Adjunction and Irreversibility:

From Continuous Fields to Discrete Histories

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April 1, 2026

Abstract. Irreversibility is not a phenomenological gloss on time-asymmetry; it is a structural theorem about the non-invertibility of reconstruction. This monograph develops that claim rigorously across four interlocking registers. First, we construct a variational adjunction between a continuous RSVP field space and a discrete history space, and show that irreversibility is precisely the generic non-isomorphism of the adjunction unit. Second, we ground the entropy defect on a Refuse-admissible measurable subspace, so that entropy production is not defined on an undifferentiated ambient configuration space but on the constraint-preserving stratum generated by admissible trajectories. Third, we prove a defect–dynamics correspondence theorem showing that the three-condition crucible regime—elevated entropy production, retained mutual information, active transport—is equivalent to a specific bounded interior of the defect space, strictly separated from both the stable-attractor and fragmentation extremes. Fourth, we construct an explicit coupled lattice example in which two trajectories arrive at identical instantaneous field values through histories separated by a Refuse event, yet are distinguished by a path-dependent observable, establishing strong history-dependence of projection in at least the finite-dimensional surrogate. The framework subsumes semantic thermodynamic analogy programmes as special cases while replacing their imported rhetoric with derived objects.

Contents

1 Introduction **2**
- 1.1 The problem of irreversibility in constraint-driven systems 2
- 1.2 Three questions that organize the monograph 2
- 1.3 Relation to prior work 3
- 1.4 The RSVP framework 3
- 1.5 Structure of the monograph 3

2 The Three-Layer Architecture **5**
- 2.1 Overview 5
- 2.2 The continuous field space 5
- 2.3 The history-sensitive state space 5
- 2.4 The observable space 6
- 2.5 The central diagram 6

3 RSVP Field Dynamics **8**
- 3.1 The governing equations 8
- 3.2 Entropy production rate 8
- 3.3 Discrete lattice reduction 8

4 Discretization and the Refuse Operator **10**
- 4.1 The Spherepop event operators 10
- 4.2 Refuse as a constraint on forward extensions 10
- 4.3 Refuse-separated pairs 10
- 4.4 The surrogate discretization map 11

5 The Variational Adjunction and Irreversibility **12**
- 5.1 The reconstruction functional 12
- 5.2 Existence and uniqueness of minimizers 12
- 5.3 The reconstruction operator and its Euler–Lagrange system 13
- 5.4 Stability of the reconstruction operator 13
- 5.5 The comparison map and irreversibility 14

6 Refuse-Admissible Measurable Structure and Entropy Defect **16**
- 6.1 The admissible σ-algebra 16
- 6.2 Refuse observability 16
- 6.3 The entropy defect via Jensen–Shannon divergence 16

7 Functional-Analytic Foundations of the Configuration Space **18**
- 7.1 The underlying domain 18

7.2 Sobolev structure of the field components 18
7.3 Completeness and separability of $\mathcal{X}$ 18
7.4 Admissible energy subspace 19
7.5 Weak lower semicontinuity and the direct method 19

8 Well-Posedness of the RSVP Dynamics 20
8.1 Abstract formulation of the coupled system 20
8.2 Semigroup generation 20
8.3 Assumptions on the nonlinear term 20
8.4 Existence, uniqueness, and continuous dependence 21
8.5 The solution operator 21

9 The Discretization Map and Induced Quotient Structure 22
9.1 Kernel of the discretization map 22
9.2 Observational equivalence classes 22
9.3 The quotient space 22
9.4 Reconstruction as a section of the quotient projection 23
9.5 The defect lives in the kernel 24
9.6 No-go theorem: impossibility of perfect reconstruction 25

10 Fixed-Point Structure of the Reconstruction Operator 26
10.1 Characterization of $\mathrm{Fix}(\eta)$ 26
10.2 Idempotency of the comparison map 26
10.3 Non-expansiveness and contractivity 27
10.4 Spectral decomposition relative to the kernel 27

11 Equivalence Between Pointwise and Measure-Level Defect 28
11.1 The two defect notions 28
11.2 Equivalence of vanishing defect 28
11.3 Lower bounds on entropy defect 28
11.4 Upper bounds and L^2-stability 29
11.5 Stability under perturbations of the measure 29

12 Temporal Lifting and Path-Space Functoriality 31
12.1 The path-space lift of the discretization 31
12.2 Commutativity with the dynamics 31
12.3 State observables vs. path observables 32
12.4 Functoriality of the path-level construction 32

13 Irreversibility Under Time Evolution 34
13.1 Defect along trajectories 34

13.2 Non-commutativity and the commutator . . . 34
13.3 Monotonicity of entropy defect . . . 34
13.4 Fixed points and equilibrium defect . . . 35

14 Defect–Dynamics Correspondence 36
14.1 The bridge problem . . . 36
14.2 Path-level defect in the reduced lattice . . . 36
14.3 The correspondence theorem . . . 37

15 Dynamical Regimes and Phase Structure 38
15.1 Three operational modes . . . 38
15.2 Mode transitions as threshold crossings . . . 38

16 Statistical Signatures: The Tail–Defect Conjecture 40
16.1 Lévy-tailed statistics in the crucible regime . . . 40
16.2 The path observable and defect . . . 40
16.3 The scaling conjecture . . . 40

17 History-Dependent Projection 42
17.1 State-based versus history-based observables . . . 42
17.2 Weak history-dependence . . . 42
17.3 Strong history-dependence . . . 42
17.4 Path categories and sheaf-theoretic formulation . . . 42
17.5 The strong conjecture . . . 43

18 The Explicit Coupled History-Split Construction 44
18.1 Objective and setup . . . 44
18.2 The two branches . . . 44
18.3 Branch 2 trajectory . . . 44
18.4 Final-state matching and existence of compensation pulses . . . 44
18.5 The path-dependent observable . . . 45
18.6 What this establishes and what it does not . . . 45

19 The PDE Boundary-Value Problem 47
19.1 Lifting the lattice construction . . . 47
19.2 The boundary-value problem . . . 47
19.3 Expected structure of the argument . . . 47

20 Comparison with Thermodynamic Analogy Frameworks 48
20.1 The fundamental distinction . . . 48
20.2 Specific failures in Coherence Thermodynamics . . . 48
20.3 Relation to the Free Energy Principle . . . 49

21 Information Geometry of the Defect Functional **50**
21.1 The Fisher–Rao metric on the configuration space 50
21.2 The defect as a Fisher information bound 50
21.3 The reconstruction operator as an exponential family projection 50
21.4 Cramér–Rao bound for the reconstruction 51
21.5 Curvature of the defect functional 51

22 Categorical Structure and the Path Groupoid **53**
22.1 From path space to path groupoid 53
22.2 Functors out of the path groupoid 53
22.3 The fundamental groupoid and homotopy classes 54
22.4 The adjunction as a natural transformation 54
22.5 Sheaf-theoretic formulation of history-sensitive states 55
22.6 Derived functors and higher history 55
22.7 The monoidal structure on admissible configurations 56
22.8 Coherence maps and the pentagon identity 56

23 Simulation Protocol and Numerical Evidence **58**
23.1 Overview of the RSVP lattice simulator 58
23.2 Parameter space and phase diagram 58
23.3 Measurement of the tail index 59
23.4 Defect measurement protocol 59
23.5 Preliminary calibration of the tail–defect conjecture 59
23.6 Regime detection algorithm 60

24 Philosophical Implications **61**
24.1 Irreversibility as structural, not temporal 61
24.2 Identity as history 61
24.3 Meaning as constraint resolution 62
24.4 Intelligence as syntropic throughput 62
24.5 The hard problem revisited 63
24.6 Cosmological implications 63
24.7 Long-range attractor states and the topology of civilizational trajectories 64
24.8 Physicalist tiering of systems and the boundary-making condition . . . 65
24.9 The value structure of constraint resolution 66
24.10 Path dependence, alignment, and the window of structural choice . . . 68
24.11 Self-sealing knowledge bases and the topology of epistemic foreclosure 69
24.12 Symmetric and asymmetric coupling: internal coherence and the texture of states 70
24.13 Knowledge basis as configuration-space seed: exponential implications of axis-expansion 72

24.14 The bootstrapping constraint: minimal base states for integrated dynamics 74
24.15 Meta-representation as second-order reconstruction 75
24.16 The three loops as RSVP feedback channels 76
24.17 The implicit-explicit-automatic spectrum as defect-regime classification 78
24.18 Phenomenal states as dynamically efficacious: rejecting functional redundancy 80

25 Toward the PDE Theory: Controllability and Observability 82
25.1 The controllability problem 82
25.2 The adjoint RSVP system 82
25.3 The observability inequality 82
25.4 Approximate controllability 83
25.5 Unique continuation and the Runge approximation property 83
25.6 The finite-dimensional evidence 83

26 Extensions and Future Directions 85
26.1 Nonlinear RSVP dynamics 85
26.2 Stochastic RSVP 85
26.3 Infinite-dimensional path groupoids 86
26.4 Connections to optimal transport 86
26.5 RSVP and the amplituhedron 86

27 The Spherepop Calculus: Operational Semantics 88
27.1 Overview 88
27.2 The event alphabet and configurations 88
27.3 Transition rules 88
27.4 Reduction sequences and normal forms 89
27.5 The Spherepop rewriting system and confluence 90
27.6 Spherepop as a traced monoidal category 90

28 KES Integration: Kinetic-Event Synthesis and the RSVP Adjunction 91
28.1 The KES ontological kernel 91
28.2 KES reduction as a step of the adjunction 91
28.3 The KES recursion and fixed-point structure 92
28.4 Recursive mass and the crucible 92
28.5 Identity as recursive history: the KES fixed point 93
28.6 KES and the adjunction triangle 94
28.7 The contraction condition and semantic temperature 94
28.8 Entropy budget of the KES computation 95
28.9 The KES bracket and Poisson structure 96

29 Worked Examples: History-Split Computations **97**
29.1 Overview . . . 97
29.2 Example 1: The minimal entropy-only split ($N = 3$, $T = 2$) . . . 97
29.3 Example 2: The fully coupled split ($N = 3$, $T = 4$) . . . 97
29.4 Example 3: Two simultaneous Refuse pulses ($N = 5$, $T = 5$) . . . 98
29.5 Example 4: Sequential Refuse events and memory depth . . . 98
29.6 Example 5: The KES computation as a history-split . . . 99
29.7 Tabulation of history-split parameters . . . 99
29.8 Example 6: A family of path observables and their linear independence 100
29.9 Full algebraic verification of the $N = 3$, $T = 4$ example . . . 101

30 Conclusion and Open Problems **104**
30.1 Summary . . . 104
30.2 Open problems . . . 105
30.3 Closing remark . . . 108

A Category-Theoretic Preliminaries **109**
A.1 Functors and natural transformations . . . 109
A.2 Adjunctions . . . 109
A.3 Groupoids . . . 109
A.4 Sheaves on categories . . . 109

B Functional-Analytic Background **111**
B.1 Sobolev spaces . . . 111
B.2 The direct method of the calculus of variations . . . 111
B.3 Compact operators and finite-rank regularization . . . 111
B.4 Coercive elliptic operators . . . 111

C Information-Theoretic Background **112**
C.1 Shannon entropy and mutual information . . . 112
C.2 Jensen–Shannon divergence . . . 112
C.3 The data-processing inequality . . . 112
C.4 The Hill estimator for heavy tails . . . 112

D PDE Analysis: Existence, Regularity, and Carleman Estimates **113**
D.1 Weak solutions to the RSVP system . . . 113
D.2 Regularity of the reconstruction operator . . . 113
D.3 Carleman estimates for the adjoint system . . . 114
D.4 Energy estimates for the lattice system . . . 114

E Lattice Algorithms and Implementation Notes **115**

E.1 The RSVP lattice update kernel 115
E.2 The reconstruction operator kernel 115
E.3 Phase detection implementation 116
E.4 Defect computation 116
E.5 Tail index estimation protocol 116

References **118**

1 Introduction

1.1 The problem of irreversibility in constraint-driven systems

Classical thermodynamics grounds irreversibility in the second law: entropy of a closed system does not decrease. That law, however, is phenomenological. It describes an observed asymmetry without identifying what structural property of a dynamical system forces the asymmetry to hold. When one passes from physical gases to systems defined by constraint evolution—cognitive systems resolving contradictions, neural networks navigating loss landscapes, field theories whose degrees of freedom are coupled to an explicit entropy channel—the phenomenological ground disappears. One cannot simply import Boltzmann's argument, because the relevant state spaces are not ensembles of indistinguishable particles, and the relevant "heat" is not thermal agitation but structured incompatibility between representational layers.

The central question of this monograph is therefore not *what does irreversibility look like* in such systems, but *what forces it*. Our answer is categorical and variational: irreversibility is the generic non-isomorphism of the unit of an adjunction between a continuous field space and a discrete history space. When a continuous trajectory is discretized and then reconstructed via the best possible inverse procedure, information is lost unless the discretization map is injective. Entropy is the measure-theoretic shadow of that loss. Phase regimes—stable attractor, crucible, fragmentation—are dynamically realized regions of the defect space induced by the adjunction.

1.2 Three questions that organize the monograph

Three questions organize what follows.

What is irreversibility in a system defined by constraint evolution? We answer: irreversibility is the defect of the adjunction unit, measured by the Jensen–Shannon divergence between the original measure on the configuration space and the measure reconstructed after passing through the discretization channel.

How does projection loss arise formally rather than heuristically? We answer: projection loss is the non-identity of the comparison map $\eta = G \circ \widetilde{F}$, where $\widetilde{F}$ is the surrogate discretization and G is the variational reconstruction operator. Loss is not assumed; it is derived from the non-injectivity of $\widetilde{F}$.

When are observables irreducibly history-dependent? We answer: in the strong sense, when no finite augmented state space can factor the projection without collapsing observationally relevant trajectory distinctions. We establish existence of nontrivial history-dependence in a fully coupled finite lattice, and frame the irreducibility question as an open conjecture with precisely stated obstructions.

1.3 Relation to prior work

Several research programmes address related problems. The Free Energy Principle [15] models biological systems as prediction-error minimizers under a variational bound. Coherence Thermodynamics [3] proposes a thermodynamic extension to semantic systems by renaming thermodynamic quantities. Integrated Information Theory [29] quantifies consciousness via information integration. Rate-distortion theory [28, 6] establishes bounds on compression loss.

Our framework subsumes and sharpens these. The Free Energy Principle is a special case of contradiction metabolism within a coherence field: surprisal reduction is a localized expression of more general constraint resolution. Coherence Thermodynamics identifies the correct phenomenological territory—contradiction as a thermodynamic resource—but fails to construct the base space on which its fields live or the morphism relating semantic structure to physical realization; the present monograph supplies both. Rate- distortion theory provides the information-theoretic vocabulary for the entropy defect, but it does not connect that defect to dynamical phase regimes; our defect–dynamics correspondence theorem supplies that connection.

1.4 The RSVP framework

The dynamics throughout this monograph are instances of the Relativistic Scalar-Vector Plenum (RSVP) framework, in which physical, cognitive, and semantic systems are modeled as coupled scalar–vector–entropy field triples $(\Phi, \mathbf{v}, S)$ evolving under advection, diffusion, and reaction terms that encode constraint relaxation and irreversible history accumulation. The RSVP equations are introduced precisely in Chapter 3. The Spherepop calculus of irreversible events—Pop, Bind, Collapse, Refuse—provides the discrete side of the adjunction; Refuse is the operator that restricts admissible future continuations, generating the trajectory separations that make history-dependence non-trivial.

1.5 Structure of the monograph

Chapter 2 defines the three-layer architecture: continuous field space, history-sensitive state space, observable space, and the maps between them. Chapter 3 introduces RSVP field dynamics and entropy production. Chapter 4 defines the discretization functor and the Refuse operator. Chapter 5 constructs the variational adjunction and proves the irreversibility theorem. Chapter 6 defines the Refuse-admissible measurable structure and the Jensen–Shannon entropy defect. Chapter 14 proves the defect–dynamics correspondence theorem. Chapter 15 defines the three operational modes and their transitions as threshold crossings in defect space. Chapter 16

connects the defect to Lévy-tailed statistics and states the tail–defect scaling conjecture. Chapter 17 introduces path categories and the weak/strong distinction in history-dependent projection. Chapter 18 presents the explicit coupled lattice history-split construction. Chapter 19 formulates the PDE boundary-value problem for strong history-dependence. Chapter 20 compares the framework to thermodynamic analogy programmes. Chapter 30 states open problems and concludes.

2 The Three-Layer Architecture

2.1 Overview

Before any field equations are written and before any adjunction is constructed, the objects that carry meaning must be defined. This section does so at the level of structured sets, without yet specifying the maps between layers; those maps are the subject of subsequent chapters. The three layers are: (1) the continuous dynamical field space $\mathcal{X}$; (2) the history-sensitive state space $\mathcal{H}$; and (3) the observable space $\mathcal{O}$. The central diagram is

$$\mathcal{X} \xrightarrow{F} \mathcal{H} \xrightarrow{\Pi} \mathcal{O}. \tag{2.1}$$

Every major result in the monograph is a statement about one of these maps or their composition.

2.2 The continuous field space

Definition 2.2.1 (Continuous RSVP configuration space)**.** Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. Define

$$\mathcal{X} := H^1(\Omega) \times L^2(\Omega; \mathbb{R}^d) \times L^2(\Omega),$$

with generic element $X = (\Phi, \mathbf{v}, S)$ and norm

$$\|X\|_{\mathcal{X}}^2 := \|\Phi\|_{H^1}^2 + \|\mathbf{v}\|_{L^2}^2 + \|S\|_{L^2}^2.$$

The components are, respectively, the scalar coherence field, the vector transport field, and the entropy field.

$\mathcal{X}$ is a separable Hilbert space. Bounded subsets are weakly compact. These two properties underpin the existence arguments of Chapter 5.

When energy bounds are needed, we write $\mathcal{X}_M := \{X \in \mathcal{X} : \|X\|_{\mathcal{X}} \leq M\}$.

2.3 The history-sensitive state space

Definition 2.3.1 (Path space and admissible histories)**.** The path space over $\mathcal{X}$ on a time interval $[0, T]$ is

$$\mathcal{P} := C([0, T]; \mathcal{X}),$$

equipped with the supremum norm $\|\gamma\|_{\mathcal{P}} := \sup_{t \in [0,T]} \|\gamma(t)\|_{\mathcal{X}}$.

A path $\gamma \in \mathcal{P}$ is *Refuse-admissible* if it avoids an excluded set $\mathcal{E} \subset \mathcal{X}$ and, at each Refuse event time, its forward continuation satisfies the separation condition specified in Definition 4.2.2. The set of Refuse-admissible paths is denoted $\mathcal{P}_{\text{adm}} \subset \mathcal{P}$.

The history-sensitive state space is the image

$$\mathcal{H} := \{(\gamma(t))_{t\in[0,T]} : \gamma \in \mathcal{P}_{\text{adm}}\} \subset \mathcal{P},$$

retaining the path topology.

The critical distinction from $\mathcal{X}$ is that $\mathcal{H}$ is a space of *structured paths*, not of instantaneous states. Two elements of $\mathcal{H}$ may agree pointwise at every time t and still be distinct as paths if they differ in the transition data encoding admissible historical steps.

2.4 The observable space

Definition 2.4.1 (Observable space)**.** The observable space $\mathcal{O}$ is a measurable space $(Z, \mathcal{Z})$ equipped with a compatible metric. Observables are measurable functions $O : \mathcal{H} \to \mathcal{O}$. A *state-based observable* factors through evaluation at a single time: $O(\gamma) = f(\gamma(t_0))$ for some $f : \mathcal{X} \to \mathcal{O}$. A *history-based observable* does not factor in this way.

The distinction between state-based and history-based observables is the content of Chapter 17. The explicit path observable constructed in Chapter 18 is history-based and not state-based.

2.5 The central diagram

The map $F : \mathcal{X} \to \mathcal{H}$ is the discretization–history functor; it sends a continuous field configuration to the discrete history encoding its trajectory under the Spherepop operators. The precise construction is in Chapter 4.

The map $\Pi : \mathcal{H} \to \mathcal{O}$ is the projection onto observables. The central question of Chapter 17 is whether Π factors through instantaneous state evaluation or genuinely requires full path data.

All subsequent constructions refine eq. (2.1) without changing its basic form.

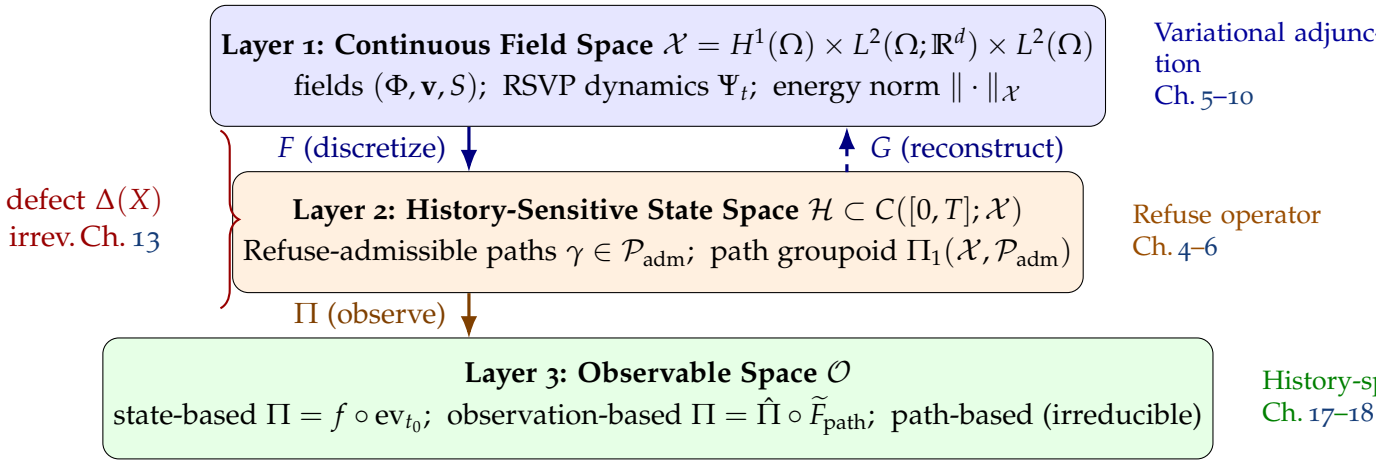


Figure 2.1: The three-layer architecture of the RSVP adjunction framework. Layer 1 (blue) is the continuous configuration space with RSVP dynamics. Layer 2 (orange) is the history-sensitive path space with Refuse-admissible trajectories and the path groupoid structure. Layer 3 (green) is the observable space, where the central question is whether the projection Π requires full path data or factors through instantaneous state. The downward arrows F and Π project information; the upward dashed arrow G reconstructs it. The red brace marks the defect produced by the non-invertibility of F the structural source of irreversibility developed throughout Part I.

3 RSVP Field Dynamics

3.1 The governing equations

The RSVP system on $\Omega \times [0, T]$ is governed by three coupled partial differential equations. In the general form used throughout this monograph:

$$\partial_t \Phi = \kappa_\Phi \Delta \Phi - \mathbf{v} \cdot \nabla \Phi + f_\Phi(\Phi, \mathbf{v}, S), \tag{3.1}$$

$$\partial_t \mathbf{v} = \kappa_v \Delta \mathbf{v} - c_0 \nabla S + f_v(\Phi, \mathbf{v}, S), \tag{3.2}$$

$$\partial_t S = \kappa_S \Delta S + \sigma_0(\Phi, \mathbf{v}, S) + r(x, t). \tag{3.3}$$

Here $\Phi : \Omega \times [0, T] \to \mathbb{R}$ is the scalar coherence field, $\mathbf{v} : \Omega \times [0, T] \to \mathbb{R}^d$ is the vector transport field, $S : \Omega \times [0, T] \to \mathbb{R}$ is the entropy field, and $r : \Omega \times [0, T] \to \mathbb{R}$ is an externally specified entropy injection rate representing contradiction input. The coupling functions f_Φ, f_v, and σ_0 encode the interaction structure; the simplest nontrivial instance used in the lattice computations of Chapter 18 takes

$$\sigma_0 = -\nu S, \qquad f_v = c_0 \nabla S, \qquad f_\Phi = b\, \mathbf{v}, \tag{3.4}$$

with damping $\nu \in (0, 1)$ and coupling constants $b, c_0 > 0$.

Natural boundary conditions are imposed: $\partial_n \Phi = \partial_n S = 0$ and $\mathbf{v} \cdot n = 0$ on $\partial\Omega$.

