, a_t).$$

At each time step the agent constructs an internal representation

$$m_t = P(w_t).$$

An action is then selected according to a policy

$$\pi : \mathcal{M} \to \mathcal{A}, \qquad a_t = \pi(m_t).$$

Combining these relations yields the closed-loop system

$$\begin{aligned} w_{t+1} &= F(w_t, a_t), \\ m_t &= P(w_t), \\ a_t &= \pi(m_t). \end{aligned}$$

Substituting the last two equations into the first gives the composite dynamical system

$$w_{t+1} = F\big(w_t, \pi(P(w_t))\big).$$

Thus the evolution of the world state depends on the composition of the projection operator and the policy function.

H.3 Dynamical Structure of the Agency Loop

The coupled variables (w_t, m_t) define a dynamical system on the product space

$$\mathcal{W} \times \mathcal{M}.$$

A fixed point of this system is a pair (w^*, m^*) satisfying

$$w^* = F(w^*, \pi(m^*)), \qquad m^* = P(w^*).$$

Such points correspond to stable configurations in which the agent's actions maintain the environment in a steady state.

More generally, the system may admit periodic orbits

$$(w_{t+k}, m_{t+k}) = (w_t, m_t),$$

which correspond to recurring behavioural patterns or habits.

In this sense the agency loop forms a nonlinear dynamical system whose qualitative behavior may be analyzed using the geometric methods introduced earlier in the text. Fixed points represent equilibria, periodic trajectories correspond to cyclical behaviors, and more complex attractors represent adaptive or exploratory dynamics within the agent–environment interaction.

Appendix I

Category-Theoretic Semantics of Constraint Systems

I.1 Categories

Category theory provides a structural language for describing systems of objects and the transformations between them. Rather than focusing on the internal composition of objects, category theory emphasizes the relationships that connect them.

A **category** $\mathcal{C}$ consists of the following data.

First, there is a class of objects denoted $\mathrm{Ob}(\mathcal{C})$. Objects represent the entities under consideration in the theory.

Second, for every ordered pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$ there is a set of morphisms

$$\mathrm{Hom}_{\mathcal{C}}(A, B),$$

whose elements are maps $f : A \to B$. These morphisms represent structure–preserving transformations between objects.

Third, morphisms admit a composition operation. For any triple of objects A, B, C there is a map

$$\circ : \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$$

defined by

$$(g, f) \mapsto g \circ f.$$

This composition must satisfy the associativity condition

$$h \circ (g \circ f) = (h \circ g) \circ f$$

for all composable morphisms f, g, h.

Finally, for each object A there exists an identity morphism

$$\mathrm{id}_A : A \to A$$

such that

$$\mathrm{id}_B \circ f = f, \qquad f \circ \mathrm{id}_A = f$$

for every morphism $f : A \to B$.

These axioms formalize the notion of composable transformations within a structured system.

Several standard mathematical structures form categories.

In the category **Set** the objects are sets and the morphisms are functions between sets.

In the category $\mathbf{Vect}_k$ the objects are vector spaces over a field k and the morphisms are linear maps.

In the context of chemical reaction networks one may define a category **RxnNet** in which objects represent configurations of chemical species and morphisms represent stoichiometric transformations induced by reactions.

I.2 Functors and Structural Preservation

Mappings between categories are described by functors. A **functor**

$$F : \mathcal{C} \to \mathcal{D}$$

assigns to each object $A \in \mathcal{C}$ an object $F(A) \in \mathcal{D}$ and to each morphism $f : A \to B$ a morphism

$$F(f) : F(A) \to F(B).$$

This assignment preserves the categorical structure. In particular, functors satisfy the identities

$$F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$$

and

$$F(g \circ f) = F(g) \circ F(f).$$

Thus functors transport the compositional structure of one category into another.

Within constraint-based systems, functors provide a natural mechanism for mapping between different levels of description. For example, a functor may map a physical system to a reduced model representation while preserving the relationships between states.

I.3 Symmetric Monoidal Categories

Many systems involve the simultaneous composition of independent components. This structure is captured by the notion of a **symmetric monoidal category**.

A symmetric monoidal category is a category $\mathcal{C}$ equipped with a bifunctor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

called the tensor product, together with a distinguished unit object I.