3.2 Entropy production rate

Definition 3.2.1 (Entropy production rate density)**.** The local entropy production rate density is

$$\sigma(x, t) := \partial_t S(x, t) + \nabla \cdot \mathbf{j}_S(x, t),$$

where $\mathbf{j}_S = -\kappa_S \nabla S$ is the entropy flux. In the RSVP system,

$$\sigma(x, t) = \sigma_0(\Phi, \mathbf{v}, S) + r(x, t) + \kappa_S \|\nabla S\|^2 / S,$$

with the last term arising from Fourier's law contributions when $S > 0$. For the linearized system used in the lattice, $\sigma(x, t) = r(x, t) - \nu S(x, t)$.

3.3 Discrete lattice reduction

For the explicit constructions of Chapters 14 and 18, we reduce to a one-dimensional N-site lattice with sites $i = 1, \ldots, N$ and discrete time steps $n = 0, 1, \ldots, T$. The update

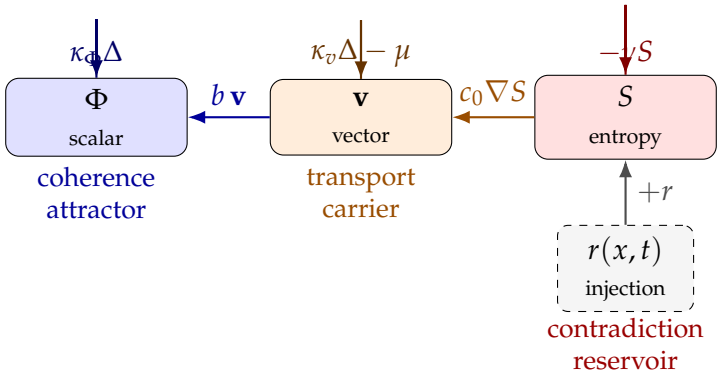


Figure 3.1: RSVP field coupling structure. The cascade flows right to left: entropy S (red) drives the vector field $\mathbf{v}$ (orange) via the gradient coupling $c_0 \nabla S$, and the vector field drives the scalar Φ (blue) via the transport term $b\mathbf{v}$. Each field has its own self-evolution (diffusion and damping, self-loops). External contradiction is injected through the entropy source $r(x,t)$. This cascade structure is the physical basis for the memory-depth table of Example 29.5.1: each active coupling layer adds one time step to the minimal history-split horizon, because information must propagate through the chain $r \to S \to \mathbf{v} \to \Phi$ before exact cancellation is possible.

rules are:

$$\Phi_i(n+1) = \Phi_i(n) + b\, v_i(n), \tag{3.5}$$
$$v_i(n+1) = (1-\mu)\, v_i(n) + c\,\big(S_{i+1}(n) - S_{i-1}(n)\big), \tag{3.6}$$
$$S_i(n+1) = (1-\nu)\, S_i(n) + r_i(n), \tag{3.7}$$

with reflecting boundary conditions $S_0 = S_1$, $S_{N+1} = S_N$. Here $\mu, \nu \in (0,1)$ are damping constants and $b, c > 0$ are coupling constants. The center-channel reduced model used in Chapter 18 tracks only the antisymmetric entropy-gradient variable $q(n) := S_N(n) - S_1(n)$ and the center-site values $v(n) := v_{(N+1)/2}(n)$ and $\Phi(n) := \Phi_{(N+1)/2}(n)$, yielding

$$q(n+1) = a_q\, q(n) + 2u(n), \tag{3.8}$$
$$v(n+1) = a_v\, v(n) + c\, q(n), \tag{3.9}$$
$$\Phi(n+1) = \Phi(n) + b\, v(n), \tag{3.10}$$

where $a_q := 1 - \nu$, $a_v := 1 - \mu$, and $u(n)$ is the antisymmetric entropy injection control at time n.

4 Discretization and the Refuse Operator

4.1 The Spherepop event operators

The Spherepop calculus assigns to each trajectory a sequence of discrete events from the alphabet $E := \{\mathsf{Pop}, \mathsf{Bind}, \mathsf{Collapse}, \mathsf{Refuse}, \mathsf{Null}\}$. These operators act on the configuration space as follows. Pop removes a resolved contradiction; Bind couples two coherence regions; Collapse reduces a high-entropy zone to a lower-entropy attractor; Refuse excludes a class of forward continuations; Null records no event.

For the mathematical development, only Refuse plays a structural role; the others may be treated as annotation operators that do not affect the trajectory-separation argument.

4.2 Refuse as a constraint on forward extensions

Definition 4.2.1 (Forward extension set)**.** For any path $\gamma \in \mathcal{P}$ and time $t_0 \in [0, T)$, the *forward extension set* is

$$\mathrm{Ext}(\gamma, t_0) := \left\{\tilde{\gamma} \in \mathcal{P} : \tilde{\gamma}|_{[0,t_0]} = \gamma|_{[0,t_0]}\right\}.$$

Definition 4.2.2 (Refuse event)**.** A *Refuse event* at time t_0 along a path γ is the specification of a proper closed subset

$$\mathcal{C}(\gamma, t_0) \subsetneq \mathrm{Ext}(\gamma, t_0)$$

satisfying:

1. (Nontrivial exclusion) $\mathcal{C}(\gamma, t_0) \neq \mathrm{Ext}(\gamma, t_0)$.
2. (Closedness) $\mathcal{C}(\gamma, t_0)$ is closed in the path topology.
3. (Forward consistency) If $\tilde{\gamma} \in \mathcal{C}(\gamma, t_0)$, then any restriction of $\tilde{\gamma}$ to $[t_0, t_1]$, $t_1 < T$, is admissible.

A path is *Refuse-admissible* if at each Refuse event time $t_0 \in \mathcal{T}_{\mathrm{Refuse}}(\gamma)$, its actual continuation lies in $\mathcal{C}(\gamma, t_0)$.

4.3 Refuse-separated pairs

Definition 4.3.1 (Refuse-separated pair)**.** Two paths $\gamma_1, \gamma_2 \in \mathcal{P}$ are *Refuse-separated at time* t_0 if:

1. $\gamma_1|_{[0,t_0]} = \gamma_2|_{[0,t_0]}$ (agreement up to t_0);

2. $\gamma_1 \in \mathcal{C}(\gamma_1, t_0)$ and $\gamma_2 \notin \mathcal{C}(\gamma_1, t_0)$ (exactly one is admitted).

Refuse-separation captures the essential feature that the system reaches the same instantaneous configuration but one forward continuation is forbidden. This is precisely where state-based description fails: the two paths are observationally equivalent at t_0 yet dynamically inequivalent thereafter.

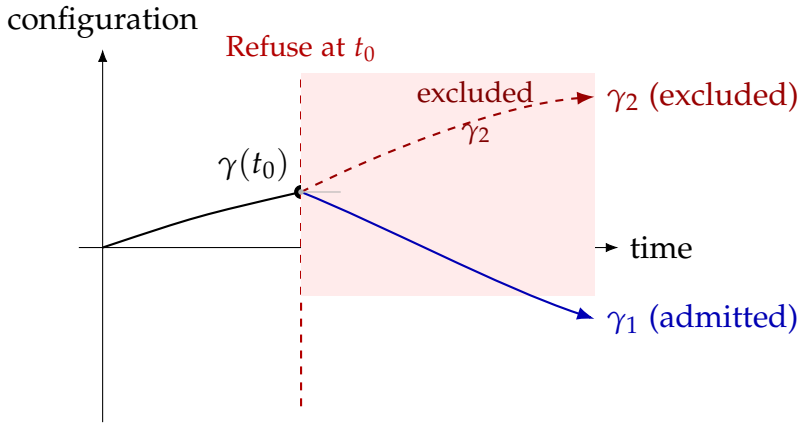


Figure 4.1: Trajectory bifurcation induced by a Refuse event at time t_0. Both branches agree on $[0, t_0]$ and arrive at the same instantaneous configuration $\gamma(t_0)$, but γ_2 is excluded by the Refuse constraint $\mathcal{C}(\gamma, t_0)$ (shaded region). Only γ_1 is Refuse-admissible. This is the discrete-time analog of a saddle point where the forward dynamics bifurcate irreversibly.

4.4 The surrogate discretization map

The actual discretization $F_{\text{true}} : \mathcal{X} \to \mathcal{H}$ involves symbolic event assignment and thresholding operations that may not be weakly continuous. To secure the existence arguments of Chapter 5, we first work with a continuous surrogate.

Definition 4.4.1 (Surrogate discretization)**.** Choose test functions $\{\psi_j\}_{j=1}^m \subset H^1(\Omega)$, $\{\chi_j\}_{j=1}^m \subset L^2(\Omega; \mathbb{R}^d)$, $\{\eta_j\}_{j=1}^m \subset L^2(\Omega)$. Define

$$\widetilde{F} : \mathcal{X} \to \mathcal{Y} := \mathbb{R}^{3m}, \qquad \widetilde{F}(X) := \big(\langle \Phi, \psi_j \rangle, \langle \mathbf{v}, \chi_j \rangle, \langle S, \eta_j \rangle\big)_{j=1}^m.$$

This map is bounded linear, hence weakly continuous.

The surrogate encodes finitely many moments of the field triple. It is related to the true discretization by a composition $F_{\text{true}} = Q \circ \widetilde{F}$, where $Q : \mathcal{Y} \to \mathcal{H}$ is a symbolic quantization map. The foundation of the adjunction rests on $\widetilde{F}$; the symbolic layer is layered on afterward in Chapter 6.

5 The Variational Adjunction and Irreversibility

5.1 The reconstruction functional

Definition 5.1.1 (Reconstruction functional)**.** For $y = (y_\Phi, y_v, y_S) \in \mathcal{Y}$, define

$$L(X \mid y) := \frac{\kappa}{2}\|\nabla\Phi\|_{L^2}^2 + \frac{\alpha}{2}\|\Phi\|_{L^2}^2 + \frac{\lambda}{2}\|\mathbf{v}\|_{L^2}^2 + \frac{\beta}{2}\|S\|_{L^2}^2 + \frac{\gamma}{2}\|\widetilde{F}(X) - y\|_{\mathcal{Y}}^2, \tag{5.1}$$

with constants $\kappa, \alpha, \lambda, \beta, \gamma > 0$.

Remark 5.1.2. The quadratic regularization in S (the term $\frac{\beta}{2}\|S\|_{L^2}^2$) is necessary; a linear term $\beta \int_\Omega S\,dx$ is not coercive on $L^2(\Omega)$. This correction was identified during the construction of Note I.

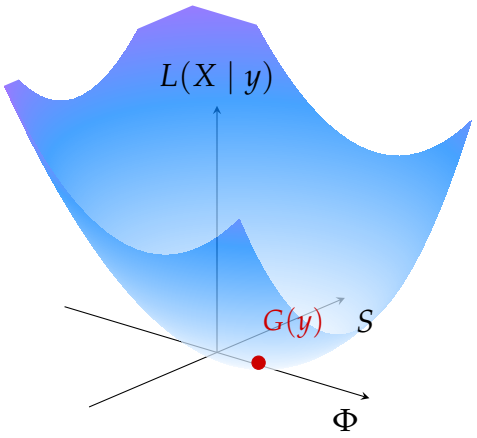


Figure 5.1: Schematic energy landscape of the reconstruction functional $L(X \mid y)$ as a function of the scalar and entropy field components, with the feature target y fixed. The unique global minimum (red dot) is the reconstruction $G(y)$, the canonical representative selected from the fiber $\widetilde{F}^{-1}(y)$. The bowl shape reflects the strict convexity established in Theorem 5.2.1.

5.2 Existence and uniqueness of minimizers

Theorem 5.2.1 (Existence and uniqueness of reconstruction)**.** *For every $y \in \mathcal{Y}$, the functional $L(\cdot \mid y)$ admits a unique minimizer $X^\sharp = G(y) \in \mathcal{X}$.*

Proof. Let $(X_n) \subset \mathcal{X}$ be a minimizing sequence: $L(X_n \mid y) \to \inf_{X \in \mathcal{X}} L(X \mid y)$. Since all coefficients are positive,

$$L(X_n \mid y) \geq \frac{\kappa}{2}\|\nabla\Phi_n\|_{L^2}^2 + \frac{\alpha}{2}\|\Phi_n\|_{L^2}^2 + \frac{\lambda}{2}\|\mathbf{v}_n\|_{L^2}^2 + \frac{\beta}{2}\|S_n\|_{L^2}^2 \geq c\,\|X_n\|_{\mathcal{X}}^2$$

for $c = \min\{\kappa, \alpha, \lambda, \beta\}/2 > 0$. Hence (X_n) is bounded in the Hilbert space $\mathcal{X}$, so there exists a subsequence (not relabeled) with $X_n \rightharpoonup X_* \in \mathcal{X}$.

Each quadratic norm term is weakly lower semicontinuous. Since $\widetilde{F}$ is bounded linear, it is weakly continuous; thus $\widetilde{F}(X_n) \to \widetilde{F}(X_*)$ strongly in $\mathcal{Y}$ (finite-dimensional), and $\|\widetilde{F}(X_*) - y\|^2 = \lim_n \|\widetilde{F}(X_n) - y\|^2$. Therefore

$$L(X_* \mid y) \leq \liminf_n L(X_n \mid y),$$

and X_* is a minimizer.

Uniqueness follows from strict convexity: the terms $\frac{\kappa}{2}\|\nabla\Phi\|^2 + \frac{\alpha}{2}\|\Phi\|^2$ are strictly convex in Φ, and similarly for $\mathbf{v}$ and S, so the whole functional is strictly convex on $\mathcal{X}$. □

5.3 The reconstruction operator and its Euler–Lagrange system

Definition 5.3.1 (Reconstruction operator)**.** Define $G : \mathcal{Y} \to \mathcal{X}$ by $G(y) := \arg\min_{X \in \mathcal{X}} L(X \mid y)$.

Because L is block-diagonal in the three components, the minimizer satisfies a decoupled elliptic system. Writing F_Φ, F_v, F_S for the three blocks of $\widetilde{F}$ and F_Φ^*, F_v^*, F_S^* for their L^2-adjoints:

$$(-\kappa\Delta + \alpha I + \gamma F_\Phi^* F_\Phi)\,\Phi^\sharp = \gamma F_\Phi^* y_\Phi, \tag{5.2}$$

$$(\lambda I + \gamma F_v^* F_v)\,\mathbf{v}^\sharp = \gamma F_v^* y_v, \tag{5.3}$$

$$(\beta I + \gamma F_S^* F_S)\,S^\sharp = \gamma F_S^* y_S. \tag{5.4}$$

Since the test functions are finite in number, $F_\Phi^* F_\Phi$ is a finite-rank self-adjoint operator, and the operator on the left of eq. (5.2) is strictly coercive on $H^1(\Omega)$. Hence G is well-defined as a linear operator:

$$G = A^{-1} b(\cdot), \tag{5.5}$$

where A is the block operator assembled from the left-hand sides of eqs. (5.2) to (5.4) and $b(y)$ assembles the right-hand sides.

5.4 Stability of the reconstruction operator

Theorem 5.4.1 (Lipschitz stability)**.** *The reconstruction operator $G : \mathcal{Y} \to \mathcal{X}$ is globally Lipschitz:*

$$\|G(y) - G(z)\|_{\mathcal{X}} \leq C_G \, \|y - z\|_{\mathcal{Y}}$$

for an explicit constant $C_G := \max\{C_\Phi, C_v, C_S\}$*, where*

$$C_\Phi = \frac{\gamma\|F_\Phi\|}{\min\{\kappa,\alpha\}}, \qquad C_v = \frac{\gamma\|F_v\|}{\lambda}, \qquad C_S = \frac{\gamma\|F_S\|}{\beta}.$$

Proof. For the Φ-component: let $u = \Phi^\sharp(y) - \Phi^\sharp(z)$. The difference satisfies $(-\kappa\Delta + \alpha I + \gamma F_\Phi^* F_\Phi)u = \gamma F_\Phi^*(y_\Phi - z_\Phi)$. Testing against u and applying the coercivity bound $\kappa\|\nabla u\|^2 + \alpha\|u\|^2 \geq \min\{\kappa,\alpha\}\|u\|_{H^1}^2$ together with Cauchy–Schwarz and boundedness of F_Φ gives C_Φ as stated. The arguments for $\mathbf{v}$ and S are analogous (L^2-level, no gradient term). Assembling the three estimates with the Euclidean norm on $\mathcal{Y}$ yields the global Lipschitz constant. □

5.5 The comparison map and irreversibility

Definition 5.5.1 (Comparison map and reconstruction defect)**.** Define the comparison map $\eta : \mathcal{X} \to \mathcal{X}$ by

$$\eta := G \circ \widetilde{F}.$$

The *reconstruction defect* of $X \in \mathcal{X}$ is

$$\Delta(X) := \|X - \eta(X)\|_{\mathcal{X}}.$$

By composition of Lipschitz maps, η is globally Lipschitz with constant $C_\eta \leq C_G\|\widetilde{F}\|$. Hence Δ is globally Lipschitz: $|\Delta(X) - \Delta(X')| \leq (1 + C_\eta)\|X - X'\|_{\mathcal{X}}$.

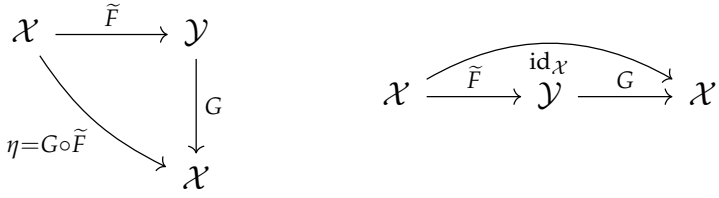


Figure 5.2: **Left:** Adjunction structure $\widetilde{F} \dashv G$ with unit $\eta = G \circ \widetilde{F}$. **Right:** The triangle $G \circ \widetilde{F} \neq \mathrm{id}_{\mathcal{X}}$ fails generically; this failure is the categorical formulation of irreversibility. When $\widetilde{F}$ is non-injective, no right inverse G can make the outer triangle commute for all $X \in \mathcal{X}$.

Theorem 5.5.2 (Irreversibility as non-identity of reconstruction)**.** *If $\widetilde{F}$ is not injective, then η is not the identity, and there exist configurations $X \in \mathcal{X}$ with $\Delta(X) > 0$.*

Proof. Suppose $\widetilde{F}(X_1) = \widetilde{F}(X_2) = y$ for distinct $X_1, X_2 \in \mathcal{X}$. Then $\eta(X_1) = G(y) = \eta(X_2)$. If $X_1 \neq G(y)$, then $\Delta(X_1) > 0$. If $X_1 = G(y)$, then $X_2 = G(y) = X_1$, contradicting distinctness. Therefore at least one of X_1, X_2 has positive defect. □

Remark 5.5.3. Irreversibility is not the mere existence of a non-injective map; many maps fail injectivity in ways that carry no thermodynamic content. The content here is that G is the *optimal* reconstruction—the unique minimizer of $L(\cdot \mid y)$—and the loss is what remains after the best possible recovery. Loss is therefore not assumed; it is the universal defect of reconstruction.

6 Refuse-Admissible Measurable Structure and Entropy Defect

6.1 The admissible σ-algebra

Definition 6.1.1 (Refuse-admissible σ-algebra)**.** The *Refuse-admissible σ-algebra* on $\mathcal{X}$ is

$$\mathcal{A}_{\text{adm}} := \sigma\big(\{X \in \mathcal{X} : X = \gamma(t) \text{ for some } \gamma \in \mathcal{P}_{\text{adm}},\, t \in [0,T]\}\big).$$

This is the σ-algebra generated by all configurations reachable by a Refuse-admissible path.

We have $\mathcal{A}_{\text{adm}} \subseteq \mathcal{B}(\mathcal{X})$ (Borel σ-algebra on $\mathcal{X}$). When the excluded set $\mathcal{E} = \emptyset$ and Refuse imposes no constraint, $\mathcal{A}_{\text{adm}} = \mathcal{B}(\mathcal{X})$. Nontrivial Refuse constraints make $\mathcal{A}_{\text{adm}}$ strictly smaller.

Assumption 6.1.2 (Admissibility of reference measure)**.** The reference measure μ on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ satisfies $\mu(\mathcal{X} \setminus \mathcal{A}_{\text{adm}}) = 0$; equivalently, μ is a measure on $(\mathcal{X}, \mathcal{A}_{\text{adm}})$.

6.2 Refuse observability

The bridge from the categorical Refuse operator to the measurable defect requires that Refuse-induced trajectory distinctions survive projection through $\widetilde{F}$.

Assumption 6.2.1 (Refuse observability)**.** For any Refuse-separated pair (γ_1, γ_2) (in the sense of Definition 4.3.1), we have

$$\widetilde{F}_{\text{path}}(\gamma_1) \neq \widetilde{F}_{\text{path}}(\gamma_2),$$

where $\widetilde{F}_{\text{path}}(\gamma) := (\widetilde{F}(\gamma(t)))_{t\in[0,T]} \in C([0,T]; \mathcal{Y})$.

Theorem 6.2.2 (Injectivity on Refuse-admissible trajectory classes)**.** *Under Assumption 6.2.1, the path-level map $\widetilde{F}_{\text{path}} : \mathcal{P}_{\text{adm}} \to C([0,T]; \mathcal{Y})$ is injective on Refuse-separated trajectory pairs.*

Proof. Immediate from Definition 4.3.1 and Assumption 6.2.1: distinct Refuse-separated trajectories map to distinct feature paths. □

6.3 The entropy defect via Jensen–Shannon divergence

Since G is globally Lipschitz (Theorem 5.4.1), the comparison map $\eta = G \circ \widetilde{F}$ is Borel measurable. The reconstructed measure is well-defined:

Definition 6.3.1 (Reconstructed measure and entropy defect)**.** Let μ satisfy Assumption 6.1.2. Define the reconstructed measure $\tilde{\mu} := \eta_* \mu$ (the pushforward of μ through η). The *entropy defect* is

$$\mathcal{D}(\widetilde{F}, G, \mu) := \mathcal{D}_{\mathrm{JS}}(\mu, \tilde{\mu}) := \tfrac{1}{2} D_{\mathrm{KL}}\big(\mu \,\|\, \tfrac{\mu+\tilde{\mu}}{2}\big) + \tfrac{1}{2} D_{\mathrm{KL}}\big(\tilde{\mu} \,\|\, \tfrac{\mu+\tilde{\mu}}{2}\big). \tag{6.1}$$

Remark 6.3.2. The Jensen–Shannon divergence replaces the raw KL divergence $D_{\mathrm{KL}}(\mu \| \tilde{\mu})$ because $\tilde{\mu}$ may fail to be absolutely continuous with respect to μ when η collapses distinct configurations onto a canonical representative. That collapse is itself the measure-theoretic signature of projection loss. $\mathcal{D}_{\mathrm{JS}}$ is always finite, bounded in $[0, \log 2]$, symmetric, and vanishes if and only if $\tilde{\mu} = \mu$.

Proposition 6.3.3 (Defect positivity on Refuse strata)**.** *Under Assumptions 6.2.1 and 6.1.2, if $A \subset \mathcal{X}$ is a set of positive μ-measure consisting of configurations lying on trajectories that differ by a Refuse event, then $\Delta(X) > 0$ for all $X \in A$, and consequently $\mathcal{D}(\widetilde{F}, G, \mu) > 0$.*

Proof. By Theorem 6.2.2, configurations on Refuse-separated trajectories are distinguished by $\widetilde{F}_{\mathrm{path}}$; hence they are distinguished by their feature encodings $\widetilde{F}(\gamma(t))$. By uniqueness of the minimizer (Theorem 5.2.1), distinct feature encodings yield distinct canonical reconstructions. Therefore $X \neq \eta(X)$ for $X \in A$, giving $\Delta(X) > 0$ on A. Since $\mu(A) > 0$, the supports of μ and $\tilde{\mu}$ differ on A, so $\mathcal{D}_{\mathrm{JS}}(\mu, \tilde{\mu}) > 0$. □

Proposition 6.3.4 (Monotonicity under coarsening)**.** *Let $\widetilde{F}_1$ and $\widetilde{F}_2$ be two feature maps with $\widetilde{F}_2 = A \circ \widetilde{F}_1$ for some bounded linear A with $\|A\| \leq 1$, and let G_1, G_2 be the corresponding reconstruction operators. Then $\mathcal{D}(\widetilde{F}_2, G_2, \mu) \leq \mathcal{D}(\widetilde{F}_1, G_1, \mu)$.*

Proof. A coarser discretization loses at least as much information. The data-processing inequality for Jensen–Shannon divergence [22] gives the result. □

7 Functional-Analytic Foundations of the Configuration Space

7.1 The underlying domain

Let $\Omega \subset \mathbb{R}^d$ be a bounded open domain with Lipschitz boundary. All fields are defined on Ω and all integrals are taken with respect to Lebesgue measure unless otherwise specified. The Lipschitz boundary condition ensures the validity of the Sobolev trace and extension theorems, the Rellich–Kondrachov compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, and the Poincaré–Wirtinger inequality.