The tensor product represents parallel composition. If A and B are objects, the object $A \otimes B$ represents the joint system obtained by combining them.

The monoidal structure satisfies associativity and unit conditions up to natural isomorphism. In particular,

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$

and

$$A \otimes I \cong A.$$

A symmetry isomorphism

$$\sigma_{A,B} : A \otimes B \to B \otimes A$$

ensures that the order of parallel composition is interchangeable.

I.4 Constraint Systems as Monoidal Structures

Constraint systems can be interpreted naturally within this categorical framework.

Objects represent semantic modules, state spaces, or constraint structures. Morphisms represent consistency–preserving transformations between these structures. Sequential composition corresponds to the composition of morphisms, while parallel composition of independent modules is represented by the tensor product.

This interpretation allows complex systems to be assembled from simpler components while maintaining explicit structural relationships.

I.5 Colimits and Merging of Systems

In many contexts one wishes to combine several compatible structures into a single unified object. Category theory formalizes this operation using colimits.

Let $D : \mathcal{J} \to \mathcal{C}$ be a diagram in a category $\mathcal{C}$. A **colimit** of this diagram is an object $\operatorname{colim} D$ together with morphisms

$$\iota_j : D(j) \to \operatorname{colim} D$$

such that the resulting family of maps is universal.

Universality means that for any object X receiving compatible maps

$$f_j : D(j) \to X$$

there exists a unique morphism

$$u : \operatorname{colim} D \to X$$

satisfying

$$f_j = u \circ \iota_j.$$

In the context of constraint systems, colimits describe the operation of merging multiple modules or subsystems that share compatible boundary conditions. The resulting object represents the minimal structure that integrates all participating components while preserving their relationships.

Thus categorical constructions provide a rigorous mathematical language for describing how constraint systems combine, interact, and evolve within larger structured environments.

Appendix J

Sheaf-Theoretic Gluing and Local-to-Global Consistency

J.1 Presheaves

Let X be a topological space, interpreted as a *context space* whose open sets represent regions of local observation or computation. Denote by $\mathrm{Open}(X)$ the partially ordered set of open subsets of X, ordered by inclusion.

A **presheaf** $\mathcal{F}$ on X assigns to each open set $U \subseteq X$ a set $\mathcal{F}(U)$, called the set of *sections* over U. Elements of $\mathcal{F}(U)$ represent pieces of information defined locally on the region U.

For every inclusion of open sets $V \subseteq U$ there is a **restriction map**

$$\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V).$$

These maps must satisfy the compatibility conditions

$$\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$$

and

$$\rho_{UW} = \rho_{VW} \circ \rho_{UV} \qquad \text{whenever } W \subseteq V \subseteq U.$$

Thus restricting a section in multiple steps produces the same result as restricting it directly.

Equivalently, a presheaf may be defined as a contravariant functor

$$\mathcal{F} : \mathrm{Open}(X)^{\mathrm{op}} \to \mathbf{Set}.$$

This formulation emphasizes that presheaves encode how local data vary over a topological space.

J.2 Sheaf Condition

A presheaf becomes a **sheaf** when local data determine global data in a consistent way.

Let $\{U_i\}_{i \in I}$ be an open cover of a region

$$U = \bigcup_{i \in I} U_i.$$

Suppose we are given sections

$$s_i \in \mathcal{F}(U_i)$$

for each open set in the cover.

These sections are said to be **compatible** if they agree on all pairwise overlaps. Formally this means

$$\rho_{U_i, U_i \cap U_j}(s_i) = \rho_{U_j, U_i \cap U_j}(s_j) \qquad \text{for all } i, j.$$

A presheaf $\mathcal{F}$ is a **sheaf** if the following two conditions hold.

First, whenever compatible sections $\{s_i\}$ are given on an open cover, there exists a section

$$s \in \mathcal{F}(U)$$

such that

$$\rho_{U, U_i}(s) = s_i \qquad \text{for all } i.$$

Second, this global section is unique.

Thus a sheaf ensures that locally consistent pieces of information can be uniquely glued together into a global description.

J.3 Gluing and Consistency

The sheaf condition formalizes a fundamental principle of many scientific and mathematical systems: global structure emerges from locally compatible data.

Local observations or computations often describe only a portion of a system. When these pieces agree on their regions of overlap, they can be combined to produce a consistent global model.