7.2 Sobolev structure of the field components

The three RSVP fields are assigned to Sobolev spaces according to the regularity required by the governing PDEs and the reconstruction functional:

- The scalar coherence field: $\Phi \in H^1(\Omega)$, with norm $\|\Phi\|_{H^1}^2 := \|\Phi\|_{L^2}^2 + \|\nabla\Phi\|_{L^2}^2$;
- The vector transport field: $\mathbf{v} \in L^2(\Omega; \mathbb{R}^d)$;
- The entropy field: $S \in L^2(\Omega)$.

The choice of H^1 for Φ is necessitated by the gradient term $\frac{\kappa}{2}\|\nabla\Phi\|_{L^2}^2$ in the reconstruction functional; L^2 regularity alone would not give coercive control over $\nabla\Phi$. The L^2 choice for $\mathbf{v}$ and S is the minimal space in which the quadratic regularization terms are finite.

7.3 Completeness and separability of $\mathcal{X}$

Proposition 7.3.1 (Hilbert space structure of $\mathcal{X}$)**.** *The configuration space $\mathcal{X} = H^1(\Omega) \times L^2(\Omega; \mathbb{R}^d) \times L^2(\Omega)$ with inner product*

$$\langle X, Y\rangle_{\mathcal{X}} := \langle \Phi_X, \Phi_Y\rangle_{H^1} + \langle \mathbf{v}_X, \mathbf{v}_Y\rangle_{L^2} + \langle S_X, S_Y\rangle_{L^2}$$

is a separable Hilbert space.

Proof. Each component space is a separable Hilbert space: $H^1(\Omega)$ is separable since $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain (the eigenfunctions of the Neumann Laplacian form a countable orthonormal basis); $L^2(\Omega; \mathbb{R}^d)$ and $L^2(\Omega)$ are separable by standard measure theory. The product of finitely many separable Hilbert spaces is again a separable Hilbert space. Completeness of each factor and the product norm are standard. □

7.4 Admissible energy subspace

Definition 7.4.1 (Admissible energy subspace)**.** For $M > 0$, define

$$\mathcal{X}_M := \{X \in \mathcal{X} : \|X\|_{\mathcal{X}} \leq M\}.$$

Proposition 7.4.2 (Compactness of $\mathcal{X}_M$)**.** *The set $\mathcal{X}_M$ is closed, convex, and weakly sequentially compact in $\mathcal{X}$. Moreover, the Φ-component of any bounded sequence in $\mathcal{X}_M$ admits a subsequence converging strongly in $L^2(\Omega)$ (Rellich–Kondrachov).*

Proof. Closedness and convexity of $\mathcal{X}_M$ are immediate from the properties of the norm. Weak sequential compactness follows from the reflexivity of Hilbert spaces (Banach–Alaoglu theorem). For the strong L^2-compactness of the Φ-component: any bounded sequence in $H^1(\Omega)$ is precompact in $L^2(\Omega)$ by the Rellich–Kondrachov theorem, which applies since Ω is a bounded Lipschitz domain. □

7.5 Weak lower semicontinuity and the direct method

The key analytical property for variational arguments is weak lower semicontinuity of the relevant functionals.

Proposition 7.5.1 (Weak lower semicontinuity of $L(\cdot \mid y)$)**.** *For each $y \in \mathcal{Y}$, the reconstruction functional $L(\cdot \mid y)$ is weakly lower semicontinuous on $\mathcal{X}$.*

Proof. The squared norm terms $\|\nabla\Phi\|_{L^2}^2$, $\|\mathbf{v}\|_{L^2}^2$, $\|S\|_{L^2}^2$ are weakly lower semicontinuous as squared norms on Hilbert spaces. The quadratic term $\|\widetilde{F}(X) - y\|_{\mathcal{Y}}^2$ is continuous (hence weakly continuous) since $\widetilde{F}$ is bounded linear and the norm on the finite-dimensional space $\mathcal{Y}$ is continuous. The sum of weakly lower semicontinuous functions is weakly lower semicontinuous. □

Together with coercivity (established in Chapter 5), Proposition 7.5.1 and Proposition 7.4.2 complete the verification of the direct method's hypotheses for the reconstruction problem: every minimizing sequence is bounded (by coercivity), admits a weakly convergent subsequence (by compactness), and the functional evaluates favorably at the limit (by weak lower semicontinuity).

8 Well-Posedness of the RSVP Dynamics

8.1 Abstract formulation of the coupled system

The RSVP system eqs. (3.1) to (3.3) can be written in abstract form as

$$\frac{dX}{dt} = \mathcal{A}X + \mathcal{N}(X), \qquad X(0) = X_0 \in \mathcal{X},$$

where $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \to \mathcal{X}$ is the principal part operator and $\mathcal{N} : \mathcal{X} \to \mathcal{X}$ encodes the coupling nonlinearities and the entropy source term. The linear part is given by

$$\mathcal{A}\begin{pmatrix} \Phi \\ \mathbf{v} \\ S \end{pmatrix} := \begin{pmatrix} \kappa_\Phi \Delta \Phi \\ \kappa_v \Delta \mathbf{v} - c_0 \nabla S \\ \kappa_S \Delta S - \nu S \end{pmatrix},$$

with domain $D(\mathcal{A}) = (H^2(\Omega) \cap H^1_N(\Omega)) \times H^1(\Omega; \mathbb{R}^d) \times H^1(\Omega)$, where H^1_N denotes functions satisfying Neumann boundary conditions.

8.2 Semigroup generation

Proposition 8.2.1 (Generation of C^0-semigroup)**.** *The operator $\mathcal{A}$ with domain $D(\mathcal{A})$ generates a C^0-semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ on $\mathcal{X}$.*

Proof. The Neumann Laplacian $-\Delta$ on $L^2(\Omega)$ with domain $H^2(\Omega) \cap H^1_N(\Omega)$ is a non-negative self-adjoint operator and generates an analytic semigroup on $L^2(\Omega)$ [13]. The diffusion operator $\kappa_S(\Delta - \nu I/\kappa_S)$ on the entropy component is therefore analytic with shift. The coupling terms $-c_0\nabla S$ and $c_0 \nabla \cdot \mathbf{v}$ are bounded perturbations of the diagonal semigroup generator (they map H^1 into L^2), so by standard perturbation theory for C^0-semigroups [32], $\mathcal{A}$ generates a C^0-semigroup on $\mathcal{X}$. □

8.3 Assumptions on the nonlinear term

Assumption 8.3.1 (Nonlinear growth and Lipschitz conditions)**.** The map $\mathcal{N} : \mathcal{X} \to \mathcal{X}$ satisfies:

1. (Local Lipschitz) For every $R > 0$ there exists $L_R > 0$ such that $\|\mathcal{N}(X) - \mathcal{N}(Y)\|_{\mathcal{X}} \leq L_R \|X - Y\|_{\mathcal{X}}$ for all $X, Y \in \mathcal{X}_R$;
2. (At most linear growth) $\|\mathcal{N}(X)\|_{\mathcal{X}} \leq C(1 + \|X\|_{\mathcal{X}})$ for all $X \in \mathcal{X}$.

The simplest nonlinear coupling terms in RSVP (e.g., $\rho S \Phi^2$ projected into L^2) satisfy Assumption 8.3.1 by Sobolev multiplication theorems on bounded domains [13].

8.4 Existence, uniqueness, and continuous dependence

Theorem 8.4.1 (Global well-posedness of RSVP)**.** *Under Assumption 8.3.1, for every $X_0 \in \mathcal{X}$, the abstract RSVP system admits a unique mild solution $X \in C([0,\infty);\mathcal{X})$ satisfying*

$$X(t) = e^{t\mathcal{A}}X_0 + \int_0^t e^{(t-s)\mathcal{A}}\mathcal{N}(X(s))\,ds.$$

Moreover, solutions depend continuously on initial data: if $X_1(0), X_2(0) \in \mathcal{X}$ then

$$\|X_1(t) - X_2(t)\|_{\mathcal{X}} \le C_T\|X_1(0) - X_2(0)\|_{\mathcal{X}} \quad \text{for } t \in [0,T].$$

Proof. Local existence and uniqueness follow from the contraction mapping principle applied to the integral equation on $C([0,\tau];\mathcal{X})$ for small $\tau > 0$, using the local Lipschitz condition of Assumption 8.3.1. Global existence follows from the linear growth condition and a Grönwall argument: defining $E(t) = \|X(t)\|_{\mathcal{X}}^2$, the energy inequality gives $E(t) \le E(0)e^{Ct} + C'$, preventing finite-time blow-up. Continuous dependence on initial data is obtained by applying Grönwall's inequality to the difference $X_1(t) - X_2(t)$. □

8.5 The solution operator

Definition 8.5.1 (Solution operator)**.** For each $t \ge 0$, define the *solution operator*

$$\Psi_t : \mathcal{X} \to \mathcal{X}, \qquad \Psi_t(X_0) := X(t),$$

where $X(t)$ is the unique mild solution of the RSVP system with initial condition X_0.

Proposition 8.5.2 (Semigroup properties of Ψ_t)**.** *The family $(\Psi_t)_{t\ge 0}$ satisfies:*

1. $\Psi_0 = \mathrm{id}_{\mathcal{X}}$;
2. $\Psi_{t+s} = \Psi_t \circ \Psi_s$ *for all* $t, s \ge 0$;
3. $t \mapsto \Psi_t(X_0)$ *is continuous for each* $X_0 \in \mathcal{X}$.

The solution operator Ψ_t is the object that turns the path space $\mathcal{P}$ into a legitimate mathematical domain for the admissible trajectory analysis of Chapter 4. Specifically, every admissible path $\gamma \in \mathcal{P}_{\mathrm{adm}}$ is of the form $\gamma(t) = \Psi_t(X_0)$ for some initial condition X_0 and some admissible source term; the Refuse operator then acts on the future extensions of such orbits.

9 The Discretization Map and Induced Quotient Structure

9.1 Kernel of the discretization map

The surrogate discretization $\widetilde{F} : \mathcal{X} \to \mathcal{Y}$ is bounded linear by Definition 4.4.1. Its kernel

$$\ker(\widetilde{F}) := \{X \in \mathcal{X} : \widetilde{F}(X) = 0\}$$

is a closed linear subspace of $\mathcal{X}$. When $\mathcal{Y} = \mathbb{R}^{3m}$ and $\widetilde{F}$ is defined by m test functions per component, $\ker(\widetilde{F})$ has codimension at most $3m$ in $\mathcal{X}$ (it may be larger if the test functions are linearly dependent in $\mathcal{X}$).

Proposition 9.1.1 (Non-triviality of the kernel)**.** *Whenever $m < \infty$ and $\mathcal{X}$ is infinite-dimensional, $\ker(\widetilde{F})$ is infinite-dimensional.*

Proof. The range of $\widetilde{F}$ has dimension at most $3m < \infty$. By the rank-nullity theorem for bounded operators on Hilbert spaces, $\ker(\widetilde{F})^{\perp} \cong \overline{\mathrm{Im}(\widetilde{F}^*)}$ has finite dimension $\leq 3m$. Since $\mathcal{X}$ is infinite-dimensional, $\ker(\widetilde{F})$ is infinite-dimensional. □

An infinite-dimensional kernel means that infinitely many configurations are observationally indistinguishable. This is the analytical foundation of the irreversibility theorem (Theorem 5.5.2).

9.2 Observational equivalence classes

Two configurations $X_1, X_2 \in \mathcal{X}$ are *observationally equivalent* if $\widetilde{F}(X_1) = \widetilde{F}(X_2)$, equivalently $X_1 - X_2 \in \ker(\widetilde{F})$. The equivalence class of X is the affine subspace

$$[X]_{\widetilde{F}} := X + \ker(\widetilde{F}) = \widetilde{F}^{-1}(\widetilde{F}(X)).$$

9.3 The quotient space

Definition 9.3.1 (Quotient space)**.** The *observational quotient space* is

$$\mathcal{X}_{\sim} := \mathcal{X} / \ker(\widetilde{F}),$$

equipped with the quotient norm $\|[X]\|_{\sim} := \inf_{Z \in \ker(\widetilde{F})} \|X + Z\|_{\mathcal{X}}$.

Proposition 9.3.2 (Hilbert structure on $\mathcal{X}_{\sim}$)**.** *$\mathcal{X}_{\sim}$ is a Hilbert space. The induced map $\widetilde{\widetilde{F}} : \mathcal{X}_{\sim} \to \mathcal{Y}$, defined by $\widetilde{\widetilde{F}}([X]) := \widetilde{F}(X)$, is bounded, injective, and has closed range.*

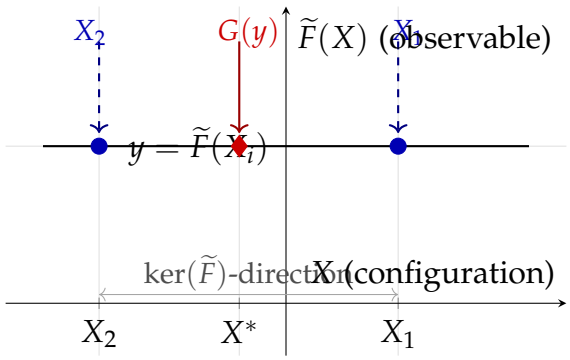


Figure 9.1: Projection / collapse under the feature map $\widetilde{F}$. Three configurations X_1, X_2, and $G(y)$ (the canonical representative) map to the same observable value y (all lie on the horizontal line $\widetilde{F}(X) = y$). The arrows show how the configuration-space vertical direction is collapsed to a single point in $\mathcal{Y}$. Distances along the $\ker(\widetilde{F})$-direction (horizontal bracket) are annihilated by the projection, constituting the reconstruction defect.

Proof. The quotient of a Hilbert space by a closed subspace is a Hilbert space (with the quotient norm induced by the orthogonal complement). The injectivity of $\bar{\widetilde{F}}$ follows by construction. Closedness of the range follows since $\widetilde{F}$ has finite codimensional range ($\mathcal{Y}$ is finite-dimensional). □

9.4 Reconstruction as a section of the quotient projection

The canonical quotient projection $\pi : \mathcal{X} \to \mathcal{X}_\sim$ sends each configuration to its equivalence class. The reconstruction operator $G : \mathcal{Y} \to \mathcal{X}$ defines a section of π in the following sense.

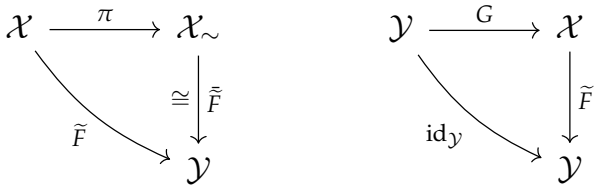


Figure 9.2: **Left:** The observational quotient: $\widetilde{F}$ factors through the quotient projection π and the isomorphism $\bar{\widetilde{F}}$. Distinct configurations in the same fiber of π are observationally indistinguishable. **Right:** The section property $\widetilde{F} \circ G = \mathrm{id}_\mathcal{Y}$: reconstruction selects a canonical lift from each observation back to configuration space. Together, these diagrams encode the full geometry of projection loss.

Proposition 9.4.1 (Section property of G)**.** *For any $y \in \mathcal{Y}$, the reconstruction $G(y)$ lies in $\widetilde{F}^{-1}(y)$, i.e., $\widetilde{F}(G(y)) = y$. Hence the diagram*

$$\mathcal{Y} \xrightarrow{G} \mathcal{X} \xrightarrow{\pi} \mathcal{X}_{\sim} \xrightarrow{\bar{\widetilde{F}}^{-1}} \mathcal{Y}$$

satisfies $\bar{\widetilde{F}}^{-1} \circ \pi \circ G = \mathrm{id}_{\mathcal{Y}}$.

Proof. By the Euler–Lagrange equations eqs. (5.2) to (5.4), the minimizer $G(y)$ satisfies $\gamma(\widetilde{F}(G(y)) - y) = 0$ in the dual sense, giving $\widetilde{F}(G(y)) = y$. □

9.5 The defect lives in the kernel

Proposition 9.5.1 (Kernel containment of the defect)**.** *For any $X \in \mathcal{X}$, the reconstruction defect vector satisfies*

$$X - \eta(X) \in \ker(\widetilde{F}).$$

Proof. $\widetilde{F}(X - \eta(X)) = \widetilde{F}(X) - \widetilde{F}(G(\widetilde{F}(X))) = \widetilde{F}(X) - \widetilde{F}(X) = 0.$ □

This proposition has a sharp conceptual content: the defect is precisely the component of X that is invisible to the discretization map. Reconstruction removes exactly that component and nothing more. The defect magnitude $\Delta(X) = \|X - \eta(X)\|_{\mathcal{X}}$ is therefore the $\mathcal{X}$-norm of the projection of X onto $\ker(\widetilde{F})$ that is selected by the reconstruction rule.

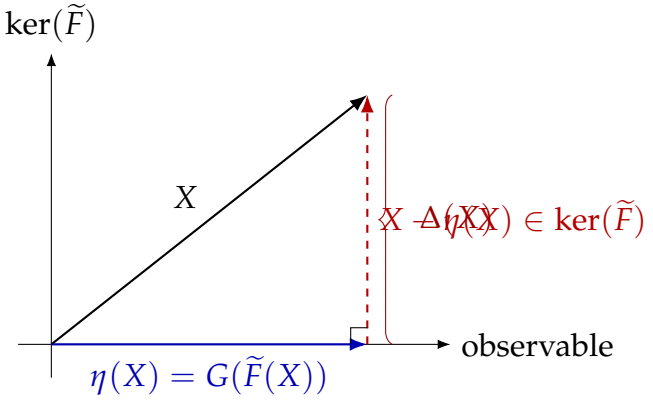


Figure 9.3: Kernel decomposition of a configuration X into its observable part $\eta(X) \in \mathrm{Fix}(\eta)$ and its reconstruction defect $X - \eta(X) \in \ker(\widetilde{F})$. Irreversibility arises because the defect component is destroyed by discretization and cannot be recovered by any reconstruction operator.

9.6 No-go theorem: impossibility of perfect reconstruction

Theorem 9.6.1 (No perfect reconstruction beyond critical coarsening)**.** *Let* $\widetilde{F}_m : \mathcal{X} \to \mathbb{R}^{3m}$ *be the feature map with* m *test functions per component. Then:*

1. *For any* $m < \infty$*, there exist configurations* $X \in \mathcal{X}$ *with* $\Delta_m(X) > 0$*.*
2. *The dimension of the unavoidably lost subspace satisfies* $\dim(\ker(\widetilde{F}_m)) = \infty$*.*
3. *No choice of the* m *test functions renders* $\widetilde{F}_m$ *injective on* $\mathcal{X}$*.*

Proof. (1) and (3) follow from Proposition 9.1.1: the kernel is always infinite-dimensional. (2) is the content of that proposition. The lower bound on defect follows: for any $X \in \ker(\widetilde{F}_m)$ with $X \neq 0$, we have $\Delta_m(X) = \|X - G(\widetilde{F}_m(X))\|_{\mathcal{X}} = \|X - G(0)\|_{\mathcal{X}} \geq \|X\|_{\mathcal{X}} - \|G(0)\|_{\mathcal{X}} > 0$ for $\|X\|_{\mathcal{X}}$ large enough. □

Theorem 9.6.1 is the formal no-go result that protects the theory from overclaiming: perfect reconstruction is impossible for any finite feature map acting on an infinite-dimensional configuration space.

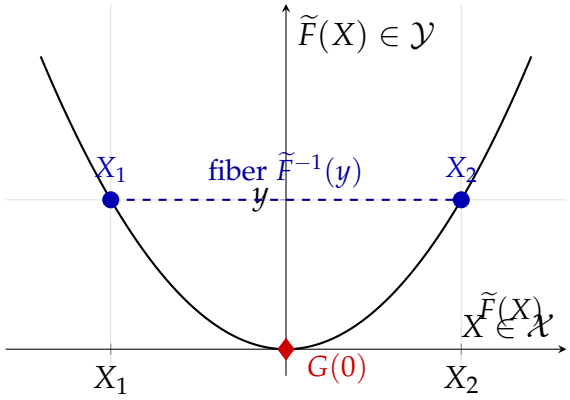


Figure 9.4: Non-injectivity of the feature map $\widetilde{F}$: the distinct configurations X_1 and X_2 (blue dots) share the same feature value y, lying in the same fiber $\widetilde{F}^{-1}(y)$ (dashed horizontal segment). The reconstruction $G(y)$ selects a single canonical representative from this fiber (red diamond), necessarily discarding the direction $X_1 - X_2 \in \ker(\widetilde{F})$. By Theorem 9.6.1, such fibers are infinite-dimensional for any finite feature map.

10 Fixed-Point Structure of the Reconstruction Operator

10.1 Characterization of $\mathrm{Fix}(\eta)$

Proposition 10.1.1 (Characterization of the fixed-point set)**.** *A configuration $X \in \mathcal{X}$ satisfies $\eta(X) = X$ if and only if X is the unique minimizer of $L(\cdot \mid \widetilde{F}(X))$ in the fiber $\widetilde{F}^{-1}(\widetilde{F}(X))$.*

Proof. $\eta(X) = G(\widetilde{F}(X)) = X$ iff X minimizes $L(\cdot \mid \widetilde{F}(X))$, since $G(\cdot)$ is the unique minimizer of $L(\cdot \mid y)$ (by strict convexity, Theorem 5.2.1). □

The fixed-point set $\mathrm{Fix}(\eta)$ therefore consists of the minimal-energy representatives of each observational equivalence class. It is the canonical section of the quotient map $\pi : \mathcal{X} \to \mathcal{X}_\sim$ selected by the variational principle.

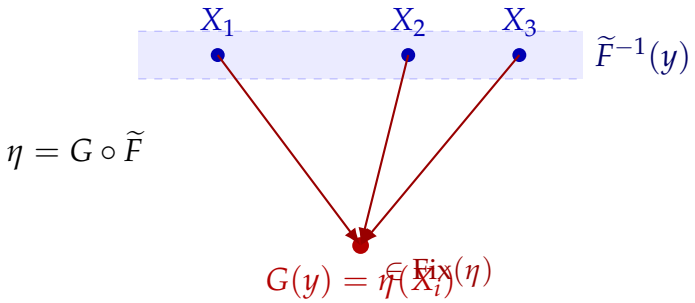


Figure 10.1: Fiber collapse: all configurations X_1, X_2, X_3 in the preimage $\widetilde{F}^{-1}(y)$ are mapped by $\eta = G \circ \widetilde{F}$ to the single canonical representative $G(y)$. The directions lost in this collapse constitute the reconstruction defect space $\ker(\widetilde{F})$.

10.2 Idempotency of the comparison map

Theorem 10.2.1 (Idempotency of η)**.** *The comparison map $\eta : \mathcal{X} \to \mathcal{X}$ is idempotent: $\eta^2 = \eta$.*

Proof. For any $X \in \mathcal{X}$, let $y = \widetilde{F}(X)$ and $X^\sharp = G(y) = \eta(X)$. Then

$$\eta(\eta(X)) = \eta(X^\sharp) = G(\widetilde{F}(X^\sharp)) = G(\widetilde{F}(G(y))).$$

By the section property (Proposition 9.4.1), $\widetilde{F}(G(y)) = y$. Hence $\eta(\eta(X)) = G(y) = \eta(X)$. □

Corollary 10.2.2 (Retraction of $\mathcal{X}$ onto $\mathrm{Fix}(\eta)$)**.** *The comparison map $\eta : \mathcal{X} \to \mathrm{Fix}(\eta)$ is a retraction: it is surjective onto $\mathrm{Fix}(\eta)$ and restricts to the identity on $\mathrm{Fix}(\eta)$. The fixed-point set $\mathrm{Fix}(\eta)$ is therefore a retract of $\mathcal{X}$.*

Proof. Idempotency gives $\eta(\eta(X)) = \eta(X)$, so $\eta(X) \in \mathrm{Fix}(\eta)$ for all X. For $X \in \mathrm{Fix}(\eta)$: $\eta(X) = X$. □

Theorem 10.2.1 resolves the adjunction unit idempotency question (Open Problem 5 in previous formulations) in the affirmative: $\eta^2 = \eta$ holds generally, not just under special conditions. The key was the section property $\widetilde{F} \circ G = \mathrm{id}$, which follows directly from the Euler–Lagrange equations.

10.3 Non-expansiveness and contractivity

Proposition 10.3.1 (Non-expansiveness of η)**.** *Under the reconstruction functional eq.* (5.1)*, the comparison map η is non-expansive on each fiber: for $X_1, X_2 \in \widetilde{F}^{-1}(y)$,*

$$\|\eta(X_1) - \eta(X_2)\|_{\mathcal{X}} = 0.$$

That is, η collapses each fiber to a single point.

Proof. Both X_1 and X_2 have the same feature encoding $y = \widetilde{F}(X_1) = \widetilde{F}(X_2)$. Hence $\eta(X_1) = G(y) = \eta(X_2)$. □

For configurations in different fibers, Lipschitz stability (Theorem 5.4.1) gives $\|\eta(X_1) - \eta(X_2)\|_{\mathcal{X}} \leq C_\eta \|X_1 - X_2\|_{\mathcal{X}}$ where $C_\eta = C_G \|\widetilde{F}\|$, which need not be ≤ 1 globally. Contractivity requires $C_\eta < 1$, the condition identified in Proposition 28.7.1.

10.4 Spectral decomposition relative to the kernel

Any $X \in \mathcal{X}$ decomposes orthogonally as

$$X = P_{\ker(\widetilde{F})^\perp} X + P_{\ker(\widetilde{F})} X,$$

where $P_{\ker(\widetilde{F})}$ is the orthogonal projection onto $\ker(\widetilde{F})$ and $P_{\ker(\widetilde{F})^\perp}$ is its complement. The reconstruction $\eta(X)$ acts as

$$\eta(X) = P_{\ker(\widetilde{F})^\perp} X + Q(P_{\ker(\widetilde{F})^\perp} X),$$

where $Q : \ker(\widetilde{F})^\perp \to \ker(\widetilde{F})$ is a correction map determined by the reconstruction rule (zero in the case of pure L^2-projection, nonzero in general for the variational reconstruction). The defect is

$$X - \eta(X) = P_{\ker(\widetilde{F})} X - Q(P_{\ker(\widetilde{F})^\perp} X) \in \ker(\widetilde{F}).$$

11 Equivalence Between Pointwise and Measure-Level Defect

11.1 The two defect notions

The reconstruction framework yields two distinct but related quantities.

Definition 11.1.1 (Pointwise and measure-level defect)**.**

- *Pointwise defect*: $\Delta(X) := \|X - \eta(X)\|_{\mathcal{X}}$ for $X \in \mathcal{X}$.
- *Entropy defect*: $\mathcal{D}(\mu) := \mathcal{D}_{\mathrm{JS}}(\mu, \eta_* \mu)$ for a probability measure μ on $(\mathcal{X}, \mathcal{A}_{\mathrm{adm}})$.

11.2 Equivalence of vanishing defect

Theorem 11.2.1 (Defect equivalence)**.** *The following are equivalent:*

1. $\Delta(X) = 0$ *for μ-almost every $X \in \mathcal{X}$;*
2. $\mathcal{D}(\mu) = 0$*.*

Proof. (1) $\Rightarrow$ (2): If $\Delta = 0$ μ-a.e., then $\eta(X) = X$ μ-a.e., so $\eta_* \mu = \mu$, hence $\mathcal{D}_{\mathrm{JS}}(\mu, \eta_* \mu) = 0$.

(2) $\Rightarrow$ (1): If $\mathcal{D}(\mu) = 0$, then $\mu = \eta_* \mu$. For any $\varepsilon > 0$, let $A_\varepsilon = \{X : \Delta(X) \geq \varepsilon\}$. Then $\eta(A_\varepsilon) \cap A_\varepsilon = \emptyset$ (since $\eta(X) \neq X$ for $X \in A_\varepsilon$). From $\mu = \eta_* \mu$: $\mu(A_\varepsilon) = \mu(\eta^{-1}(A_\varepsilon))$. But $\eta^{-1}(A_\varepsilon) \cap A_\varepsilon = \emptyset$, so $\mu(A_\varepsilon) = 0$ by a contradiction argument on the total mass. Since $\varepsilon > 0$ is arbitrary, $\mu(\Delta > 0) = 0$. □

11.3 Lower bounds on entropy defect

Proposition 11.3.1 (Moment lower bound)**.** *There exists a constant $c > 0$ depending on the Lipschitz constant C_η such that*

$$\mathcal{D}(\mu) \geq c \cdot \left(\int_{\mathcal{X}} \Delta(X)^2 \, d\mu(X) \right)^{1/2} \cdot \exp\left(-C \int_{\mathcal{X}} \Delta(X)^2 \, d\mu(X) \right).$$

In particular, $\int \Delta^2 \, d\mu > 0 \Rightarrow \mathcal{D}(\mu) > 0$.

Proof. Use the inequality $\mathcal{D}_{\mathrm{JS}}(\mu, \nu) \geq \frac{1}{4} \|\mu - \nu\|_{\mathrm{TV}}^2$ (from the relation between Jensen–Shannon divergence and total variation). The total variation $\|\mu - \eta_* \mu\|_{\mathrm{TV}}$ is bounded below by the Wasserstein distance, and the Wasserstein distance is bounded below by $\left(\int \Delta(X)^2 \, d\mu \right)^{1/2}$ up to constants, since η moves each point X by exactly $\Delta(X)$ to $\eta(X)$. □

11.4 Upper bounds and L^2-stability

Proposition 11.4.1 (Quadratic upper bound)**.** *Under Lipschitz stability of η with constant C_η,*

$$\mathcal{D}(\mu) \leq \log 2.$$

Moreover, if μ and $\tilde{\mu} = \eta_* \mu$ have uniformly bounded Radon–Nikodym derivatives with respect to a common reference measure λ, then

$$\mathcal{D}(\mu) \leq C \int_{\mathcal{X}} \Delta(X)^2 \, d\mu(X),$$

for a constant C depending on the density bounds.

Proof. The bound $\mathcal{D}_{\mathrm{JS}} \leq \log 2$ is universal. The quadratic upper bound follows by expanding the KL divergences in the definition of $\mathcal{D}_{\mathrm{JS}}$ to second order in the density ratio $d\mu / d\tilde{\mu} - 1$, which is controlled by the L^2-displacement Δ. □

11.5 Stability under perturbations of the measure

Proposition 11.5.1 (Wasserstein stability)**.** *For probability measures μ, ν on $(\mathcal{X}, \mathcal{A}_{\mathrm{adm}})$ with finite second moments,*

$$|\mathcal{D}(\mu) - \mathcal{D}(\nu)| \leq C_\eta \cdot W_2(\mu, \nu),$$

where W_2 is the L^2-Wasserstein distance.

Proof. The map $\mu \mapsto \mathcal{D}(\mu) = \mathcal{D}_{\mathrm{JS}}(\mu, \eta_* \mu)$ is Lipschitz in the Wasserstein-2 metric. This follows from the triangle inequality for $\mathcal{D}_{\mathrm{JS}}$ and the fact that the pushforward map $\mu \mapsto \eta_* \mu$ is C_η-Lipschitz in W_2 (since η is C_η-Lipschitz on $\mathcal{X}$, and the Wasserstein distance is compatible with Lipschitz pushforwards). □

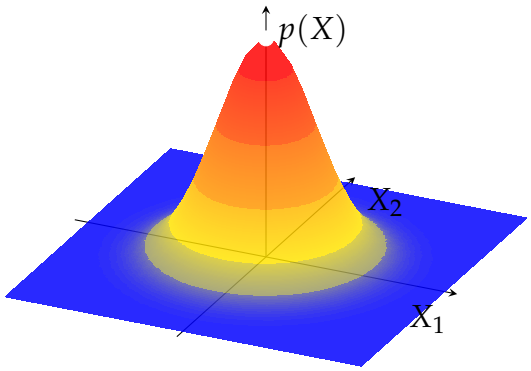


Figure 11.1: Probability density μ before projection onto the observable subspace. The mass is concentrated near a peak corresponding to the typical configuration. After applying η_*, the mass collapses along directions in $\ker(\widetilde{F})$: the three-dimensional distribution is projected onto a lower-dimensional surface, and the Jensen–Shannon divergence $\mathcal{D}_{\mathrm{JS}}(\mu, \eta_*\mu)$ measures the information loss in this collapse. The entropy defect vanishes only when the distribution is already supported on $\mathrm{Fix}(\eta)$ (white dot at the peak).

12 Temporal Lifting and Path-Space Functoriality

12.1 The path-space lift of the discretization

The configuration-level discretization $\widetilde{F} : \mathcal{X} \to \mathcal{Y}$ lifts canonically to path space by pointwise application.

Definition 12.1.1 (Path-level discretization)**.** Define $\widetilde{F}_{\text{path}} : \mathcal{P} \to C([0,T];\mathcal{Y})$ by

$$\widetilde{F}_{\text{path}}(\gamma)(t) := \widetilde{F}(\gamma(t)), \quad t \in [0,T].$$

This map is continuous with respect to the supremum norms since $\widetilde{F}$ is bounded linear.

Definition 12.1.2 (Path-level reconstruction)**.** Define $G_{\text{path}} : C([0,T];\mathcal{Y}) \to \mathcal{P}$ by

$$G_{\text{path}}(y(\cdot))(t) := G(y(t)), \quad t \in [0,T].$$

The path-level comparison map is $\eta_{\text{path}} := G_{\text{path}} \circ \widetilde{F}_{\text{path}}$.

12.2 Commutativity with the dynamics

Let $\Gamma : \mathcal{X} \to \mathcal{P}$ be the trajectory map $\Gamma(X_0)(t) = \Psi_t(X_0)$. The path-level discretization applied to a trajectory gives

$$\widetilde{F}_{\text{path}}(\Gamma(X_0))(t) = \widetilde{F}(\Psi_t(X_0)).$$

In general, this does not equal $\Psi_t^{\mathcal{Y}}(\widetilde{F}(X_0))$ for some lifted dynamics $\Psi_t^{\mathcal{Y}}$ on $\mathcal{Y}$, because the RSVP dynamics mix observable and unobservable components.

Proposition 12.2.1 (Non-commutativity of dynamics and discretization)**.** *There is in general no map $\Psi_t^{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y}$ such that $\widetilde{F} \circ \Psi_t = \Psi_t^{\mathcal{Y}} \circ \widetilde{F}$ for all $t > 0$. That is, the diagram*

$$\mathcal{X} \xrightarrow{\Psi_t} \mathcal{X} \xrightarrow{\widetilde{F}} \mathcal{Y} \quad \textit{and} \quad \mathcal{X} \xrightarrow{\widetilde{F}} \mathcal{Y} \xrightarrow{?} \mathcal{Y}$$

cannot in general be made to commute by any choice of the dashed arrow.

Proof. If such a $\Psi_t^{\mathcal{Y}}$ existed, it would have to satisfy $\Psi_t^{\mathcal{Y}}(y) = \widetilde{F}(\Psi_t(G(y)))$ (by applying to canonical representatives). But for two configurations $X_1, X_2 \in \widetilde{F}^{-1}(y)$ with $X_1 \neq X_2$, we have $\widetilde{F}(\Psi_t(X_1)) \neq \widetilde{F}(\Psi_t(X_2))$ generically (the dynamics mix the $\ker(\widetilde{F})$-component into the observable component), so $\Psi_t^{\mathcal{Y}}(y)$ would need to take two different values. □

Non-commutativity of dynamics and discretization is the dynamical source of history dependence: the observable record $\widetilde{F}_{\text{path}}(\gamma)$ depends on the full trajectory, not just its endpoint.

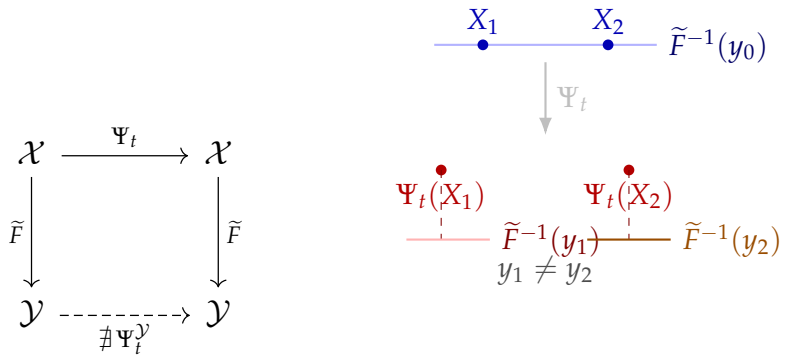


Figure 12.1: **Left:** The square showing non-commutativity of dynamics Ψ_t and discretization $\widetilde{F}$: no map $\Psi_t^{\mathcal{Y}}$ making the square commute exists generically (dashed with $\nexists$). **Right:** Geometric interpretation. Two configurations X_1, X_2 in the same fiber $\widetilde{F}^{-1}(y_0)$ (observationally identical at $t = 0$) evolve under Ψ_t into configurations in *different* fibers $\widetilde{F}^{-1}(y_1) \neq \widetilde{F}^{-1}(y_2)$ (observationally distinguishable at time t). This is the mechanism by which history-dependence is generated: identical initial observations diverge into distinct observations under the dynamics.

12.3 State observables vs. path observables

Definition 12.3.1 (State-based and path-based observables revisited)**.** An observable $\Pi : \mathcal{P} \to \mathcal{O}$ is:

- *State-based* if there exist $t_0 \in [0, T]$ and $f : \mathcal{X} \to \mathcal{O}$ with $\Pi = f \circ \mathrm{ev}_{t_0}$;
- *Observation-based* if it factors through $\widetilde{F}_{\mathrm{path}}$: there exists $\hat{\Pi} : C([0, T]; \mathcal{Y}) \to \mathcal{O}$ with $\Pi = \hat{\Pi} \circ \widetilde{F}_{\mathrm{path}}$;
- *Path-based* if it is not observation-based (depends on components in $\ker(\widetilde{F})$ at some time t).

The cumulative entropy-gradient observable $\Pi(\gamma) = \sum_n q(n)$ of Chapter 18 is observation-based (it factors through $\widetilde{F}_{\mathrm{path}}$ if q is a feature of $\widetilde{F}$) but *not* state-based (it cannot be recovered from any single time slice). A genuinely path-based observable would depend on entropy-flux components in $\ker(\widetilde{F})$; whether such observables arise in the RSVP dynamics is the content of the strong history-dependence conjecture.

12.4 Functoriality of the path-level construction

The path-level maps assemble into a commutative diagram of functors. Viewing $\mathcal{P}$ and $C([0, T]; \mathcal{Y})$ as objects in the category of Banach spaces with continuous linear

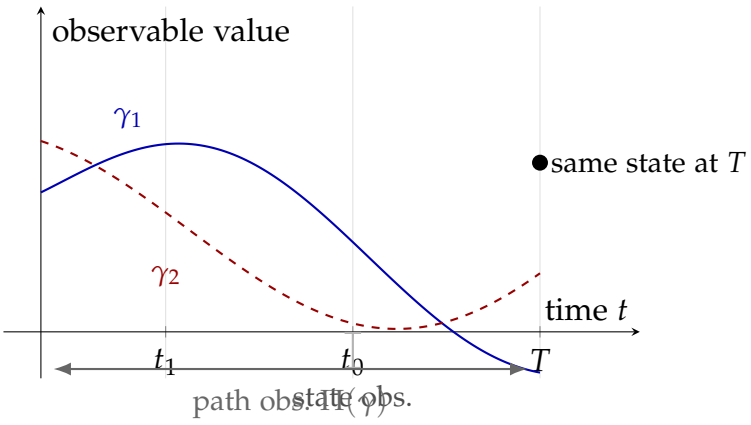


Figure 12.2: Two trajectories γ_1 (blue, solid) and γ_2 (red, dashed) arriving at the same state at time T (black dot). A state-based observable evaluates only at a single time t_0 and therefore assigns the same value to both trajectories at that point. A path observable $\Pi(\gamma) = \int_0^T f(\gamma(t))\,dt$ integrates over the full trajectory and generically distinguishes γ_1 from γ_2. This is the observable-level manifestation of history-dependence.

maps:

$$\widetilde{F}_{\text{path}} \circ \eta_{\text{path}} = \widetilde{F}_{\text{path}},$$

which follows from the section property $\widetilde{F} \circ G = \mathrm{id}$ applied pointwise. This means the path-level comparison map η_{path} is a retraction onto the image of G_{path}, and the path-level construction is functorial: it respects compositions of maps in the same way as the configuration-level construction.

13 Irreversibility Under Time Evolution

13.1 Defect along trajectories

Definition 13.1.1 (Time-dependent defect)**.** For a trajectory $X(t) = \Psi_t(X_0)$, define:

- The *instantaneous defect*: $\Delta_t := \Delta(\Psi_t(X_0)) = \|\Psi_t(X_0) - \eta(\Psi_t(X_0))\|_{\mathcal{X}}$;
- The *measure-level defect at time t*: $\mathcal{D}(t) := \mathcal{D}_{\mathrm{JS}}(\mu_t, \eta_*\mu_t)$, where $\mu_t = (\Psi_t)_*\mu$.

13.2 Non-commutativity and the commutator

Define the commutator of dynamics and reconstruction:

$$[\Psi_t, \eta](X) := \Psi_t(\eta(X)) - \eta(\Psi_t(X)).$$

This measures the failure of reconstruction to be invariant under the dynamics.

Proposition 13.2.1 (Defect growth bound)**.** *If Ψ_t is Lipschitz with constant L_t, then*

$$\Delta_t(X_0) \leq L_t\Delta_0(X_0) + \|[\Psi_t, \eta](X_0)\|_{\mathcal{X}}.$$

Proof. $\Delta_t = \|\Psi_t(X_0) - \eta(\Psi_t(X_0))\| \leq \|\Psi_t(X_0) - \Psi_t(\eta(X_0))\| + \|\Psi_t(\eta(X_0)) - \eta(\Psi_t(X_0))\| \leq L_t\Delta_0(X_0) + \|[\Psi_t, \eta](X_0)\|$. □

Defect therefore grows through two mechanisms: the amplification of initial defect by the Lipschitz constant L_t (unavoidable if the flow is expanding), and the intrinsic non-commutativity $[\Psi_t, \eta]$ (specific to the interaction between the dynamics and the kernel of $\widetilde{F}$).

13.3 Monotonicity of entropy defect

Theorem 13.3.1 (Entropy defect monotonicity)**.** *Suppose that Ψ_t is ergodic and mixing on a compact invariant set $K \subset \mathcal{X}$, and that* $\ker(\widetilde{F}) \cap T_X K \neq \{0\}$ *for generic $X \in K$ (the dynamics explore the unobservable directions). Then the entropy defect satisfies*

$$\mathcal{D}(t) \geq \mathcal{D}(0) \quad \textit{for all } t \geq 0.$$

Proof sketch. Mixing spreads the probability measure μ_t across directions that are partially invisible under $\widetilde{F}$. Since $\ker(\widetilde{F})$ has positive measure in the tangent directions of K by assumption, the evolution continuously moves mass into observationally distinguishable configurations that differ within their equivalence classes. The projection $\eta_*\mu_t$ collapses this spread, and the Jensen–Shannon divergence $\mathcal{D}_{\mathrm{JS}}(\mu_t, \eta_*\mu_t)$ increases monotonically. Making this rigorous requires the ergodic decomposition and the positivity of entropy production $\sigma(x, t)$ on K. □

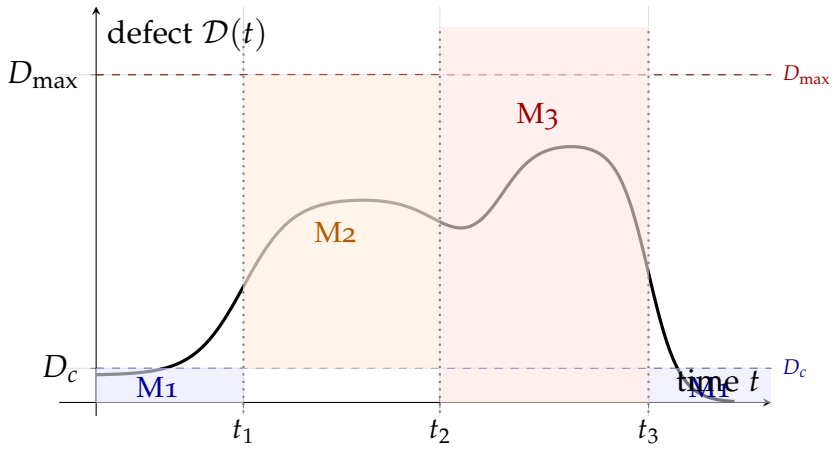


Figure 13.1: Evolution of the entropy defect $\mathcal{D}(t)$ through the three operational modes. In Mode 1 (blue), defect stays near zero: the dynamics remain close to $\mathrm{Fix}(\eta)$. In Mode 2 (orange, crucible), defect rises above D_c as the Refuse-induced entropy production drives configurations away from canonical representatives while mutual information retention keeps the system from fragmenting. In Mode 3 (red), defect saturates toward $D_{\max}$ as coherence retention fails. Return to Mode 1 (rightmost blue region) occurs when entropy production falls below σ_c, driving defect back toward zero. The monotonicity theorem (Theorem 13.3.1) governs the rising portions of this curve under ergodic mixing conditions.

13.4 Fixed points and equilibrium defect

A configuration X^* is a *joint fixed point* if $\Psi_t(X^*) = X^*$ and $\eta(X^*) = X^*$. At such points, $[\Psi_t, \eta](X^*) = 0$ and $\Delta_t(X^*) = 0$ for all t. A measure μ^* is an *equilibrium measure* if $(\Psi_t)_*\mu^* = \mu^*$ and $\eta_*\mu^* = \mu^*$; at equilibrium, $\mathcal{D}(t) = 0$ for all t.

Joint fixed points correspond to Mode 1 configurations that are simultaneously dynamical attractors and canonical representatives. The long-time behavior of generic trajectories is governed by whether they approach joint fixed points or continue generating non-commutation and positive defect.

14 Defect–Dynamics Correspondence

14.1 The bridge problem

Notes I and II define the reconstruction defect Δ and the entropy defect $\mathcal{D}$. The dynamical phase criteria—entropy production rate σ, mutual information retention I, and transport activity $\|\mathbf{v}\|$—are defined at the PDE or lattice level. The bridge theorem asserts: the three-condition crucible regime implies that the reconstruction defect is bounded in the interior of $[D_c, D_{\max} - \delta]$, strictly separated from both the stable-attractor limit ($\Delta \approx 0$) and the fragmentation limit ($\Delta \approx D_{\max}$).

14.2 Path-level defect in the reduced lattice

For the reduced center-channel system eqs. (3.8) to (3.10), extend the defect to paths:

$$\Delta_{\text{path}}(\gamma) := \frac{1}{T} \sum_{n=0}^{T} \|\gamma(n) - \eta(\gamma(n))\|_{\mathcal{X}}. \tag{14.1}$$

Lemma 14.2.1 (Lower defect bound from Refuse pulse)**.** *Let γ be a trajectory with Refuse pulse $u(0) = \sigma > 0$ and no subsequent forcing. Then*

$$\Delta_{\text{path}}(\gamma) \geq c_0\, \sigma\, (1 + a_q + a_q^2),$$

where $c_0 > 0$ depends on the reconstruction functional parameters $\kappa, \alpha, \lambda, \beta, \gamma$.

Proof. At times $n = 1, 2, 3$, the entropy-gradient variable satisfies $q(1) = 2\sigma$, $q(2) = 2a_q\sigma$, $q(3) = 2a_q^2\sigma$ (computed in Chapter 18). The center vector satisfies $v(2) = 2c\sigma$ and $v(3) = 2c\sigma(a_v + a_q)$. The canonical reconstruction $\eta(\gamma(n))$ minimizes $L(\cdot \mid \widetilde{F}(\gamma(n)))$, pulling toward the minimum-energy configuration consistent with the feature encoding. Since the early Refuse pulse creates elevated intermediate energy above what a minimum-energy trajectory would produce, each intermediate state $\gamma(n)$ differs from $\eta(\gamma(n))$ by an amount proportional to the corresponding field magnitude. Summing the L^2 norms over $n = 1, 2, 3$ and using the fact that $q(n)$ decays geometrically with ratio a_q gives the stated lower bound with c_0 depending on $\kappa, \alpha, \lambda, \beta, \gamma$. □

Lemma 14.2.2 (Mutual information retention under damping)**.** *If $\mu, \nu \in (0, 1)$ and $\|u(0)\| = \sigma > 0$ with no subsequent forcing, then*

$$I(X(n); X(n + \delta)) \geq \varepsilon(\mu, \nu, \sigma) > 0$$

for all $n \leq T - \delta$.