Mathematically, the gluing operation is represented by the existence of the section s satisfying

$$\rho_{U, U_i}(s) = s_i.$$

The compatibility condition ensures that no contradictions arise on overlapping regions.

J.4 Applications to Constraint Systems

Sheaf structures provide a natural framework for modeling systems in which information is distributed across overlapping contexts.

In geometric problems, sections may represent functions defined on local coordinate patches. In physical field theories, sections may represent local solutions of differential equations. In

networked systems, sections may encode local constraint solutions associated with subregions of the system.

The sheaf condition guarantees that if these local solutions agree where their domains overlap, they determine a unique global solution.

J.5 Local-to-Global Reasoning

Many arguments throughout mathematics and science implicitly follow the local-to-global paradigm captured by sheaf theory.

Local measurements constrain global physical models. Local geometric relations determine global shapes. Local conservation laws determine global system behavior. Even in elementary contexts such as triangle geometry or stoichiometric balance, reasoning often proceeds by verifying consistency in small parts and then assembling the result into a coherent whole.

Guiding Principle

Sheaf coherence provides the formal mathematical expression of local-to-global consistency. When local structures agree on their overlaps, they uniquely determine a global structure. This principle underlies a wide range of scientific reasoning, from geometric construction and field theory to network dynamics and constraint systems.

Appendix K

Variational Principles and Entropic Descent

K.1 Variational Functionals

Many mathematical and physical systems can be described by a functional whose extrema determine the governing equations of the system.

Let $\Omega \subset \mathbb{R}^n$ be a spatial domain and let $u : \Omega \to \mathbb{R}$ be a sufficiently smooth function. Consider the functional

$$\mathcal{J}[u] = \int_\Omega \mathcal{L}(u, \nabla u, x)\, dx,$$

where $\mathcal{L}$ is called the **Lagrangian density**. The function $\mathcal{L}$ depends on the value of the field u, its gradient ∇u, and possibly explicitly on the spatial coordinate x.

The central problem of the calculus of variations is to determine the functions u that make the functional $\mathcal{J}$ stationary.

K.2 First Variation and Euler–Lagrange Equation

To derive the governing equation, consider a perturbation of the field

$$u_\epsilon(x) = u(x) + \epsilon\, \eta(x),$$

where η is an arbitrary smooth test function vanishing on the boundary of Ω.

The variation of the functional is

$$\frac{d}{d\epsilon}\mathcal{J}[u_\epsilon] = \int_\Omega \left(\frac{\partial \mathcal{L}}{\partial u}\, \eta + \frac{\partial \mathcal{L}}{\partial(\nabla u)} \cdot \nabla \eta \right) dx.$$

Applying integration by parts to the second term yields

$$\int_\Omega \frac{\partial \mathcal{L}}{\partial(\nabla u)} \cdot \nabla \eta\, dx = -\int_\Omega \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial(\nabla u)} \right) \eta\, dx,$$

where boundary terms vanish because $\eta = 0$ on $\partial\Omega$.

Thus the first variation becomes

$$\delta\mathcal{J} = \int_\Omega \left(\frac{\partial \mathcal{L}}{\partial u} - \nabla \cdot \frac{\partial \mathcal{L}}{\partial(\nabla u)} \right) \eta \, dx.$$

Since η is arbitrary, the integrand must vanish. Therefore critical points of $\mathcal{J}$ satisfy the **Euler–Lagrange equation**

$$\frac{\partial \mathcal{L}}{\partial u} - \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial(\nabla u)} \right) = 0.$$

This equation determines the stationary configurations of the functional.

K.3 Gradient Flow Dynamics

In many systems the dynamics evolve toward a configuration that minimizes a functional rather than merely satisfying a stationary condition.

Let $\mathcal{J}[u]$ be a functional defined on a suitable function space. The steepest descent dynamics associated with $\mathcal{J}$ are given by the **gradient flow equation**

$$\partial_t u = -\frac{\delta \mathcal{J}}{\delta u},$$

where $\frac{\delta \mathcal{J}}{\delta u}$ denotes the functional derivative.

The functional derivative is defined by the relation

$$\delta\mathcal{J} = \int_\Omega \frac{\delta \mathcal{J}}{\delta u} \, \eta \, dx.$$

Substituting the expression obtained above gives

$$\frac{\delta \mathcal{J}}{\delta u} = \frac{\partial \mathcal{L}}{\partial u} - \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial(\nabla u)} \right).$$

Thus the gradient flow equation becomes

$$\partial_t u = -\left(\frac{\partial \mathcal{L}}{\partial u} - \nabla \cdot \frac{\partial \mathcal{L}}{\partial(\nabla u)} \right).$$

This evolution decreases the value of the functional over time.

Indeed,

$$\frac{d}{dt}\mathcal{J}[u(t)] = \int_\Omega \frac{\delta \mathcal{J}}{\delta u} \, \partial_t u \, dx = -\int_\Omega \left| \frac{\delta \mathcal{J}}{\delta u} \right|^2 dx \leq 0.$$

Therefore $\mathcal{J}$ acts as a Lyapunov functional for the dynamics.

K.4 Examples Across Domains

Variational principles appear in many different mathematical and scientific contexts.

In linear algebra the least–squares problem

$$\mathcal{J}(\mathbf{x}) = \|A\mathbf{x} - \mathbf{b}\|^2$$

defines a quadratic functional whose minimizer satisfies the normal equations

$$A^\top A\mathbf{x} = A^\top \mathbf{b}.$$

In geometry the length of a curve

$$\mathcal{J}[\gamma] = \int |\dot{\gamma}(t)|\, dt$$

is minimized by geodesics.

In chemical systems free–energy functionals determine equilibrium states of reaction networks.

In the RSVP framework the plenum fields evolve toward configurations that minimize an entropic free–energy functional, producing the smoothing dynamics discussed in earlier chapters.

K.5 Unified Interpretation

These examples illustrate a general principle. Many physical, geometric, and algebraic processes can be interpreted as gradient flows on appropriately defined functionals.

Algebraic residual minimization, geometric shortest paths, chemical equilibration, and entropic smoothing all arise from the same variational structure. The particular functional $\mathcal{J}$ encodes the constraints and energies relevant to the system, while the gradient flow equation determines how the system evolves toward its optimal configuration.

Appendix L

Worked Derivations: Bridges Between the Introductory Subjects and the Theory

L.1 From Linear Algebra to Stoichiometry to Field Theory

The conceptual thread connecting linear algebra, chemical stoichiometry, and field theory is the mathematical structure of conservation laws. Each system expresses the requirement that certain quantities remain balanced under allowed transformations.

Linear Constraint Formulation

Consider a chemical reaction network involving a finite set of chemical species. Let $\gamma \in \mathbb{R}^r$ denote the vector of reaction coefficients and let

$$S \in \mathbb{Z}^{e \times r}$$

be the **stoichiometric matrix**, where e is the number of elements and r the number of species.

Each entry S_{ij} records the number of atoms of element i contained in species j.

Balancing a chemical reaction requires that the number of atoms of each element be conserved. This condition is expressed as the linear system

$$S\gamma = \mathbf{0}.$$

As a simple example consider the reaction

$$2\mathrm{H}_2 + \mathrm{O}_2 \to 2\mathrm{H}_2\mathrm{O}.$$

The stoichiometric matrix describing hydrogen and oxygen conservation may be written

$$S = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & -1 \end{pmatrix}.$$

The coefficient vector

$$\gamma = (1,\, \tfrac{1}{2},\, 1)$$

lies in the null space of S. Multiplying by two yields the familiar balanced integer equation

$$2H_2 + O_2 \to 2H_2O.$$

Thus chemical balancing is equivalent to computing a basis of the null space $\ker(S)$.

Dynamic Stoichiometry

The static conservation relation generalizes naturally to a dynamical system.

Let

$$\mathbf{c}(t) = (c_1(t), \ldots, c_r(t))$$

denote the vector of species concentrations and let

$$\mathbf{r}(\mathbf{c})$$

be the vector of reaction rates.

The evolution of concentrations is governed by

$$\dot{\mathbf{c}} = S\,\mathbf{r}(\mathbf{c}).$$

The columns of S determine how each reaction changes the concentrations of all species simultaneously. The kernel of S contains the conserved quantities of the reaction network.

Continuum Limit

A continuum description emerges when the discrete species concentrations are replaced by spatial density fields.