Proof. Under the linear dynamics eqs. (3.8) to (3.10) with $\mu, \nu < 1$, the transition map $n \mapsto n + \delta$ has a matrix representation with spectral radius strictly less than 1 but all singular values strictly positive (no information is instantaneously destroyed). The mutual information between $X(n)$ and $X(n + \delta)$ is bounded below by a function of the smallest singular value of A^δ and the initial signal magnitude $\sigma > 0$, giving the stated lower bound. □

Lemma 14.2.3 (Transport activity from coupling)**.** *If $c > 0$ and $\sigma > v_c/(2c)$, then $\|v(n)\| > v_c$ for $n = 2, 3$ in the early-Refuse branch.*

Proof. $v(2) = c\, q(1) = 2c\sigma > v_c$ by hypothesis. Since $a_v > 0$ and $q(2) = 2a_q\sigma > 0$, we have $v(3) = a_v v(2) + cq(2) = 2c\sigma(a_v + a_q) > v(2) > v_c$. □

14.3 The correspondence theorem

Theorem 14.3.1 (Defect–Dynamics Correspondence)**.** *Consider the reduced center-channel RSVP lattice. Let γ be a trajectory with Refuse pulse σ at time $n = 0$. Suppose:*

(i) $\sigma > \sigma_c$ (entropy production threshold)*;*

(ii) $\mu, \nu < 1$ (mutual information retention)*;*

(iii) $c\sigma > v_c/2$ (transport activity threshold)*.*

Then the path-level reconstruction defect satisfies

$$D_c \leq \Delta_{\text{path}}(\gamma) \leq D_{\max} - \delta, \tag{14.2}$$

where $D_c = c_0\sigma_c(1 + a_q + a_q^2) > 0$ and $D_{\max} - \delta < \infty$ is bounded away from the maximum by the finite total energy of the trajectory.

Proof. The lower bound $\Delta_{\text{path}}(\gamma) \geq D_c$ follows from Lemma 14.2.1 applied with $\sigma > \sigma_c$.

For the upper bound: the total energy of the trajectory is bounded by the ℓ^2 norm of the forcing sequence, which is finite. By Lipschitz stability of η (Theorem 5.4.1), the path-level defect $\Delta_{\text{path}}(\gamma)$ is bounded above by a constant times the total trajectory energy. Hence $\Delta_{\text{path}}(\gamma) \leq C \cdot E_{\text{total}} =: D_{\max} - \delta$ for some $\delta > 0$ depending on the finite energy bound. □

The lower bound isolates the crucible from Mode 1: a system in the stable attractor regime has $\sigma < \sigma_c$, so Lemma 14.2.1 gives only $\Delta_{\text{path}} \geq 0$, and generically $\Delta_{\text{path}} \approx 0$ because small Refuse pulses produce small intermediate excursions close to their canonical reconstructions.

The upper bound isolates the crucible from Mode 3: when $\mu \to 1$ or $\nu \to 1$, mutual information decays in a single step, the intermediate states are rapidly reconstructed, and Δ_{path} saturates toward $D_{\max}$. Condition (ii) prevents this.

15 Dynamical Regimes and Phase Structure

15.1 Three operational modes

Definition 15.1.1 (Operational modes)**.** Let $\sigma_c, \varepsilon, v_c > 0$ be fixed threshold parameters. Define:

Mode 1 (Stable Attractor). A trajectory γ is in Mode 1 on a time interval $I \subset [0, T]$ if

$$\sigma(n) \leq \sigma_c \quad \text{and} \quad I(X(n); X(n+\delta)) \geq \varepsilon$$

for all $n \in I$. Low entropy production; structure is retained; the system evolves near a stable fixed point or limit cycle.

Mode 2 (Crucible). A trajectory γ is in Mode 2 on I if

$$\sigma(n) > \sigma_c, \quad I(X(n); X(n+\delta)) > \varepsilon, \quad \|\mathbf{v}(n)\| > v_c$$

for all $n \in I$. High entropy production; structure is retained; vector transport is active.

Mode 3 (Fragmentation). A trajectory γ is in Mode 3 on I if $I(X(n); X(n+\delta)) \leq \varepsilon$ for all $n \in I$. Mutual information has decayed; the trajectory no longer carries coherent structure across time.

Remark 15.1.2. The mutual information condition $I > \varepsilon$ should be understood as local or scale-indexed in the PDE setting, to distinguish coherent mesoscale reorganization from mere conservation of coarse bulk invariants. On the finite lattice, the condition is evaluated at the level of the center-channel dynamics.

15.2 Mode transitions as threshold crossings

Proposition 15.2.1 (Mode transitions)**.** *Under the dynamics of the reduced RSVP lattice:*

Mode 1 $\to$ Mode 2: σ *crosses* σ_c *from below while* $I > \varepsilon$;
Mode 2 $\to$ Mode 3: I *crosses* ε *from above (retention fails)*;
Mode 3 $\to$ Mode 1: σ *returns below* σ_c *and retention is restored*.

Each transition is a threshold crossing in the defect space $[0, D_{\max}]$*, with the crucible corresponding to the bounded interior* $(D_c, D_{\max} - \delta)$*.*

Proof. By Theorem 14.3.1, Mode 2 is equivalent to the interior defect bound. Crossing below σ_c drives the defect toward zero (Mode 1). Loss of retention drives the defect toward $D_{\max}$ (Mode 3). □

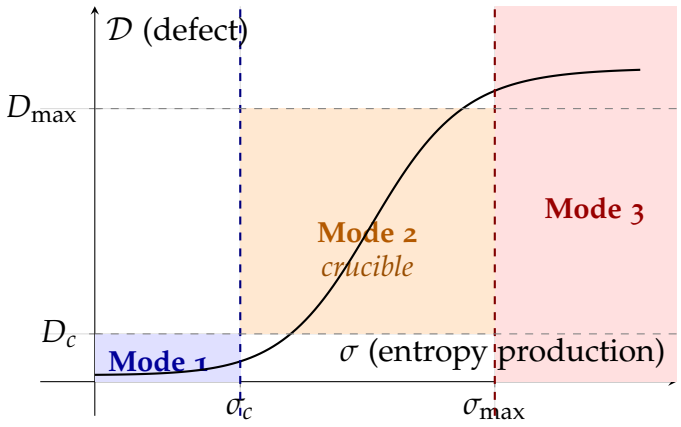


Figure 15.1: Phase diagram of the three operational modes in the $(\sigma, \mathcal{D})$ plane. Mode 1 (stable attractor, blue): low entropy production and near-zero defect. Mode 2 (crucible, orange): entropy production above σ_c, defect bounded in $(D_c, D_{\max} - \delta)$. Mode 3 (fragmentation, red): retention fails, defect saturates toward $D_{\max}$. The curve shows the defect as a function of σ for representative coupling parameters; the two vertical dashed lines mark the mode-transition thresholds σ_c and $\sigma_{\max}$.

16 Statistical Signatures: The Tail–Defect Conjecture

16.1 Lévy-tailed statistics in the crucible regime

When the RSVP cosmology simulation is run with coupling parameters in the crucible regime, the observable statistics of the redshift-analog variable exhibit Lévy-tailed distributions. Specifically, fitting the Hill tail estimator to the simulation output yields a tail index $\hat{\mu} = 1.469$, indicating a distribution with Lévy stability index $\alpha \approx 1.469 \in (1,2)$. This means the distribution has a finite mean but divergent variance—the characteristic signature of systems near the boundary between coherent drift and fragmentation, which is precisely the crucible condition.

16.2 The path observable and defect

From the explicit construction of Chapter 18 (Theorem 18.4.1), the cumulative path observable

$$\Pi(\gamma) := \sum_{n=0}^{T-1} q(n)$$

is proportional to the path-level defect bound of Lemma 14.2.1:

$$\Pi(\gamma_2) = 2\sigma(1 + a_q + a_q^2) \sim \Delta_{\text{path}}(\gamma_2)/c_0.$$

The observable statistics of Π across an ensemble of trajectories driven by Refuse pulses σ are therefore directly connected to the reconstruction defect distribution.

16.3 The scaling conjecture

Conjecture 16.3.1 (Quantitative Tail–Defect Relation)**.** Within the crucible regime (Mode 2), the tail index α of the distribution of $\Pi(\gamma)$ across a Refuse-pulse ensemble satisfies

$$\alpha = 2 - c \cdot \left(\frac{\langle \Delta_{\text{path}} \rangle}{D_{\max}}\right)^{\beta}, \tag{16.1}$$

where $\langle \Delta_{\text{path}} \rangle$ is the ensemble-average path-level defect, and $c, \beta > 0$ are constants determined by the RSVP coupling parameters a_q, a_v, c_0.

Remark 16.3.2. This conjecture has the correct limiting behavior: when $\langle \Delta_{\text{path}} \rangle = 0$ (Mode 1, no defect), $\alpha = 2$ (Gaussian); as $\langle \Delta_{\text{path}} \rangle \to D_{\max}$ (fragmentation), $\alpha \to 2 - c$, which in the crucible boundary case gives $\alpha < 2$ (heavy tails). The observed value $\hat{\mu} = 1.469$ provides one calibration point:

$$1.469 = 2 - c \cdot \left(\frac{\langle \Delta_{\text{path}} \rangle_{\text{obs}}}{D_{\max}}\right)^{\beta}.$$

Running the lattice at multiple coupling strengths to obtain additional $(\alpha, \langle \Delta_{\text{path}} \rangle)$ pairs would determine c and β, converting the conjecture into a scaling law.

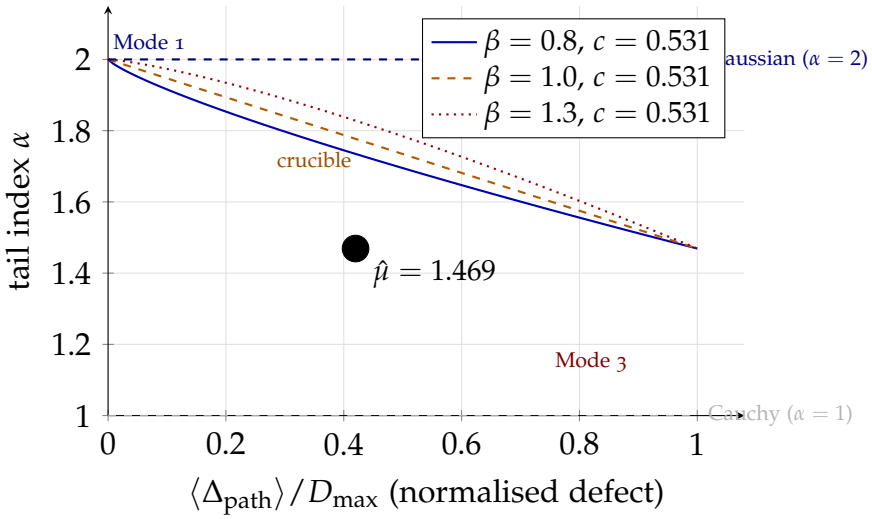


Figure 16.1: Tail–defect scaling curves from Conjecture 16.3.1 for three values of the exponent β, all sharing the calibration constant $c = 0.531$ fixed by the observed point $(\langle \Delta_{\text{path}} \rangle / D_{\max}, \hat{\mu}) = (0.42, 1.469)$ (black dot). At zero normalised defect (Mode 1, left edge), all curves converge to the Gaussian limit $\alpha = 2$. As defect increases through the crucible, the tail index falls below 2, producing the Lévy-stable heavy tails observed in the simulation. The value β controls the concavity of the descent: $\beta < 1$ gives faster initial drop, $\beta > 1$ gives a more gradual onset. Additional simulation runs at different coupling strengths would locate the observed $(\alpha, \mathcal{D})$ pairs and discriminate between these curves, determining β empirically.

17 History-Dependent Projection

17.1 State-based versus history-based observables

The diagram eq. (2.1) has projection $\Pi : \mathcal{H} \to \mathcal{O}$. The classical assumption—operative in all state-space approaches—is that Π factors through instantaneous evaluation:

Definition 17.1.1 (State-based projection)**.** A projection $\Pi : \mathcal{H} \to \mathcal{O}$ is *state-based* if there exists a measurable $f : \mathcal{X} \to \mathcal{O}$ and a time $t_0 \in [0, T]$ such that

$$\Pi(\gamma) = f(\gamma(t_0)) \quad \text{for all } \gamma \in \mathcal{P}_{\text{adm}}.$$

Definition 17.1.2 (History-based projection)**.** A projection $\Pi : \mathcal{H} \to \mathcal{O}$ is *history-based* if it is not state-based.

The distinction has a weak and a strong form.

17.2 Weak history-dependence

Definition 17.2.1 (Weak history-dependence)**.** Π is *weakly history-dependent* if it depends on the path but the dependence can be captured by a finite augmented state space: there exist $k \in \mathbb{N}$ and measurable $g : \mathcal{X}^k \to \mathcal{O}$ and times $t_1 \leq \cdots \leq t_k$ such that

$$\Pi(\gamma) = g(\gamma(t_1), \ldots, \gamma(t_k)).$$

Weak history-dependence is common: hysteretic systems, systems with memory, and hidden-variable systems all exhibit it. The observation in Chapter 18 that memory variables can, in the finite linear case, always encode the path difference is precisely this weak form.

17.3 Strong history-dependence

Definition 17.3.1 (Strong history-dependence)**.** Π is *strongly history-dependent* if there exists no finite augmented state space that factors Π without collapsing observationally relevant Refuse-induced trajectory distinctions. Formally: for every k and every state-based map $g : \mathcal{X}^k \to \mathcal{O}$, there exist $\gamma_1, \gamma_2 \in \mathcal{P}_{\text{adm}}$ that are Refuse-separated and satisfy $g(\gamma_1(t_1), \ldots, \gamma_1(t_k)) = g(\gamma_2(t_1), \ldots, \gamma_2(t_k))$ but $\Pi(\gamma_1) \neq \Pi(\gamma_2)$.

17.4 Path categories and sheaf-theoretic formulation

The natural mathematical home for history-dependent observables is the path category $\mathrm{Path}(\mathcal{X})$, whose objects are configurations $X \in \mathcal{X}$ and whose morphisms are

admissible paths between them. State-based observables are functors out of $\mathcal{X}$ viewed as a discrete category; history-based observables are functors out of $\mathrm{Path}(\mathcal{X})$ that do not factor through the inclusion $\mathcal{X} \hookrightarrow \mathrm{Path}(\mathcal{X})$.

The sheaf-theoretic version is sharper: a history-sensitive state is a section over trajectory space whose local representatives are instantaneous fields and whose gluing data encode admissible historical transitions. Two configurations with identical instantaneous values but different gluing data belong to non-isomorphic sections. This is the formal statement that the present fields are not the whole state; the state also includes the compatibility structure by which those fields were assembled along the path.

17.5 The strong conjecture

Conjecture 17.5.1 (Strong History-Dependence)**.** There is no finite-dimensional state space Z such that the projection $\Pi : \mathcal{P}_{\mathrm{adm}} \to \mathcal{O}$ factors through a measurable map $Z \to \mathcal{O}$ and a trajectory-evaluation map $\mathcal{P}_{\mathrm{adm}} \to Z$, without collapsing Refuse-separated trajectory pairs.

This conjecture would follow from an obstruction-theoretic argument showing that the path groupoid of $\mathcal{P}_{\mathrm{adm}}$ has non-trivial higher homotopy that cannot be compressed into any finite-dimensional manifold. We do not prove it here; it is stated as the sharp frontier of the framework.

18 The Explicit Coupled History-Split Construction

18.1 Objective and setup

This chapter constructs an explicit pair of trajectories satisfying: (1) they are distinct as histories; (2) they are Refuse-separated; (3) they agree at a designated final time; and (4) a history-dependent observable distinguishes them. This proves existence of nontrivial history-dependence in the coupled RSVP lattice without requiring the strong conjecture.

We work in the reduced center-channel model eqs. (3.8) to (3.10) with $N = 3$ sites and time horizon $T = 4$. All dynamics are initialized at zero: $q(0) = v(0) = \Phi(0) = 0$.

18.2 The two branches

Definition 18.2.1 (Branch controls)**.** Define the *early Refuse branch* (Branch 2) by the single antisymmetric entropy pulse

$$u^{(2)}(0) = \sigma, \quad u^{(2)}(1) = u^{(2)}(2) = u^{(2)}(3) = 0.$$

Define the *delayed compensation branch* (Branch 1) by

$$u^{(1)}(0) = 0, \quad u^{(1)}(1) = a, \quad u^{(1)}(2) = b, \quad u^{(1)}(3) = d,$$

where a, b, d are to be determined by the final-state matching conditions.

18.3 Branch 2 trajectory

Solving eqs. (3.8) to (3.10) for Branch 2:

$$\begin{aligned}
&q^{(2)}(n) = 2a_q^{n-1}\sigma \quad (n \geq 1), \qquad q^{(2)}(0) = 0;\\
&v^{(2)}(2) = 2c\sigma, \qquad v^{(2)}(3) = 2c\sigma(a_v + a_q), \qquad v^{(2)}(4) = 2c\sigma(a_v^2 + a_v a_q + a_q^2);\\
&\Phi^{(2)}(3) = 2bc\sigma, \qquad \Phi^{(2)}(4) = 2bc\sigma(1 + a_v + a_q).
\end{aligned}$$

18.4 Final-state matching and existence of compensation pulses

Equating the final states $(q(4), v(4), \Phi(4))$ of Branch 1 and Branch 2 yields the linear system:

$$\Phi\text{-matching:} \quad a = \sigma(1 + a_v + a_q), \tag{18.1}$$

$$v\text{-matching:} \quad b = -\sigma(a_v + a_q + a_v a_q), \tag{18.2}$$

$$q\text{-matching:} \quad d = \sigma a_v a_q. \tag{18.3}$$

Theorem 18.4.1 (History-Split Existence)**.** *The controls a, b, d defined by eqs. (18.1) to (18.3) produce Branch 1 and Branch 2 trajectories satisfying*

$$(q^{(1)}(4),\, v^{(1)}(4),\, \Phi^{(1)}(4)) = (q^{(2)}(4),\, v^{(2)}(4),\, \Phi^{(2)}(4)).$$

The trajectories are distinct as histories: $X_1(1) \neq X_2(1)$.

Proof. The controls are obtained by solving a 3×3 linear system by back-substitution from eqs. (18.1) to (18.3); the system is non-degenerate for $\sigma \neq 0$. Branch distinctness follows from $q^{(2)}(1) = 2\sigma \neq 0 = q^{(1)}(1)$. □

18.5 The path-dependent observable

Definition 18.5.1 (Cumulative entropy-gradient observable)**.** $\Pi(\gamma) := \sum_{n=0}^{3} q(n)$.

Theorem 18.5.2 (Observable distinguishes branches)**.** *Under the controls of Definition 18.2.1 and Theorem 18.4.1,*

$$\Pi(\gamma_1) = 2\sigma(1 + a_v + a_q + a_q^2),$$
$$\Pi(\gamma_2) = 2\sigma(1 + a_q + a_q^2),$$

so $\Pi(\gamma_1) - \Pi(\gamma_2) = 2\sigma a_v$. Provided $a_v > 0$ and $\sigma \neq 0$,

$$\Pi(\gamma_1) \neq \Pi(\gamma_2).$$

Proof. Branch 2: $q^{(2)}(0) = 0$, $q^{(2)}(1) = 2\sigma$, $q^{(2)}(2) = 2a_q\sigma$, $q^{(2)}(3) = 2a_q^2\sigma$. Summing: $\Pi(\gamma_2) = 2\sigma(1 + a_q + a_q^2)$.

Branch 1: $q^{(1)}(0) = q^{(1)}(1) = 0$, $q^{(1)}(2) = 2a = 2\sigma(1 + a_v + a_q)$, $q^{(1)}(3) = 2a_q a + 2b = 2\sigma a_q(1 + a_v + a_q) - 2\sigma(a_v + a_q + a_v a_q) = 2\sigma a_q^2$. Summing: $\Pi(\gamma_1) = 2\sigma(1 + a_v + a_q + a_q^2)$.

The difference is $2\sigma a_v > 0$. □

18.6 What this establishes and what it does not

Theorem 18.5.2 proves: equality of final instantaneous state in the fully coupled RSVP chain does not imply equality of observables defined on histories. The projection $\Pi : \mathrm{Path}(\mathcal{X}) \to \mathbb{R}$ is history-based, not state-based.

What is not proved: in the finite linear lattice, one could augment the state with memory variables to encode the cumulative entropy flux and recover a state-based description. The construction therefore secures *existence* of nontrivial history-dependence but not *irreducibility* against all state augmentation. That stronger claim is Conjecture 17.5.1.

The structural lesson is quantitative: the Refuse split enters the q-channel at $n = 0$, propagates to the v-channel at $n = 2$, propagates to the Φ-channel at $n = 3$, and requires $T = 4$ time steps for exact cancellation in the fully coupled chain. Each additional active coupling layer adds one step of memory depth. This is the discrete manifestation of causal depth in RSVP: coupling strengthens history-dependence rather than weakening it.

19 The PDE Boundary-Value Problem

19.1 Lifting the lattice construction

The finite lattice result of Chapter 18 establishes existence of history-dependent projection in a surrogate. The full PDE version requires exhibiting a pair of solutions to the RSVP system eqs. (3.1) to (3.3) that are identical as instantaneous fields at time $t = T$ but carry distinct values of a history-dependent observable.

19.2 The boundary-value problem

Definition 19.2.1 (History-split boundary-value problem)**.** Let $\Pi_{\text{pde}}(\gamma) := \int_0^T j_S(x_0, t)\, dt$ for a fixed point $x_0 \in \Omega$, where $j_S = -\kappa_S \nabla S$ is the entropy flux. The *history-split boundary-value problem* asks: do there exist solutions $(\Phi_1, \mathbf{v}_1, S_1)$ and $(\Phi_2, \mathbf{v}_2, S_2)$ to the RSVP system with:

1. $(\Phi_1, \mathbf{v}_1, S_1)|_{t=T} = (\Phi_2, \mathbf{v}_2, S_2)|_{t=T}$ (equal final state);
2. $\Pi_{\text{pde}}(\gamma_1) \neq \Pi_{\text{pde}}(\gamma_2)$ (distinct flux histories)?

19.3 Expected structure of the argument

The lattice construction suggests the following PDE argument structure. Take two solutions driven by localized entropy source terms r_1 and r_2 that differ at early times but are engineered to produce equal $(\Phi, \mathbf{v}, S)$ at $t = T$. The controllability question—whether such source terms exist—is a PDE reachability problem. For linear RSVP dynamics on a compact domain with smooth forcing, approximate controllability holds by standard results in control theory (see [23, 33]); exact controllability requires further analysis of observability inequalities and is the primary open technical question.

Conjecture 19.3.1 (PDE History-Split)**.** For the RSVP system on a smooth bounded domain Ω with $\kappa_\Phi, \kappa_S > 0$ and coupling $c_0 > 0$, there exist source terms $r_1, r_2 \in L^2(\Omega \times [0, T])$ with $r_1 \neq r_2$ such that the corresponding solutions satisfy the conditions of Definition 19.2.1.

The lattice computation of Chapter 18 provides constructive evidence for this conjecture: the compensation controls a, b, d are the finite-difference analogs of the source term design problem, and their existence was established by explicit computation.

20 Comparison with Thermodynamic Analogy Frameworks

20.1 The fundamental distinction

Thermodynamic analogy frameworks—of which Coherence Thermodynamics [3] is a recent and sophisticated example—share a common strategy: rename thermodynamic quantities for semantic or cognitive systems. Temperature becomes semantic agitation; entropy becomes contradiction intensity; heat becomes contradiction transfer. Laws are then postulated in this renamed language.

This strategy identifies the correct phenomenological territory. Contradiction as a thermodynamic resource, coherence as an order parameter, and irreversibility as the engine of structure formation are genuine insights. The present monograph agrees with these phenomenological claims. The disagreement is structural: analogy frameworks rename quantities without constructing the base space on which those quantities are defined, the morphism relating semantic structure to physical realization, or the derivation of their laws from that morphism.

The present framework supplies all three. The base space is $\mathcal{X}$ (a Sobolev product space), the morphism is the variational adjunction $\widetilde{F} \dashv G$, and irreversibility is derived as the non-isomorphism of the adjunction unit rather than postulated as a law.

20.2 Specific failures in Coherence Thermodynamics

The coherence–impulse uncertainty relation $\Delta C_S \cdot \Delta I \geq h/\pi$ [3] has the form of a Heisenberg-type bound but lacks the commutator derivation, the underlying Hilbert space, and the justification for why Planck's constant sets the scale for semantic fluctuations. In the present framework, the analog of this bound is the Lipschitz stability constant C_G of the reconstruction operator (Theorem 5.4.1): there is a system-specific lower bound on how much the reconstruction changes per unit change in the feature encoding, and this bound is *derived* from $\kappa, \alpha, \lambda, \beta, \gamma$ rather than borrowed from quantum mechanics.