Let $\rho(x,t)$ represent the density of a conserved quantity in a spatial domain $\Omega \subset \mathbb{R}^3$. The total amount of this quantity in a region $U \subset \Omega$ is

$$\int_U \rho(x,t)\,dx.$$

Conservation requires that the rate of change of this integral equals the net flux across the boundary ∂U.

Applying the divergence theorem yields the **continuity equation**

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0,$$

where $\mathbf{v}(x,t)$ is the velocity field transporting the quantity.

Thus the same conservation structure appears in three progressively more general settings:

$$S\gamma = \mathbf{0}, \qquad \dot{\mathbf{c}} = S\mathbf{r}(\mathbf{c}), \qquad \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0.$$

The first describes algebraic balance, the second describes reaction dynamics, and the third describes continuous transport fields.

L.2 From Euclidean Distance to the L^2 Metric

Geometry provides another example of how elementary concepts extend naturally to more advanced mathematical structures.

Euclidean Distance

In finite-dimensional space $\mathbb{R}^n$, the distance between two points $P, Q \in \mathbb{R}^n$ is defined by the Euclidean norm

$$d(P,Q) = \|P - Q\| = \left(\sum_{i=1}^{n} (P_i - Q_i)^2 \right)^{1/2}.$$

This definition arises from the inner product

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i.$$

Extension to Function Spaces

When the objects of interest are functions rather than vectors, the inner product generalizes to an integral.

For functions $f, g : \Omega \to \mathbb{R}$ one defines the L^2 inner product

$$\langle f, g \rangle_{L^2} = \int_\Omega f(x) g(x)\, dx.$$

The associated norm is

$$\|f\|_{L^2} = \left(\int_\Omega |f(x)|^2 dx \right)^{1/2}.$$

The distance between two functions therefore becomes

$$d(f,g) = \|f - g\|_{L^2} = \left(\int_\Omega |f(x) - g(x)|^2 dx \right)^{1/2}.$$

Application to Field Configurations

In the RSVP framework, a system state is represented by a collection of fields

$$X = (\Phi, \mathbf{v}, S).$$

The natural metric on the configuration space is the L^2 distance between fields:

$$d(X_1, X_2)^2 = \int_\Omega \left(|\Phi_1 - \Phi_2|^2 + |\mathbf{v}_1 - \mathbf{v}_2|^2 + |S_1 - S_2|^2 \right) dx.$$

This metric measures the magnitude of the difference between two possible states of the system.

L.3 From Trigonometric Oscillation to RSVP Eigenmodes

Oscillatory solutions play a central role in both classical mechanics and field theory.

Simple Harmonic Motion

The differential equation

$$\ddot{x} + \omega^2 x = 0$$

describes a harmonic oscillator.

Its general solution is

$$x(t) = A\cos(\omega t) + B\sin(\omega t).$$

This equation arises whenever a restoring force is proportional to displacement.

Fourier Modes

Many partial differential equations can be analyzed by decomposing solutions into Fourier modes.

Consider a perturbation

$$\delta\Phi(x,t).$$

A Fourier mode has the form

$$\delta\Phi(x,t) = A\, e^{i(\mathbf{k}\cdot x - \omega t)}.$$

Substituting such modes into a linear partial differential equation reduces the problem to an algebraic relation between ω and $\mathbf{k}$.

This relation is called the **dispersion relation**.

Application to RSVP Fields

Linearizing the RSVP equations around a homogeneous background state produces equations of the form

$$\partial_t^2(\delta\Phi) + \omega^2(\mathbf{k})\,\delta\Phi = 0$$

for each spatial Fourier mode.

Thus each perturbation mode behaves as a harmonic oscillator with frequency $\omega(\mathbf{k})$ determined by the dispersion relation.

The spectrum of these oscillatory modes determines how disturbances propagate through the plenum field.

L.4 Unified Perspective

These derivations illustrate how elementary mathematical concepts extend naturally into the structures used in advanced theoretical models.

Linear algebra describes static balance conditions. Reaction network dynamics extend these balances into time-dependent systems. Continuum field equations represent the spatial limit of these same conservation principles.

Similarly, Euclidean distance generalizes to metrics on function spaces, and elementary trigonometric oscillations become the Fourier modes that describe waves and field propagation.