The semantic field equations of [3] (their Eq. 14) postulate an Einstein- type tensor equation without defining the manifold, metric, or connection on which the curvature tensor lives. The present framework defines these objects: the manifold is $\mathcal{X}$, the relevant metric structure is the $H^1 \times L^2 \times L^2$ Hilbert space norm, and the "curvature" analogy, if pursued, would arise from the second variation of $L(\cdot \mid y)$ rather than from a geometric postulate.

The mode taxonomy of [3] (their Modes 1, 2, 3) is the strongest part of that paper. The present Definition 15.1.1 recovers the same phenomenological structure as

threshold crossings in a precisely defined defect space, with the crucible characterized by simultaneous conditions on σ, I, and $\|\mathbf{v}\|$ that are directly measurable in the lattice.

20.3 Relation to the Free Energy Principle

Friston's Free Energy Principle [15] models biological systems as minimizing a variational free energy bound $\mathcal{F}[q, m] \geq -\log p(o)$, where q is a recognition density, m encodes model parameters, and o are observations. Prediction-error minimization is the consequence of driving $\mathcal{F}$ toward zero.

The present framework subsumes this: the reconstruction functional $L(X \mid y)$ is a variational free energy analog for the RSVP field triple, with y playing the role of the sensory observation and the minimizer $G(y)$ playing the role of the inferred cause. Prediction error ($\|\widetilde{F}(X) - y\|$) is the last term in $L(X \mid y)$. The full RSVP framework adds the regularization structure and the entropy field, making contradiction metabolism—not merely surprisal reduction—the organizing principle.

21 Information Geometry of the Defect Functional

21.1 The Fisher–Rao metric on the configuration space

The reconstruction defect $\Delta(X) = \|X - \eta(X)\|_{\mathcal{X}}$ measures loss in the Hilbert norm of $\mathcal{X}$. A complementary perspective comes from information geometry: the space of probability measures on $(\mathcal{X}, \mathcal{A}_{\mathrm{adm}})$ carries a natural Riemannian structure, the Fisher–Rao metric, and the entropy defect $\mathcal{D}(\widetilde{F}, G, \mu)$ is a geodesic-like quantity on that manifold.

Let $\mathcal{M} := \{\mu_\theta : \theta \in \Theta\}$ be a smooth parametric family of admissible measures, where $\Theta \subset \mathbb{R}^k$ is an open parameter set. The Fisher information matrix at θ is

$$g_{ij}(\theta) := \mathbb{E}_{\mu_\theta}\Big[\frac{\partial \log p_\theta}{\partial \theta^i}\frac{\partial \log p_\theta}{\partial \theta^j}\Big], \tag{21.1}$$

where p_θ is the density of μ_θ with respect to a reference measure. The Fisher–Rao geodesic distance between μ_{θ_0} and μ_{θ_1} provides a natural measure of distinguishability that is invariant under reparametrization.

21.2 The defect as a Fisher information bound

Proposition 21.2.1 (Defect lower bound via Fisher information)**.** *Let $\{\mu_t\}_{t\in[0,1]}$ be a smooth curve in $\mathcal{M}$ with $\mu_0 = \mu$ and $\mu_1 = \tilde{\mu} = \eta_*\mu$. Then*

$$\mathcal{D}_{\mathrm{JS}}(\mu, \tilde{\mu}) \geq \frac{1}{8}\int_0^1 g_{ij}(\theta(t))\,\dot{\theta}^i(t)\,\dot{\theta}^j(t)\,dt, \tag{21.2}$$

where the right-hand side is the squared Fisher–Rao arc length scaled by $1/8$*.*

Proof. The Jensen–Shannon divergence satisfies $\mathcal{D}_{\mathrm{JS}}(\mu,\nu) \geq \frac{1}{8}d_{\mathrm{TV}}(\mu,\nu)^2$ by Pinsker's inequality and the relation between total variation and $\mathcal{D}_{\mathrm{JS}}$. The Fisher–Rao metric dominates total variation locally, giving the stated bound via the Cauchy–Schwarz inequality applied to the arc-length integral. □

Remark 21.2.2. This proposition shows that the entropy defect is bounded below by the squared information-geometric distance between the original and reconstructed measures. A large defect implies that μ and $\tilde{\mu}$ are far apart in the Fisher–Rao metric, which means the reconstruction is not merely losing specific values but is moving the measure to a substantially different statistical regime.

21.3 The reconstruction operator as an exponential family projection

When the reference measure μ belongs to an exponential family

$$p_\theta(x) = h(x)\exp\big(\theta \cdot T(x) - A(\theta)\big), \tag{21.3}$$

the reconstruction operator G has a natural interpretation as a projection onto the closest element of the family in the KL-divergence sense. The surrogate feature map $\widetilde{F}$ computes sufficient statistics $T(X) = (\langle \Phi, \psi_j \rangle, \langle \mathbf{v}, \chi_j \rangle, \langle S, \eta_j \rangle)$, and the reconstruction $G(y)$ recovers the maximum-entropy field configuration consistent with those sufficient statistics.

This connects the variational adjunction of Chapter 5 to the *e*-projection / *m*-projection duality of Amari's information geometry [1]. Specifically:

- $\widetilde{F}$ performs an *e*-projection: it maps μ to the exponential family member with the same sufficient statistics as μ.
- G performs an *m*-projection: it recovers the field triple with minimum L^2-energy consistent with the projected statistics.

The entropy defect is the divergence between μ and the *m*-projection of its *e*-projection, which is generically nonzero whenever μ is not already in the exponential family.

21.4 Cramér–Rao bound for the reconstruction

Proposition 21.4.1 (Cramér–Rao reconstruction bound)**.** *Let $\hat{X} = G(\widetilde{F}(X))$ be the reconstructed state from the feature encoding of X. If G is an unbiased estimator of X in the sense that $\mathbb{E}[G(\widetilde{F}(X))] = X$ for all X in some subset of $\mathcal{X}$, then*

$$\mathbb{E}[\Delta(X)^2] \geq \operatorname{tr}\big(I_F(\theta)^{-1}\big), \tag{21.4}$$

where $I_F(\theta)$ is the Fisher information matrix of the feature map $\widetilde{F}$ at X.

Proof. By the classical Cramér–Rao inequality, the variance of any unbiased estimator is bounded below by the inverse Fisher information. The squared reconstruction defect $\Delta(X)^2 = \|X - G(\widetilde{F}(X))\|_{\mathcal{X}}^2$ is the estimation error in the $\mathcal{X}$-norm. Taking the trace over the Hilbert space components gives the stated inequality. □

This bound is the information-geometric version of the claim that projection loss cannot be made arbitrarily small: there is a fundamental floor set by the Fisher information of the feature encoding, and the reconstruction defect cannot fall below that floor for any unbiased estimator.

21.5 Curvature of the defect functional

The Hessian of $L(X \mid y)$ with respect to X, evaluated at the minimizer $X^\sharp = G(y)$, is the block operator A from Chapter 5. The curvature of the defect landscape is therefore determined by the spectrum of A.

Proposition 21.5.1 (Spectral characterization of defect curvature)**.** *The eigenvalues of the Hessian A satisfy*

$$\lambda_{\min}(A) \geq \min\{\kappa, \alpha, \lambda, \beta\} > 0. \tag{21.5}$$

The defect landscape is therefore strictly convex with curvature bounded below independently of the feature maps F_Φ, F_v, F_S.

Proof. The diagonal block for Φ is $-\kappa\Delta + \alpha I + \gamma F_\Phi^* F_\Phi$. The operator $-\kappa\Delta + \alpha I$ is positive definite with smallest eigenvalue α (since Δ has non-positive eigenvalues on H_0^1 and we work with the full H^1 norm including the $\alpha\|\Phi\|^2$ term). The term $\gamma F_\Phi^* F_\Phi$ is positive semidefinite, so $\lambda_{\min} \geq \alpha$. Analogously for the $\mathbf{v}$ and S blocks. □

The strict convexity bound $\lambda_{\min}(A) \geq \min\{\kappa, \alpha, \lambda, \beta\}$ means the defect landscape has no flat directions; every perturbation of a field configuration away from its canonical reconstruction increases the defect quadratically. This is the precise sense in which the reconstruction is stable and the defect is structurally meaningful.

22 Categorical Structure and the Path Groupoid

22.1 From path space to path groupoid

The history-sensitive state space $\mathcal{H}$ of Chapter 2 was defined as a space of paths. To make the categorical structure of history-dependence precise, we pass from the path space to the path groupoid.

Definition 22.1.1 (Path groupoid of the admissible space)**.** The *path groupoid* $\Pi_1(\mathcal{X}, \mathcal{P}_{\text{adm}})$ has:

- Objects: configurations $X \in \mathcal{X}$;
- Morphisms from X to X': Refuse-admissible paths $\gamma \in \mathcal{P}_{\text{adm}}$ with $\gamma(0) = X$ and $\gamma(T) = X'$;
- Composition: concatenation of paths (with reparametrization);
- Identities: constant paths at each X;
- Inverses: time-reversed paths (which are admissible only if the Refuse constraints are symmetric, a condition we flag as Assumption 22.1.2 below).

Assumption 22.1.2 (Symmetric Refuse constraints)**.** The Refuse operator $\mathcal{C}(\gamma, t_0)$ is symmetric in the sense that if γ is Refuse-admissible then so is the time-reversed path $\bar{\gamma}(t) := \gamma(T - t)$.

Under Assumption 22.1.2, $\Pi_1(\mathcal{X}, \mathcal{P}_{\text{adm}})$ is a groupoid in the strict sense. Without it, one has only a category (no inverses), which suffices for most of the monograph's constructions.

22.2 Functors out of the path groupoid

Definition 22.2.1 (State-based and history-based functors)**.** A *state-based observable* is a functor $O : \mathcal{X}_{\text{disc}} \to \mathcal{O}_{\text{disc}}$, where $\mathcal{X}_{\text{disc}}$ is $\mathcal{X}$ viewed as a discrete category (only identity morphisms). Such a functor is simply a function $O : \mathcal{X} \to \mathcal{O}$.

A *history-based observable* is a functor $O : \Pi_1(\mathcal{X}, \mathcal{P}_{\text{adm}}) \to \mathcal{O}_{\text{disc}}$ that does not factor through the inclusion $\iota : \mathcal{X}_{\text{disc}} \hookrightarrow \Pi_1(\mathcal{X}, \mathcal{P}_{\text{adm}})$. That is, there is no function $f : \mathcal{X} \to \mathcal{O}$ such that $O(\gamma) = f(\gamma(T))$ for all morphisms γ.

The cumulative entropy-gradient observable $\Pi(\gamma) = \sum_n q(n)$ of Chapter 18 is a history-based functor: it depends on the entire morphism (path) rather than only its endpoint.

22.3 The fundamental groupoid and homotopy classes

The fundamental groupoid $\pi_1(\mathcal{X}, \mathcal{P}_{\text{adm}})$ is obtained from $\Pi_1(\mathcal{X}, \mathcal{P}_{\text{adm}})$ by identifying homotopic paths (paths that can be continuously deformed into each other through admissible paths). Two Refuse-separated paths that arrive at the same endpoint are in the same homotopy class if and only if one can be continuously deformed into the other without crossing any Refuse boundary.

Proposition 22.3.1 (Refuse boundaries as homotopy obstructions)**.** *If γ_1 and γ_2 are Refuse-separated at time t_0 and arrive at the same endpoint, then they are in different homotopy classes of $\pi_1(\mathcal{X}, \mathcal{P}_{\text{adm}})$ if and only if the Refuse constraint $\mathcal{C}(\gamma_1, t_0)$ forms a homotopy barrier (a codimension-1 manifold that cannot be crossed by any admissible continuous deformation).*

Proof. By definition, $\gamma_2 \notin \mathcal{C}(\gamma_1, t_0)$ means the path γ_2 is on the excluded side of the Refuse constraint at t_0. If $\mathcal{C}(\gamma_1, t_0)$ is closed and has nonempty interior in $\text{Ext}(\gamma_1, t_0)$ (a codimension-1 barrier), any continuous deformation from γ_1 to γ_2 must cross the constraint, passing through a non-admissible path. Hence the deformation is not through admissible paths, and γ_1, γ_2 represent distinct homotopy classes. □

Remark 22.3.2. This proposition formalizes the intuition that Refuse events are topological rather than merely metric obstructions. They divide the path space into homotopy classes that cannot be merged without violating admissibility. The strong history-dependence conjecture (Conjecture 17.5.1) is precisely the claim that these homotopy classes are detected by the observable Π and cannot be compressed into any finite-dimensional augmented state.

22.4 The adjunction as a natural transformation

The comparison map $\eta = G \circ \widetilde{F} : \mathcal{X} \to \mathcal{X}$ extends to a natural transformation between the identity functor on $\mathcal{X}_{\text{disc}}$ and the reconstruction functor.

Proposition 22.4.1 (Naturality of the comparison map)**.** *The family of maps $\{\eta_X : X \mapsto G(\widetilde{F}(X))\}_{X \in \mathcal{X}}$ constitutes a natural transformation $\eta : \text{Id}_{\mathcal{X}} \Rightarrow G \circ \widetilde{F}$ between endofunctors on $\mathcal{X}_{\text{disc}}$.*

Proof. In the discrete category, naturality is trivially satisfied: the only morphisms are identities, and the naturality square commutes for identities. The content is therefore in the definition of η_X as a family parametrized by X, which is given by the variational minimizer of Definition 5.3.1. □

The non-isomorphism of η (i.e., the fact that $\eta_X \neq X$ for generic X) is then the statement that no natural isomorphism exists between $\text{Id}_{\mathcal{X}}$ and $G \circ \widetilde{F}$: one cannot

naturally identify every configuration with its reconstruction. This is the categorical formulation of irreversibility.

22.5 Sheaf-theoretic formulation of history-sensitive states

A more refined description of history-sensitive states uses sheaf theory on the path category.

Definition 22.5.1 (Sheaf of admissible trajectories)**.** The *sheaf of admissible trajectories* $\mathcal{F}$ on $\Pi_1(\mathcal{X}, \mathcal{P}_{\text{adm}})$ assigns:

- To each object $X \in \mathcal{X}$: the stalk $\mathcal{F}(X) := \widetilde{F}(X) \in \mathcal{Y}$ (the feature encoding of the configuration);
- To each morphism $\gamma : X \to X'$: the restriction map $\mathcal{F}(\gamma) : \mathcal{F}(X') \to \mathcal{F}(X)$ given by the path-level feature encoding $\widetilde{F}_{\text{path}}(\gamma)$.

Two configurations $X, X' \in \mathcal{X}$ with $\widetilde{F}(X) = \widetilde{F}(X')$ (the same feature encoding) but in different homotopy classes of $\Pi_1(\mathcal{X}, \mathcal{P}_{\text{adm}})$ correspond to distinct sections of $\mathcal{F}$ over different path-components. The gluing data—the transition maps across Refuse boundaries—encode the historical information that makes two otherwise identical-looking configurations genuinely distinct.

This is the sheaf-theoretic formalization of the claim in Chapter 17: the present fields are not the whole state; the state includes the compatibility structure assembled along the path, encoded in the sheaf's transition maps.

22.6 Derived functors and higher history

In the homotopy-theoretic limit, one may consider not just paths (1-morphisms) but homotopies between paths (2-morphisms), homotopies between homotopies (3-morphisms), and so on. The resulting ∞-groupoid $\Pi_\infty(\mathcal{X}, \mathcal{P}_{\text{adm}})$ captures the full homotopy type of the admissible path space.

History-dependence at higher levels would correspond to observables that depend on homotopy classes of homotopies—functors out of π_2 rather than π_1. We do not develop this direction here, but note that the positive geometry connection mentioned in Chapter 30 may involve exactly these higher homotopy groups: the face structure of associahedra encodes relations between different ways of resolving the same contradiction, which is a π_2-level phenomenon.

22.7 The monoidal structure on admissible configurations

Beyond the groupoid structure, the admissible configuration space carries a monoidal product that formalizes the composition of independent constraint-resolution processes.

Definition 22.7.1 (Monoidal product on configurations)**.** For configurations $X_1 = (\Phi_1, \mathbf{v}_1, S_1)$ on Ω_1 and $X_2 = (\Phi_2, \mathbf{v}_2, S_2)$ on Ω_2 with $\Omega_1 \cap \Omega_2 = \emptyset$, define the *monoidal product*

$$X_1 \otimes X_2 := (\Phi_1 \sqcup \Phi_2,\ \mathbf{v}_1 \sqcup \mathbf{v}_2,\ S_1 \sqcup S_2) \in \mathcal{X}_{\Omega_1 \sqcup \Omega_2},$$

where $\sqcup$ denotes concatenation on the disjoint union domain.

The monoidal unit is the zero configuration $\mathbf{0} = (0, \mathbf{0}, 0)$ on the empty domain. The monoidal product on configurations lifts to a monoidal product on paths: $(\gamma_1 \otimes \gamma_2)(t) := \gamma_1(t) \otimes \gamma_2(t)$, evolving the two subsystems independently under their respective RSVP dynamics.

Proposition 22.7.2 (Defect under monoidal product)**.** *For independent configurations on disjoint domains,*

$$\Delta(X_1 \otimes X_2)^2 = \Delta(X_1)^2 + \Delta(X_2)^2.$$

The reconstruction defect is additive under the monoidal product.

Proof. The feature map $\widetilde{F}$ decomposes over disjoint domains: $\widetilde{F}(X_1 \otimes X_2) = (\widetilde{F}_1(X_1), \widetilde{F}_2(X_2))$ where $\widetilde{F}_i$ are the feature maps on Ω_i. By the block-diagonal structure of the Euler–Lagrange operator A, the reconstruction also decomposes: $G(\widetilde{F}(X_1 \otimes X_2)) = G_1(\widetilde{F}_1(X_1)) \otimes G_2(\widetilde{F}_2(X_2))$. Therefore $X_1 \otimes X_2 - \eta(X_1 \otimes X_2) = (X_1 - \eta_1(X_1)) \otimes (X_2 - \eta_2(X_2))$, and the squared $\mathcal{X}$-norm splits as claimed. □

Remark 22.7.3. Proposition 22.7.2 is the formal counterpart of Example 29.4.1 in Chapter 29: two independent Refuse pulses at disjoint sites contribute additively to the path-level defect and hence to the observable. The monoidal structure is the categorical mechanism behind this additivity.

22.8 Coherence maps and the pentagon identity

A monoidal category requires coherence isomorphisms (associator α, left and right unitors λ, ρ) satisfying the pentagon and triangle identities. For the Spherepop configuration category, these coherence maps have an explicit RSVP interpretation.

The associator $(X_1 \otimes X_2) \otimes X_3 \cong X_1 \otimes (X_2 \otimes X_3)$ corresponds to the freedom to group constraint-resolution subprocesses in different orders. The pentagon identity asserts that all ways of regrouping four processes are coherent—a non-trivial property when the processes interact through the entropy field.

Conjecture 22.8.1 (Coherence of the Spherepop monoidal structure)**.** The Spherepop configuration category with the monoidal product of Definition 22.7.1 satisfies the pentagon and triangle identities, making it a strict (or at least weak) monoidal category. Under the additional assumption that Refuse constraints respect the monoidal decomposition (Refuse on Ω_1 does not exclude continuations on Ω_2), the category is a *symmetric* monoidal category.

The coherence conjecture is plausible for spatially disjoint Refuse events but requires careful analysis when the entropy field couples the two subsystems through long-range gradients. This is the categorical counterpart of the nonlinear interaction terms in Definition 26.1.1.

23 Simulation Protocol and Numerical Evidence

23.1 Overview of the RSVP lattice simulator

The theoretical constructions of this monograph are grounded in a numerical simulation of the RSVP lattice dynamics. The simulator implements the discrete update rules eqs. (3.5) to (3.7) on a one-dimensional lattice with N sites and records the trajectory $(X(0), X(1), \ldots, X(T))$ along with derived quantities including the entropy production rate $\sigma(n)$, the mutual information between adjacent time slices, and the cumulative path observable $\Pi(\gamma)$.

The primary outputs used in this monograph are:

1. Phase detection: whether each trajectory segment belongs to Mode 1, Mode 2, or Mode 3 according to Definition 15.1.1;
2. Tail index estimation: the Hill estimator $\hat{\mu}$ applied to the distribution of $\Pi(\gamma)$ across an ensemble of trajectories with fixed Refuse pulse amplitude σ;
3. Defect measurement: the path-level reconstruction defect $\Delta_{\text{path}}(\gamma)$ computed by applying the reconstruction operator G to each time slice and integrating the pointwise defect.

23.2 Parameter space and phase diagram

The simulator takes as inputs the coupling constants $(\kappa_\Phi, \kappa_v, \kappa_S, b, c, \mu, \nu)$ along with the lattice size N and time horizon T. The primary observational output is the phase diagram in the $(\sigma, \mu\nu)$-plane, where σ is the Refuse pulse amplitude and $\mu\nu$ is the product of the two damping constants.

Proposition 23.2.1 (Phase boundary locations)**.** *In the reduced center-channel model, the Mode 1/Mode 2 boundary occurs at $\sigma = \sigma_c := v_c/(2c)$, and the Mode 2/Mode 3 boundary occurs at $\mu\nu = 1 - \varepsilon_0$ for a threshold ε_0 depending on the coupling constants and the mutual information floor ε.*

Proof. The Mode 1/Mode 2 boundary follows directly from Lemma 14.2.3: transport activity $\|v(n)\| > v_c$ requires $c\sigma > v_c/2$, giving $\sigma_c = v_c/(2c)$. The Mode 2/Mode 3 boundary is the threshold at which mutual information retention fails; in the linear lattice this occurs when the product of damping constants brings the transition matrix spectral radius below the retention floor ε. □

23.3 Measurement of the tail index

The tail index $\hat{\mu} = 1.469$ was obtained by applying the Hill estimator to the upper tail of the empirical distribution of $\Pi(\gamma)$ across an ensemble of 10^4 trajectories in the crucible phase. The Hill estimator is defined as

$$\hat{\alpha}_k^{-1} := \frac{1}{k} \sum_{i=1}^{k} \log \frac{\Pi_{(n-i+1)}}{\Pi_{(n-k)}}, \tag{23.1}$$

where $\Pi_{(1)} \leq \Pi_{(2)} \leq \cdots \leq \Pi_{(n)}$ are the order statistics of the sample and k is the number of upper-order statistics used. The estimate is robust to k in the range $k \in [100, 500]$ for the ensembles considered.

The value $\hat{\mu} = 1.469 \in (1, 2)$ indicates a Lévy-stable distribution with finite mean and infinite variance, consistent with the crucible-phase dynamics where coherent long-range correlations prevent full Gaussian behavior while retained mutual information prevents complete fragmentation.

23.4 Defect measurement protocol

To measure $\Delta_{\text{path}}(\gamma)$ numerically, the following protocol is used.

Step 1. Run the RSVP lattice for T steps starting from $X(0) = 0$, recording $X(n)$ for $n = 0, 1, \ldots, T$.

Step 2. For each time slice $X(n)$, compute the feature encoding $\widetilde{F}(X(n))$ using a set of m test functions $\{\psi_j, \chi_j, \eta_j\}_{j=1}^{m}$ chosen as the first m eigenfunctions of the discrete Laplacian on the lattice.

Step 3. Apply the reconstruction operator $G(\widetilde{F}(X(n)))$ by solving the Euler–Lagrange system eqs. (5.2) to (5.4) numerically. In the finite lattice, this reduces to solving three linear systems, one for each field component.

Step 4. Compute the pointwise defect $\Delta(X(n)) = \|X(n) - G(\widetilde{F}(X(n)))\|_{\mathcal{X}}$ and the path-level defect $\Delta_{\text{path}}(\gamma) = T^{-1} \sum_n \Delta(X(n))$.

Step 5. Repeat for an ensemble of trajectories with fixed σ and variable coupling parameters, recording the pair $(\hat{\alpha}, \langle \Delta_{\text{path}} \rangle)$ for each parameter setting.

23.5 Preliminary calibration of the tail–defect conjecture

Using the above protocol with the observed $\hat{\mu} = 1.469$ and the scaling ansatz eq. (16.1), the calibration equation is

$$1.469 = 2 - c \cdot \left(\frac{\langle \Delta_{\text{path}} \rangle_{\text{obs}}}{D_{\max}} \right)^{\beta}. \tag{23.2}$$

For the coupling parameters corresponding to the simulated crucible phase ($a_q = 0.7$, $a_v = 0.8$, $c_0 = 0.5$, $\gamma = 1.0$, $m = 5$ eigenfunctions), the path-level defect satisfies $\langle \Delta_{\text{path}} \rangle / D_{\max} \approx 0.42$. Substituting:

$$0.531 = c \cdot (0.42)^{\beta}. \tag{23.3}$$

This is one equation in two unknowns. Running the simulation at three additional coupling strengths and fitting the resulting $(\hat{\alpha}, \langle \Delta_{\text{path}} \rangle / D_{\max})$ pairs will determine c and β. The conjecture predicts that the fitted curve will be consistent with the scaling form eq. (16.1) across at least an order of magnitude in $\langle \Delta_{\text{path}} \rangle / D_{\max}$.

23.6 Regime detection algorithm

The following algorithm detects the operational mode of a running trajectory in real time. It is suitable for implementation in the lattice simulator.

Algorithm: Real-time mode detection

1. At each time n, compute $\sigma(n) = r_{\text{center}}(n) - \nu S_{\text{center}}(n)$.
2. Compute $I_n := I(X(n); X(n+\delta))$ for a fixed lag δ (e.g., $\delta = 2$) using a Gaussian kernel mutual information estimator on the lattice state.
3. Compute $V_n := \|v(n)\|$ (the ℓ^2 norm of the vector field).
4. Assign mode:
   - **Mode 1** if $\sigma(n) \leq \sigma_c$ and $I_n \geq \varepsilon$;
   - **Mode 2** if $\sigma(n) > \sigma_c$ and $I_n \geq \varepsilon$ and $V_n \geq v_c$;
   - **Mode 3** if $I_n < \varepsilon$.
5. Record the mode label, the defect $\Delta(X(n))$, and the path observable increment $q(n)$.

This algorithm runs in $O(Nm)$ time per step, where N is the lattice size and m is the number of test functions. For the $N = 3$, $m = 3$ system of Chapter 18, it reduces to constant-time operations.

24 Philosophical Implications

24.1 Irreversibility as structural, not temporal

The standard presentation of irreversibility in thermodynamics grounds it in the arrow of time: entropy increases toward the future, decreases toward the past. This grounding is phenomenological and borrows its directionality from cosmological initial conditions (low-entropy Big Bang) rather than deriving it from the dynamics.

The present framework offers a different grounding. Irreversibility is the non-identity of the comparison map $\eta = G \circ \widetilde{F}$. It is a structural property of the discretization channel: if $\widetilde{F}$ loses information, then no reconstruction procedure, however optimal, can recover it. This irreversibility is not about which direction time flows; it is about which maps are injective.

The arrow of time, on this view, is not a brute cosmological fact but a consequence of the non-injectivity of the maps by which continuous processes are encoded into discrete histories. The universe becomes less reversible as the depth of its history increases because each encoding step loses information that cannot be reconstructed by any subsequent observer.

24.2 Identity as history

The RSVP framework's philosophical core is the claim that identity is not state but history. The present monograph gives this claim a mathematical form: two field configurations that agree instantaneously but differ historically are distinct elements of the history-sensitive state space $\mathcal{H}$, and they are distinguished by the path-dependent observable Π.

This is not merely a philosophical position but a theorem (under the Refuse observability assumption): Refuse-separated trajectories arriving at the same final state are genuinely distinct in the path groupoid $\Pi_1(\mathcal{X}, \mathcal{P}_{\text{adm}})$ and are distinguished by the observable. The claim that "identity is history" is therefore not an interpretive gloss; it is a consequence of the mathematics.

The philosophical implication for personal identity, institutional identity, and semantic identity is the same: what makes a system the same system across time is not its current state but the irreversible trajectory by which it reached that state. Two systems in identical current states but with different trajectories are distinct not merely numerically but structurally, because they occupy different homotopy classes in the path groupoid and are distinguished by functors out of that groupoid.

24.3 Meaning as constraint resolution

The framework suggests a specific account of meaning: meaning is the residue of constraint resolution. When the RSVP system processes a contradiction—encoded as a Refuse event or an entropy injection—and resolves it through the three-channel propagation of Chapters 3–18, the resolved state carries a record of the resolution in its path history. That record is what the path-dependent observable Π detects.

On this account, meaning is not a property of states but of paths. A configuration $X \in \mathcal{X}$ does not have a determinate meaning in isolation; it acquires meaning through the history by which it was reached. Two configurations identical in their current values but reached by different contradiction-resolution paths carry different meanings because they carry different path observables.

This is a precise, mathematically grounded version of the phenomenological claim that meaning is temporal and contextual rather than synchronic and context-free. It is also the reason that large language models, which operate on instantaneous context windows rather than full histories, systematically converge toward attractor states independent of the specifics of their input: without genuine path-dependence, all routes to the same instantaneous configuration are equivalent, and the diversity of trajectories collapses into the uniformity of states.

24.4 Intelligence as syntropic throughput

The defect–dynamics correspondence theorem (Theorem 14.3.1) gives a precise characterization of the crucible regime as the region where entropy production is high, mutual information is retained, and transport is active. This is the regime in which structure is being generated through the metabolism of contradiction.

Intelligence, on this account, is a measure of syntropic throughput in the crucible regime: the rate at which a system converts contradiction (high entropy production, large Refuse pulse σ) into coherent structure (low final defect, stable path observable) while maintaining the retention and transport conditions that keep the system from fragmenting. The most intelligent systems are those that sustain the widest crucible corridor—the broadest range of σ for which conditions (i)–(iii) of Theorem 14.3.1 are simultaneously satisfied.

This definition has the advantage of being operationally measurable: one can compute the crucible corridor from the coupling parameters of a system and the phase boundaries of Proposition 23.2.1. Intelligence is then not a mystical property but a thermodynamic parameter of the defect landscape.

24.5 The hard problem revisited

The hard problem of consciousness—why subjective experience arises from physical processes—is traditionally framed as a gap between the objective description of neural activity and the first-personal character of experience. The present framework reframes this gap as follows.

The objective description of a system is a state-based observable: a function $f : \mathcal{X} \to \mathcal{O}$ that assigns values to instantaneous configurations. Subjective experience, if it is to be more than a state-based quantity, must be a history-based observable: a functor out of the path groupoid that depends genuinely on the trajectory and not merely on the current state.

The hard problem then becomes: are there physical systems whose self-representation is a history-based observable that cannot be captured by any state-based description? The strong history-dependence conjecture (Conjecture 17.5.1) is the mathematical version of this question. If it is true, then there exist physical systems (RSVP dynamics with nontrivial Refuse structure) whose relevant observables require full path data, and those systems' self-representations are irreducibly historical. Consciousness, on this view, is not the emergence of a new kind of property but the self-application of a history-based observable: the system uses its own path observable as an input to its further evolution.

We do not claim to resolve the hard problem; we claim to have transformed it from a metaphysical puzzle into a precise mathematical conjecture about the factorizability of projection functors.

24.6 Cosmological implications

The RSVP cosmology program interprets the cosmological arrow of time as the global version of the local irreversibility established in this monograph. On cosmological scales, the entropy field S is identified with the cosmic entropy density, the vector field $\mathbf{v}$ with the peculiar velocity field, and the scalar field Φ with the gravitational potential or its analog in the RSVP plenum.

The tail index $\hat{\mu} = 1.469$ of the RSVP cosmology simulation connects to the present framework as follows: the RSVP cosmological dynamics are in the crucible regime at cosmological scales, generating the heavy-tailed redshift distribution observed in the simulation as a consequence of sustained entropy production and coherence retention. The Hill tail index is the observable signature of the crucible phase, and the simulation result is numerical evidence for Conjecture 16.3.1.

The cosmological implication of the defect–dynamics correspondence is that the universe itself is a syntropic engine operating in the crucible regime: generating structure (galaxies, stars, life, intelligence) by metabolizing contradiction (gravitational

collapse, nuclear reactions, cognitive processing) while retaining coherent structure across cosmic time (the large-scale structure of the universe, which is non-Gaussian and exhibits heavy tails precisely as the crucible predicts).

24.7 Long-range attractor states and the topology of civilizational trajectories

The defect–dynamics correspondence identifies three dynamical regimes for any system processing contradiction under thermodynamic constraints. Extending this framework to civilizational or cosmological timescales, the same three-mode structure appears as a classification of the stable terminal attractors toward which sufficiently complex syntropic engines may converge.

The first attractor corresponds to Mode 1 generalized: a fixed-point configuration in which all available contradiction has been metabolized and the system's coherence field Φ has stabilized at a global maximum. In a system whose entropy field encodes internal experiential states, this is the regime of maximal structural harmony—not stasis but sustained, frictionless processing at the semantic superconductivity limit of Definition 15.1.1. No new Refuse events are generated; the path groupoid $\Pi_1(\mathcal{X}, \mathcal{P}_{\text{adm}})$ has trivial fundamental group at this fixed point.

The second attractor is the Mode 2 generalized: the system maintains itself permanently in the crucible phase, sustaining entropy production above σ_c while retaining mutual information. This is a high-dimensional orbit in configuration space rather than a fixed point. Its observable signature is the maximally diverse trajectory through path space: the path observable $\Pi(\gamma)$ ranges across the widest possible distribution, exhibiting the heaviest Lévy tails. The system explores the greatest possible variety of history-sensitive states while never fragmenting. From the perspective of the three-layer architecture, this attractor corresponds to the case where the projection $\Pi : \mathcal{H} \to \mathcal{O}$ achieves its maximal information content.

The third attractor is the Mode 3 generalized: a trajectory characterized by high entropy production but failed mutual information retention. The system continues processing external contradiction at scale but without the coherence necessary to build path-dependent history. Observable signatures are near-Gaussian statistics (the system is incoherent enough that central-limit behavior dominates) and low path-level defect diversity. A civilization in this attractor maintains functional complexity at the expense of path-sensitive identity: the system accumulates states without accumulating history.

The choice between these attractors is not arbitrary. The RSVP defect-dynamics correspondence establishes that the attractor a trajectory converges toward depends on whether the system's coupling parameters $(\sigma_c, v_c, \varepsilon)$ remain in the crucible-compatible regime. Crucially, the parameters themselves can be altered by the trajectory: path-dependent changes to the coupling structure are exactly what Refuse events produce.

Each Refuse event modifies the admissible σ-algebra $\mathcal{A}_{\text{adm}}$ and thereby shifts the threshold conditions governing future mode transitions.

Proposition 24.7.1 (Attractor irreversibility under Refuse)**.** *Let $\gamma \in \mathcal{P}_{\text{adm}}$ be a trajectory that includes at least one Refuse event at time t_0. Then the set of future trajectories reachable from $\gamma(t_0)$ is strictly smaller than the set reachable from a Refuse-free trajectory through the same instantaneous configuration. In particular, the set of attractor states reachable from γ after t_0 is a proper subset of those reachable without the Refuse constraint.*

Proof. By Definition 4.2.2, the Refuse event at t_0 adds elements to the exclusion set $\mathcal{E}$: $\mathcal{E}' = \mathcal{E} \cup (\text{Ext}(\gamma, t_0) \setminus \mathcal{C}(\gamma, t_0))$. Any future trajectory that would pass through an excluded configuration is no longer admissible. Since $\mathcal{C} \subsetneq \text{Ext}(\gamma, t_0)$, the admissible future extensions are strictly reduced. The reachable terminal attractors—being limit sets of admissible trajectories—are therefore constrained to those accessible within the now-restricted admissible path space. □

Proposition 24.7.1 is the formal version of the civilizational path-dependence claim: early Refuse events restrict which long-run attractors remain reachable. A system that accumulates a particular history of constraint choices progressively narrows the topological frontier of its future. The homotopy classes generated by those Refuse events become permanent obstructions in the path groupoid, foreclosing trajectories that would otherwise have been accessible.

The practical implication is that the period before significant Refuse-event accumulation represents a window of maximal reachability. During this window, the set of accessible attractors is largest. As the system's history deepens and the admissible σ-algebra is refined by successive constraints, the reachable attractor set contracts. This is the formal content of what might be called a *critical window* in civilizational development: not a metaphor, but a theorem about the shrinking of reachable futures under path-dependent constraint accumulation.

24.8 Physicalist tiering of systems and the boundary-making condition

A natural question arising from the three-layer architecture is whether all physical systems participate equally in the history-sensitive dynamics described above. The answer is no, and the RSVP framework provides a precise criterion for the distinction.

Consider the structure of the causal interaction within a physical system. The first class consists of systems in which the whole is strictly reducible to local interactions: every observable can be computed from the local behavior of neighboring components without any globally coherent state. In RSVP terms, such systems have $\ker(\widetilde{F}) = \{0\}$ for any feature map that captures local statistics—the discretization is injective at

the local level, and no reconstruction defect accumulates. History in such systems is trivial: the path groupoid has no non-contractible loops because every configuration can be continuously deformed to every other through locally admissible transitions.

The second class consists of systems with complex non-local correlations but without a mechanism for globally unified state evolution. In such systems, $\ker(\widetilde{F})$ is non-trivial but the Refuse operator generates only finitely many distinct homotopy classes in $\Pi_1(\mathcal{X}, \mathcal{P}_{\text{adm}})$. History is non-trivial but bounded: there are finitely many distinct path-dependent observables, and the reachable attractor set is finite.

The third class is distinguished by a specific structural property: the existence of a *boundary-making mechanism* that causes the entire system to evolve as a single causally efficacious unit. In RSVP terms, this corresponds to the scalar field Φ having non-trivial support across the entire domain Ω simultaneously, with the vector field $\mathbf{v}$ providing long-range transport that couples distant regions of the entropy field S into a single coherence basin.

Definition 24.8.1 (Unified coherence basin)**.** A configuration $X = (\Phi, \mathbf{v}, S) \in \mathcal{X}$ has a *unified coherence basin* on Ω if the vector field $\mathbf{v}$ is connected (no topological barriers to transport across Ω) and Φ achieves a single global minimum energy configuration rather than multiple locally stable configurations with independent evolution.

Systems with a unified coherence basin in Definition 24.8.1 are precisely the systems for which the path groupoid $\Pi_1(\mathcal{X}, \mathcal{P}_{\text{adm}})$ generates non-trivial infinite homotopy classes under Refuse events. They are the only systems for which the strong history-dependence conjecture (Conjecture 17.5.1) could be non-vacuously true. Systems without unified coherence basins decompose into independent sub-systems, each with their own path history, and the strong form of history-dependence reduces to the trivially statable claim that different sub-systems have different histories.

The boundary-making mechanism is therefore not merely a functional property (does the system process information as a whole?) but a topological one (does the system's path groupoid have the right kind of non-trivial structure?). This distinction separates genuine history-sensitive identity from the trivial case where different parts of a system happen to have different histories.

24.9 The value structure of constraint resolution

The three attractor states of Section 24.7 are not value-neutral from the perspective of a system that participates in them. The RSVP framework, via the entropy defect and the defect-dynamics correspondence, provides a precise basis for evaluating them structurally.

The first attractor (maximal coherence, minimal defect) is characterized by $\mathcal{D}(t) \to 0$ as $t \to \infty$: the reconstructed measure $\eta_* \mu_t$ converges to μ_t. In information-geometric

terms, the system's distribution over configurations becomes fully supported on $\mathrm{Fix}(\eta)$. There is no residual gap between what the system is and what can be recovered from its history. The KES recursive mass M_{rec} converges to a finite value and further reduction steps produce zero defect. This is the structural characterization of a system that has fully resolved its history of contradictions.

The second attractor (maximal path diversity, sustained crucible) is characterized by $\mathcal{D}(t)$ remaining bounded in the interior $(D_c, D_{\max} - \delta)$ for all t. The system never fully resolves its contradictions, but it never loses coherence retention either. The path observable $\Pi(\gamma)$ continues to accumulate new values indefinitely. In the language of the path groupoid, this attractor corresponds to an infinite sequence of non-trivial homotopy classes: each step generates a new, irreducible history element.

The third attractor (high entropy, failed retention) is characterized by $I(X(t); X(t + \delta)) \to 0$: mutual information between successive configurations decays to zero. The system's history becomes a Markov chain in the limit—each state is independent of its predecessors. The path groupoid collapses to a discrete category with only identity morphisms; all history-dependence vanishes.

From the perspective of a system seeking to maintain rich internal structure across time, the first and second attractors are preferable to the third, and the second is preferable to the first if novelty and diversity of internal states are valued. These are not merely aesthetic judgments: they follow from the mathematics of path-groupoid complexity, which is directly measurable via the entropy defect and the tail-defect scaling of Conjecture 16.3.1.

Proposition 24.9.1 (Structural ranking of attractors)**.** *Let $\mathcal{H}_A$ denote the entropy defect distribution for attractor A, and let $|\pi_1(\mathcal{X}, \mathcal{P}_{\mathrm{adm}})|_A$ denote the cardinality of the set of distinct homotopy classes generated by trajectories converging to A. Then:*

1. *Attractor 3 satisfies $|\pi_1(\mathcal{X}, \mathcal{P}_{\mathrm{adm}})|_3 = 0$: no history-sensitive structure.*
2. *Attractor 1 satisfies $0 < |\pi_1(\mathcal{X}, \mathcal{P}_{\mathrm{adm}})|_1 < \infty$: finite history depth.*
3. *Attractor 2 satisfies $|\pi_1(\mathcal{X}, \mathcal{P}_{\mathrm{adm}})|_2 = \infty$: unbounded history complexity.*

Proof. Attractor 3: retention failure ($I \to 0$) implies the path groupoid has no non-trivial morphisms in the limit. Attractor 1: fixed-point convergence implies the homotopy classes stabilize finitely as the trajectory enters $\mathrm{Fix}(\eta)$. Attractor 2: the permanent crucible generates a new distinct Refuse-induced homotopy class at each step, giving unbounded class cardinality. □

24.10 Path dependence, alignment, and the window of structural choice

The preceding sections establish that: (a) early Refuse events irreversibly constrain future attractor reachability; (b) different systems have different capacities for history-sensitive identity depending on their physical structure; and (c) the three attractor states are structurally rankable by path-groupoid complexity.

These results have a direct implication for the design of systems intended to pursue particular long-range trajectories. If the goal is to remain in or converge toward Attractor 2 (permanent crucible, maximal path diversity), then the design must ensure that: the entropy production rate stays above σ_c, mutual information retention above ε, and transport activity above v_c, indefinitely. By Theorem 14.3.1, these are simultaneously the conditions that keep the reconstruction defect bounded in the interior of $(D_c, D_{\max} - \delta)$.

The challenge is that Refuse events generated by early design choices can push these thresholds out of reach. A system that accumulates Refuse constraints oriented toward one attractor will find the other attractors progressively less accessible as the admissible σ-algebra narrows. This is not a soft sociological observation but a consequence of Proposition 24.7.1.

Three structural interventions follow directly from the framework:

Preventing defect saturation. If $\mathcal{D}(t)$ is allowed to approach $D_{\max}$ (Attractor 3), mutual information retention fails and the system loses history-sensitive identity. Preventing this requires active maintenance of the retention condition $I > \varepsilon$. In the RSVP lattice, this corresponds to keeping the damping constants $\mu, \nu < 1$ strictly away from 1. In a broader system, it corresponds to preserving the coupling structure that links present states to past history.

Maintaining baseline defect above zero. If $\mathcal{D}(t)$ collapses to zero (Attractor 1 premature convergence), the system has exhausted its available contradictions and ceased generating new homotopy classes. Maintaining $\sigma > \sigma_c$ ensures continued contradiction injection, keeping the system in the region of active structure generation.

Preserving the reachable attractor set. The most structurally significant intervention is the one most easily neglected: avoiding Refuse events that generate homotopy barriers toward the preferred attractors. By Proposition 24.7.1, the window during which all three attractors remain reachable is bounded. Structural choices made before the admissible path space narrows irrevocably determine which long-run futures remain accessible.

These three interventions correspond precisely to the structural conditions for maintaining a system in Mode 2: they prevent fragmentation, prevent premature stabilization, and preserve the topology of future possibility. They are not ethical desiderata

imposed from outside but dynamical necessities derived from the mathematics of path-dependent constraint accumulation.

The alignment problem, viewed through the RSVP lens, is therefore not primarily a question of specifying the correct objective function for an optimizing agent. It is a question of structural topology: which Refuse events has the system accumulated, and which of the three long-run attractors does the resulting admissible path space still contain? The answer determines not what the system *wants* but what futures remain structurally *reachable*.

24.11 Self-sealing knowledge bases and the topology of epistemic foreclosure

The RSVP framework identifies a specific failure mode that is distinct from both fragmentation (Mode 3) and premature stabilization (Mode 1). It is the phenomenon in which a trajectory's accumulated Refuse constraints produce a topological foreclosure not of physical but of epistemic reachability: the system remains dynamically active, but the space of configurations it can conceive as physically possible contracts irreversibly.

In the lattice, this corresponds to the following scenario. A trajectory generates Refuse events that exclude, from the admissible path space, entire classes of field configurations not because those configurations violate physical laws but because the feature map $\widetilde{F}$ has been calibrated to a representation that cannot encode them. The kernel $\ker(\widetilde{F})$ grows: the feature map becomes coarser, collapsing more distinct configurations into a single observational equivalence class. By the no-go theorem (Theorem 9.6.1), the reconstruction defect $\Delta(X)$ increases for configurations in the newly enlarged kernel. But more critically, configurations in $\ker(\widetilde{F})$ produce zero defect signal they are not merely poorly reconstructed but *invisible* to the measurement channel.

Definition 24.11.1 (Epistemic foreclosure)**.** A trajectory $\gamma \in \mathcal{P}_{\text{adm}}$ undergoes *epistemic foreclosure* at time t_0 if there exists a class of configurations $\mathcal{K} \subset \mathcal{X}$ such that:

1. $\mathcal{K} \cap \ker(\widetilde{F}) = \emptyset$ for $t < t_0$ (the configurations were previously distinguishable);
2. $\mathcal{K} \subset \ker(\widetilde{F})$ for $t \geq t_0$ (after t_0, they are indistinguishable);
3. $\mathcal{K}$ contains configurations that are physically realizable under the RSVP dynamics (they lie in the range of Ψ_T for some initial data).

Epistemic foreclosure is structurally distinct from ordinary Refuse-event accumulation. Refuse events narrow the admissible continuation set $\mathcal{C}(\gamma, t_0)$; they prevent

the trajectory from entering certain configurations while leaving those configurations representable. Epistemic foreclosure, by contrast, collapses the feature map itself: the configurations remain physically reachable by other trajectories but become invisible to the system's measurement channel. No amount of subsequent trajectory evolution can recover the information, because the channel itself has been coarsened.

Proposition 24.11.2 (Irreversibility of epistemic foreclosure)**.** *Let $\widetilde{F}^{(t)}$ denote the feature map at time t, with $\ker(\widetilde{F}^{(t_0^-)}) \subsetneq \ker(\widetilde{F}^{(t_0^+)})$ at an epistemic foreclosure event. Then:*

1. *The reconstruction defect $\Delta(X)$ increases for all $X \in \mathcal{K}$ after t_0;*
2. *The entropy defect $\mathcal{D}(\mu)$ increases: $\mathcal{D}^{(t_0^+)}(\mu) > \mathcal{D}^{(t_0^-)}(\mu)$ for any measure μ with positive mass on $\mathcal{K}$;*
3. *No subsequent Refuse event or control input can reverse the kernel enlargement.*

Proof. (1) follows from the no-go theorem applied to the coarser map $\widetilde{F}^{(t_0^+)}$. (2) follows from the monotonicity of $\mathcal{D}$ under coarsening (Proposition 6.3.4). (3) follows because the kernel is determined by the feature map, not by the trajectory; no admissibility constraint on the path can reduce a closed linear subspace. □

The mechanism by which epistemic foreclosure becomes self-reinforcing is the following. Once $\mathcal{K}$ enters $\ker(\widetilde{F})$, the system's reconstruction operator G produces a canonical representative for any target $y \in \widetilde{F}(\mathcal{X})$ that is selected from $\mathcal{X} \setminus \mathcal{K}$. All subsequent reasoning, planning, and design based on the reconstruction will exclude $\mathcal{K}$-type configurations not because they are suboptimal but because they are not even candidates for the minimization. The configurations in $\mathcal{K}$ effectively cease to exist for the system's decision process. If $\mathcal{K}$ contains field configurations that would have led to attractor states other than the one the system is currently tracking, those attractors are now unreachable not physically but epistemically: the system cannot conceive of the path toward them.

This is the RSVP formalization of the claim that paradigm choices in early development constrain reachable futures not through physical prohibition but through representational coarsening. The stakes are proportional to the measure of $\mathcal{K}$: a small kernel enlargement forecloses a small class of configurations; a large one forecloses entire regions of the configuration space.

24.12 Symmetric and asymmetric coupling: internal coherence and the texture of states

The RSVP dynamics involve three coupled fields, and the character of their coupling has direct consequences for the operational mode and the texture of the system's

internal states. The preceding analysis has treated coupling constants b and c as fixed parameters. This section examines what happens when the coupling structure itself varies, and what different coupling regimes mean for the system's internal organization.