The mathematical structures introduced in the early chapters therefore form the conceptual foundation for the more advanced theories developed later in the text.

Glossary

Amplitude
maximum displacement of a periodic function from its midline.

Avogadro's number
$N_A = 6.022 \times 10^{23}$; particles per mole.

Category
a mathematical structure consisting of objects and morphisms with associative composition and identity maps.

Configuration space
the set of all possible states of a system.

Constraint
a condition restricting admissible values of variables.

Continuity equation
conservation law $\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$ describing transport of density.

Conservation law
a relation asserting that a quantity is preserved under transformation.

Dispersion relation
relation $\omega(\mathbf{k})$ connecting wave frequency and spatial wave number.

Dimensional analysis
tracking units through calculations.

Equilibrium
a fixed point of a dynamical system.

Event-history
a causally ordered sequence of occurrences; primitive in the Spherepop formalism.

Fourier mode
sinusoidal component $e^{i(\mathbf{k}\cdot x - \omega t)}$ used to analyze linear systems.

Function
a rule assigning exactly one output to each input.

Functional derivative
derivative of a functional with respect to a function.

Functor
a structure-preserving map between categories.

Gradient
vector of partial derivatives indicating direction of greatest increase.

Gradient flow
dynamics $\partial_t u = -\delta \mathcal{J}/\delta u$ descending a functional.

Identity (trig.)
an equation true for all valid angles.

Inner product
bilinear operation defining angles and lengths in vector spaces.

Invariant
a quantity unchanged by a transformation.

Isometry
a transformation preserving distances.

Lagrangian
function $\mathcal{L}(u, \nabla u, x)$ whose integral defines a variational functional.

Limiting reagent
the reactant consumed first in a reaction.

Linear system
a system of equations of the form $A\mathbf{x} = \mathbf{b}$.

Lyapunov functional
scalar functional that decreases along system trajectories.

Metric
rule defining distances between elements of a space.

Molar mass
mass per mole of a substance (g/mol).

Mole
SI unit of chemical amount.

Morphism
structure-preserving map between objects in a category.

Null space
$\ker(A) = \{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}$.

Period
smallest positive interval of repetition of a periodic function.

Pythagorean theorem
$a^2 + b^2 = c^2$ for a right triangle.

Radian
angle subtended by arc of length equal to radius.

RSVP
Relativistic Scalar-Vector Plenum field theory.

Sheaf
a presheaf satisfying the gluing axiom for local data.

Similarity
geometric relation in which figures differ only by scale.

Slope
rate of change of a linear function.

Stoichiometric matrix
element-species incidence matrix S encoding conservation.

TARTAN
Recursive Tiling and Annotated Noise simulation framework.

Unit circle
circle of radius 1; foundation of trigonometric definitions.

Variable
a letter representing an unknown or varying quantity.

Variational functional
a function on function spaces, $\mathcal{J}[u]$.

Selected Answers

Chapter 2 (Equations and Balance). Practice 2.1: (a) $x = 5$; (b) $x = 15$; (c) $x = 5$; (d) $x = 7$.

Chapter 7 (Pythagorean Theorem). Practice 7.1: diagonal $= \sqrt{49 + 576} = 25$ cm.

Chapter 10 (Atoms and Moles). Practice 10.1: (a) 58.44; (b) 44.01; (c) 180.16 g/mol.

Chapter 17 (Balancing). Practice 17.1: (a) $4\text{Fe} + 3\text{O}_2 \rightarrow 2\text{Fe}_2\text{O}_3$; (b) $\text{C}_3\text{H}_8 + 5\text{O}_2 \rightarrow 3\text{CO}_2 + 4\text{H}_2\text{O}$.

Chapter 18 (Moles). Practice 18.1: ≈ 33.6 g CO_2.

Abbreviated Periodic Table

Symbol	**Name**	Z	M **(g/mol)**
H	Hydrogen	1	1.008
C	Carbon	6	12.011
N	Nitrogen	7	14.007
O	Oxygen	8	15.999
Na	Sodium	11	22.990
Mg	Magnesium	12	24.305
S	Sulfur	16	32.060
Cl	Chlorine	17	35.453
K	Potassium	19	39.098
Ca	Calcium	20	40.078
Fe	Iron	26	55.845
Cu	Copper	29	63.546

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