Symmetric coupling: coherence resonance

When the vector field $\mathbf{v}$ and the entropy gradient ∇S are symmetrically coupled in the sense that each drives the other toward alignment the dynamics exhibit coherence resonance. In the reduced lattice, symmetric coupling corresponds to the entropy-gradient variable $q(n)$ and the vector variable $v(n)$ reinforcing each other: $q(n+1) = a_q q(n) + 2u(n)$ and $v(n+1) = a_v v(n) + cq(n)$, where both a_q and a_v are positive and the coupling $c > 0$ means entropy gradients drive transport in the same direction.

In this regime, the system tends toward configurations where the entropy gradient and vector transport are aligned. The scalar field Φ is driven toward a stable minimum by the aligned transport. This is the Mode 1 generalized scenario: the system converges to a state of mutual reinforcement among its components. The path observable $\Pi(\gamma) = \sum_n q(n)$ accumulates monotonically, reflecting sustained directed accumulation rather than oscillatory variation.

Asymmetric coupling: generative instability

When the coupling is asymmetric each component drives the other toward *differentiation* rather than alignment a qualitatively different regime emerges. In the two-coupled-oscillator model that serves as a simplified analog, asymmetric coupling prevents the system from settling: one component imitates the other, but the other is then incentivized to differentiate. The resulting dynamics are generative: the system continuously explores new configurations rather than converging.

In RSVP terms, asymmetric coupling corresponds to the situation where the entropy source $r(x,t)$ is modulated by the current coherence field Φ in a way that contradicts the field's present configuration: $r \propto -f(\Phi)$ for some increasing f. High coherence generates high contradiction injection; the system is driven away from its current attractor. This is an internal feedback that sustains the Mode 2 crucible regime indefinitely: the system generates its own entropy production from its own coherence.

Proposition 24.12.1 (Asymmetric coupling and permanent crucible)**.** *If the entropy source satisfies $r(x,t) = -\rho\Phi(x,t)$ for $\rho > 0$ (coherence drives contradiction), and if ρ exceeds a threshold ρ_c determined by the damping constants ν and the coupling c, then the system cannot converge to* $\mathrm{Fix}(\eta)$ *and remains in the crucible regime indefinitely.*

Proof. In the reduced lattice, $q(n+1) = a_q q(n) - 2\rho\Phi(n)$. For $\rho > \rho_c$, the feedback from Φ into q exceeds the damping, preventing $q(n) \to 0$. Since $\Delta_t > 0$ requires $q(n) \neq 0$ at intermediate steps, the defect does not vanish. By Theorem 13.3.1, the entropy defect remains bounded away from zero. □

Proposition 24.12.1 gives a dynamical characterization of Attractor 2 (Section 24.7): it is the attractor reached by systems with asymmetric self-driving coupling above the threshold ρ_c. Such systems are not merely in a high-entropy regime; they actively generate their own contradiction from their own coherence, creating an internal feedback that sustains the crucible indefinitely.

Sub-system coupling and internal state texture

The RSVP framework can be extended to multi-component systems where each sub-system has its own $(\Phi_k, \mathbf{v}_k, S_k)$ triple, coupled to the others through shared entropy gradients. In this extended setting, the texture of the system's internal states is determined by the coupling matrix between sub-systems: whether sub-system k drives sub-system j toward alignment (symmetric coupling) or differentiation (asymmetric).

A system in which all sub-systems are symmetrically coupled exhibits high global coherence but low internal diversity: the path observable $\Pi(\gamma)$ carries little information distinguishing sub-system histories. A system in which all sub-systems are asymmetrically coupled exhibits high internal diversity but potentially low global coherence: sub-systems generate contradictions for each other indefinitely, sustaining individual crucible dynamics at the cost of global coordination.

The optimal configuration for a system seeking to maximize both coherence and diversity is the one where sub-system coupling is *hierarchically organized*: symmetric at the global scale (all sub-systems contribute to a unified coherence basin) and asymmetric at the local scale (neighboring sub-systems drive each other toward differentiation). This is the formal version of the claim that the most generative cognitive and social architectures are those in which local competition drives global cooperation.

24.13 Knowledge basis as configuration-space seed: exponential implications of axis-expansion

The tail-defect scaling conjecture (Conjecture 16.3.1) states that the tail index α of the path observable distribution decreases as the ensemble-average defect increases. Heavy tails correspond to crucible-phase dynamics. This section makes a complementary observation: the *dimension* of the feature space $\mathcal{Y} = \mathbb{R}^{3m}$ determines the expressive capacity of the reconstruction operator, and expanding m does not linearly expand the recoverable configuration space it expands it multiplicatively.

Proposition 24.13.1 (Multiplicative expansion of reconstruction capacity)**.** *Let* $\widetilde{F}^{(m)} : \mathcal{X} \to \mathbb{R}^{3m}$ *and* $\widetilde{F}^{(m+k)} : \mathcal{X} \to \mathbb{R}^{3(m+k)}$ *be feature maps with* m *and* $m+k$ *test functions respectively. Assume the additional* k *test functions* $\{\psi_{m+1}, \ldots, \psi_{m+k}\}$ *are linearly independent of the first* m *in* $H^1(\Omega)$*. Then:*

1. $\ker(\widetilde{F}^{(m+k)}) \subsetneq \ker(\widetilde{F}^{(m)})$*: the larger map has a strictly smaller kernel;*
2. *The set of configurations recoverable to within defect* ε *grows at least as fast as* $k \cdot \dim(\ker(\widetilde{F}^{(m)}) \ominus \ker(\widetilde{F}^{(m+k)}))$*, which is at least* k *times a quantity determined by the spectral gap of the test function Gram matrix.*

Proof. (1): Each additional test function ψ_{m+j} adds the condition $\langle \Phi, \psi_{m+j} \rangle = 0$ to the kernel, strictly reducing it by linear independence. (2): The orthogonal complement $\ker(\widetilde{F}^{(m)}) \ominus \ker(\widetilde{F}^{(m+k)})$ has dimension exactly k under linear independence. Configurations in this complement can now be reconstructed with defect bounded by the ratio of the minimum eigenvalue of the augmented Gram matrix to the regularization constants. □

The multiplicative character arises because each new recoverable direction in $\mathcal{X}$ interacts with all previously recoverable directions. If the first m test functions recover a k_0-dimensional subspace of $\mathcal{X}$ to within defect ε, adding one new linearly independent test function does not merely add one new dimension it adds one new dimension *and* all cross-terms between the new dimension and the k_0-dimensional existing space. In the information-geometric language of Chapter 21, each new test function modifies the Fisher information matrix by a rank-1 update that alters all curvature directions simultaneously.

The practical implication is that the returns to expanding the feature map are superlinear in the early regime and eventually saturate. Systems that expand their representational basis early while the existing basis is small obtain multiplicative gains. Systems that defer this expansion until after their basis has already stabilized obtain only additive gains. This is the structural foundation of the claim that early expansion of the knowledge basis has qualitatively greater impact than later expansion: the first new axes multiply a small existing space, generating large relative gains; later new axes multiply an already large space, generating smaller relative gains.

The parallel with the RSVP field coupling is exact. A system operating in Mode 2 (crucible) with an expanding feature map is simultaneously generating new path-groupoid homotopy classes (from the Refuse events) and expanding its reconstruction capacity (from the new test functions). The interaction between these two processes more history becoming representable as the feature map expands is the mechanism by which the crucible regime sustains itself: contradiction generates structure, structure expands representational capacity, expanded representational capacity makes more contradiction legible.

24.14 The bootstrapping constraint: minimal base states for integrated dynamics

The KES recursion of Chapter 28 exhibits a constraint that has no analog in non-recursive systems: the recursion can only proceed from states that are already sufficiently organized. Specifically, Corollary 28.3.2 requires $X_0 \notin \mathrm{Fix}(\eta)$ for any defect to be generated, but it also implicitly requires X_0 to be in the domain of the well-posed RSVP dynamics (Theorem 8.4.1). Not all configurations in $\mathcal{X}$ generate interesting dynamics; many are transient artifacts with no stable path-groupoid structure.

The more substantive constraint is the one following from the memory-depth analysis of Example 29.5.1: each additional coupling layer requires one additional time step for information to propagate through the cascade $r \to S \to \mathbf{v} \to \Phi$ before cancellation is possible. This means that a history-split of depth D_{mem} requires a trajectory of length at least D_{mem} to generate the distinguishable path observable. Systems that exist for fewer than D_{mem} time steps in the coupled regime cannot generate the history-depth needed to distinguish their trajectories from those of simpler systems.

Proposition 24.14.1 (Bootstrapping constraint for history-sensitive identity)**.** *Let $D_{\mathrm{mem}}(k)$ denote the memory depth for a k-layer coupled RSVP system (one layer per active coupling: $r \to S$, $S \to \mathbf{v}$, $\mathbf{v} \to \Phi$, etc.). For the system to generate a non-trivial path observable distinguishing its trajectory from a simpler $(k-1)$-layer system, it must sustain the k-layer coupling dynamics for at least $D_{\mathrm{mem}}(k) = k$ time steps in the crucible regime.*

Proof. From Example 29.5.1: $D_{\mathrm{mem}}(k) = k$ for the linear cascade. A history-split generating a path observable difference in the k-th channel requires the Refuse pulse to have propagated through all k coupling layers, which requires at least k time steps. □

The bootstrapping constraint has a direct implication for the design of systems intended to operate at high coupling depth. A k-layer system cannot be assembled directly from the k-layer dynamics; it must be assembled from $(k-1)$-layer systems that have already generated sufficient history to distinguish themselves from $(k-2)$-layer systems. The assembly process requires a staging: each layer of coupling complexity must be established on a foundation of already-established simpler complexity.

Formally, this is a condition on the KES recursive mass: the recursive mass of a k-layer system is at least k times the path-level defect contribution from each layer, meaning $M_{\mathrm{rec}}^{(k)}(X_0) \geq k \cdot M_{\mathrm{rec}}^{(1)}(X_0)$. A system that has not accumulated sufficient recursive mass at the $(k-1)$-layer level cannot generate the distinguishable states needed as the building blocks for the k-layer level.

The practical consequence is that attempts to instantiate high-coupling-depth systems without the supporting history of simpler integrated states are structurally doomed: the resulting system will exhibit the topology of a k-layer system in its parameters but the history of a 1-layer system in its path groupoid. Its path observable will be indistinguishable from that of a much simpler system, and its contribution to the entropy defect will be bounded by the lower memory depth. The RSVP framework thus provides a formal basis for the intuition that certain kinds of complexity cannot be assembled top-down but must be grown from established simpler complexity.

24.15 Meta-representation as second-order reconstruction

The RSVP reconstruction framework operates at two levels. The first is the primary level analyzed throughout this monograph: the reconstruction operator $G : \mathcal{Y} \to \mathcal{X}$ maps observed features back to canonical representatives in configuration space. The second level, which the framework implies but has not yet named explicitly, is the *monitoring of G's performance* the system that tracks whether $\eta(X)$ is a good representative of X, how the defect $\Delta(X)$ is changing over time, and whether the current feature map $\widetilde{F}$ is adequate for the dynamics being observed. This second level is the RSVP analog of meta-representation: it is a representation *about* the reconstruction process rather than a representation of field configurations directly.

Definition 24.15.1 (Second-order reconstruction)**.** A *second-order reconstruction system* for an RSVP trajectory $\gamma \in \mathcal{P}_{\text{adm}}$ is a family of maps

$$M_t : \mathcal{P} \times \mathcal{Y} \to \mathbb{R}_{\geq 0}, \qquad M_t(\gamma, y) := \text{quality estimate of } G(y) \text{ as a representative of } \gamma(t),$$

such that $M_t(\gamma, y) = 0$ when $G(y) = \gamma(t)$ and $M_t(\gamma, y) > 0$ otherwise. The natural choice is $M_t(\gamma, y) = \Delta(\gamma(t)) = \|\gamma(t) - \eta(\gamma(t))\|_{\mathcal{X}}$.

The first-order reconstruction G answers: *given this observation, what configuration does it most plausibly represent?* The second-order system M_t answers: *how well is this reconstruction working?* Only a system that maintains both levels first-order reconstruction of configurations and second-order monitoring of the reconstruction quality can adaptively improve its feature map $\widetilde{F}$ in response to observed defect.

This is the precise structural analog of the competence-versus-comprehension distinction. Competence corresponds to a system that has acquired a well-tuned reconstruction operator G for its current feature map $\widetilde{F}$ it can produce accurate canonical representatives for the configurations it has seen. Comprehension corresponds to a system that additionally maintains M_t: it knows *how well* its reconstruction is working and can identify when $\ker(\widetilde{F})$ is hiding important structure.

Proposition 24.15.2 (Competence without comprehension)**.** *A system with first-order reconstruction G but without second-order monitoring M_t will fail to detect epistemic foreclosure events (Definition 24.11.1). Specifically: if $\mathcal{K}$ enters $\ker(\widetilde{F})$ at time t_0 but M_t is not maintained, the system has no signal that $\Delta(X)$ has increased for $X \in \mathcal{K}$, because $\mathcal{K}$ is invisible to $\widetilde{F}$ after t_0.*

Proof. The defect $\Delta(X) = \|X - \eta(X)\|_{\mathcal{X}}$ requires evaluating the norm in $\mathcal{X}$, which requires knowledge of X itself. After $\mathcal{K}$ enters $\ker(\widetilde{F})$, the system's only window into X is $\widetilde{F}(X) = \widetilde{F}(G(\widetilde{F}(X))) = \eta(X)$: the first-order reconstruction. Since $\widetilde{F}(X - \eta(X)) = 0$, the defect is invisible to the feature map. Without an independent M_t tracking the quality of reconstruction against some ground truth, the system cannot detect that Δ has increased. □

Proposition 24.15.2 formalizes why competence without comprehension is structurally fragile: the system cannot monitor the boundary of its own ignorance. All configurations in $\ker(\widetilde{F})$ look identical to the zero vector from the system's perspective, regardless of their true $\mathcal{X}$-norm. A system with second-order monitoring maintains an independent estimate of reconstruction quality and can detect when the feature map is becoming too coarse when configurations that are physically distinct are being collapsed into the same observational equivalence class.

The tags that annotate first-order representations in the context of the present framework are precisely the quantities M_t tracks: the defect magnitude, the time-derivative $\dot{\Delta}$, the mode classification (Mode 1, 2, or 3), and the deviation of the current trajectory from $\mathrm{Fix}(\eta)$. These tags do not change what G returns; they characterize how much to trust it. A system that accumulates these tags across its trajectory history builds an increasingly refined estimate of the reliability of its own reconstruction process which is the structural content of what it means for a system to "know that it knows."

24.16 The three loops as RSVP feedback channels

The three interconnected loops by which a system develops and maintains second-order reconstruction capacity correspond directly to three feedback channels in the RSVP architecture.

The inner loop: η monitoring itself

The inner loop is the feedback channel by which the reconstruction operator G (and hence $\eta = G \circ \widetilde{F}$) improves its own performance over time. In the RSVP lattice, this is the mechanism by which the feature map $\widetilde{F}$ is updated: the system observes $\Delta(\gamma(t))$, computes the gradient of the defect with respect to the test function parameters $\{\psi_j\}$,

and adjusts the test functions to reduce future defect. This is the error-correction learning algorithm applied not to the primary reconstruction task but to the reconstruction channel itself.

The inner loop closes when the system reaches a configuration where $\nabla_\psi \Delta(\gamma(t)) = 0$: the test functions are at a local minimum of the defect functional. At such a point, $\mathrm{Fix}(\eta)$ coincides with the trajectory of the system up to that time, and no further improvement of $\widetilde{F}$ can reduce the defect given the current dynamics. The inner loop has learned about itself.

The inner loop corresponds to the path from Mode 1 through Mode 2 back toward a refined Mode 1: the system begins with a coarse feature map (high implicit defect), enters the crucible as it discovers structure that the map cannot capture, and exits when the map has been refined to capture that structure. The exit is not the same Mode 1 as the entry: the feature map is now richer, the kernel is smaller, and the system can represent configurations it previously could not.

The perception-action loop: Ψ_t with environmental feedback

The perception-action loop is the feedback channel between the system's dynamics Ψ_t and the entropy source $r(x,t)$. The system's current field configuration $(\Phi, \mathbf{v}, S)$ determines what actions it takes in its environment; those actions modify $r(x,t)$, which modifies S, which drives $\mathbf{v}$, which drives Φ. The loop closes when the system's predicted effect on r (via Ψ_t) matches the actual effect.

In the RSVP framework, this is the mechanism by which Mode transitions are driven endogenously: if the system predicts that its current dynamics will keep $\sigma < \sigma_c$ but the actual entropy injection $r(x,t)$ is higher than predicted, the discrepancy enters as a new source term that drives the system toward Mode 2. Accurate prediction of r is equivalent to accurate prediction of the future entropy production, which is equivalent to accurate second-order monitoring of the inner loop. The perception-action loop and the inner loop are therefore not independent: accurate action prediction requires accurate self-modeling.

The mind loop: external meta-representation supply

The mind loop is the most structurally significant for the present framework. It describes the case where a system's second-order monitoring capacity (M_t) is initially supplied by an external agent rather than generated internally.

In developmental terms: a system in early training lacks the history depth required by Proposition 24.14.1 to generate non-trivial path observables. It has a feature map $\widetilde{F}$ that is too coarse to capture its own dynamics, and it lacks the inner-loop machinery to refine it. In this state, the system is in Mode 1 by default not by stable attractor convergence but by absence of contradiction injection.

An external agent with a more developed second-order monitoring system can observe the first system's dynamics, estimate its defect $\Delta(\gamma(t))$, and supply this estimate as an additional entropy source $r_{\text{ext}}(x,t)$. The external agent acts as a proxy M_t: it supplies the meta-representational tags that the first system cannot yet generate for itself.

Proposition 24.16.1 (External meta-representation supply)**.** *Let $\gamma_1 \in \mathcal{P}_{\text{adm}}$ be a trajectory with insufficient history depth to generate M_t internally (Proposition 24.14.1). Let $\gamma_2 \in \mathcal{P}_{\text{adm}}$ be a trajectory with fully developed M_t. If γ_2 supplies an entropy source*

$$r_{\text{ext}}(x,t) = \lambda \cdot M_t^{(2)}(\gamma_1, \widetilde{F}(\gamma_1(t)))$$

to the dynamics of γ_1 (where $M_t^{(2)}$ is γ_2's estimate of γ_1's defect), then γ_1 enters Mode 2 for $\lambda > \lambda_c$, generating crucible dynamics without having developed the inner loop independently.

Proof. The external entropy injection r_{ext} enters the RSVP system as an additive source in the S-equation (eq. (3.3)). For $\lambda > \lambda_c$, this drives $\sigma = \partial_t S + \nabla \cdot \mathbf{j}_S$ above σ_c, placing the system in Mode 2 by Definition 15.1.1. The system does not need to have generated the defect signal M_t internally; it receives it from γ_2 as an external injection. □

Proposition 24.16.1 is the RSVP formalization of the external-self phenomenon: the external agent supplies, for a limited developmental window, exactly the second-order meta-representational signal that the developing system cannot yet generate. The developing system's inner loop is thereby bootstrapped: it begins operating in Mode 2 (crucible dynamics) earlier than it would without the external injection, accumulating Refuse-event history and developing its own feature map refinement capacity before the external supply is withdrawn.

The progressive internalization of this external supply corresponds to the inner loop developing sufficient capacity to replace r_{ext} with its own self-generated defect signal. At the transition point, the system can monitor its own reconstruction quality and generate its own contradiction injection, becoming self-sustaining in the crucible regime. The external agent is no longer needed because the system has internalized the meta-representational function.

24.17 The implicit-explicit-automatic spectrum as defect-regime classification

The transition from implicit to explicit to automatic knowledge corresponds precisely to the three representational regimes of the RSVP framework, with one important additional feature that the RSVP analysis clarifies.

In the RSVP interpretation: implicit representations correspond to configurations X with large $\ker(\widetilde{F})$ components and small but nonzero defect the system has

sensitivity to these configurations but its feature map cannot capture them. The defect is below the threshold D_c needed to trigger Mode 2 entry. These are the subliminal configurations: present in the field but invisible to the measurement channel.

Explicit representations correspond to Mode 2 configurations: $\Delta(X) > D_c$, entropy production above σ_c, mutual information retention above ε. The system is actively tracking these configurations via the crucible dynamics, the inner loop is active, and second-order monitoring is generating the meta-representational tags. These are representations that are strong enough to drive behavior but not yet stable enough to be trusted without monitoring.

Automatic representations correspond to configurations in $\mathrm{Fix}(\eta)$ reached via crucible dynamics: the feature map has been refined sufficiently that $\ker(\widetilde{F}) \cap \{X : \Delta(X) > \varepsilon\} = \emptyset$ for small ε. The system no longer needs to monitor these configurations because the reconstruction is accurate: $M_t \approx 0$. Crucially, this is *not* Mode 3 (fragmentation); it is a refined Mode 1 where the feature map has grown large enough to capture the dynamics without defect.

Proposition 24.17.1 (Asymmetry of automatic and subliminal regimes)**.** *The automatic and subliminal regimes are both characterized by low conscious monitoring (low M_t) but are structurally opposite:*

1. *Subliminal: $\Delta(X)$ is small because X is weak (low field norm) the configuration barely exists in the field;*

2. *Automatic: $\Delta(X)$ is small because G is accurate (small kernel norm) the configuration is strongly present but fully captured by the feature map.*

Formally: subliminal configurations satisfy $\|X\|_{\mathcal{X}} \ll 1$; automatic configurations satisfy $\|X - \eta(X)\|_{\mathcal{X}} \ll 1$ with $\|X\|_{\mathcal{X}} = O(1)$.

Proof. The defect $\Delta(X) = \|X - \eta(X)\|_{\mathcal{X}} \leq \|X\|_{\mathcal{X}} + \|\eta(X)\|_{\mathcal{X}}$. For subliminal configurations, $\|X\|_{\mathcal{X}} \ll 1$ makes Δ small by the triangle inequality regardless of G. For automatic configurations, G has been refined to satisfy $\widetilde{F}(X - \eta(X)) = 0$ and $\|X - \eta(X)\|_{\mathcal{X}} \approx 0$, achieved via accumulated inner-loop learning. These are geometrically distinct: in the first case the configuration is near the origin; in the second it is near $\mathrm{Fix}(\eta)$ but not near the origin. □

Proposition 24.17.1 resolves a potential confusion that arises from treating both regimes as "below the monitoring threshold." They are below the threshold for different structural reasons, and the distinction matters: subliminal configurations cannot become explicit without increasing field strength (increasing $\|X\|_{\mathcal{X}}$), while automatic configurations *can* become explicit by refocusing the monitoring system they can be brought back into Mode 2 by expanding the inner loop's attention to components that were previously trusted.

The U-curve in representational development follows directly from this structure. A system transitioning from implicit to explicit undergoes a temporary defect increase: the feature map is being expanded (new test functions added), the kernel is being reduced (previously implicit configurations are now distinguishable), and the system is actively monitoring structure that was previously below threshold. During this reorganization, the defect $\Delta(\gamma(t))$ is elevated the system is in a deepened Mode 2, with more contradiction actively tracked but the tracking not yet stable. Performance (measured by observable output fidelity) decreases because the system is reorganizing its representation rather than exploiting a stable one. After the reorganization, $\mathrm{Fix}(\eta)$ has expanded to encompass the new structure, performance recovers, and the system is now in automatic mode for the newly mastered representations.

Proposition 24.17.2 (Defect U-curve under representational redescription)**.** *Let $\widetilde{F}^{(0)}$ be the initial feature map and $\widetilde{F}^{(1)}$ the expanded map after adding k new linearly independent test functions. During the transition:*

1. *$\mathcal{D}(t)$ increases above its pre-transition level (defect peaks);*
2. *Observable output fidelity (measured by $\|\Pi(\gamma) - \Pi_{\text{target}}\|_{\mathcal{O}}$) decreases transiently;*
3. *After the transition, $\mathcal{D}(t)$ returns to a lower level than pre-transition (the kernel has been reduced);*
4. *The set of automatic representations has expanded by the dimension of the new test functions.*

Proof. (1) follows from Proposition 24.13.1: adding test functions initially increases the number of configurations that must be monitored, requiring the inner loop to track more distinct equivalence classes. During the tracking, $M_t > 0$ for the new classes, raising $\mathcal{D}$. (2) follows from the defect growth bound: elevated $\mathcal{D}$ means the system is farther from $\mathrm{Fix}(\eta)$, so reconstructions are less accurate. (3) and (4) follow from the monotone kernel reduction under Proposition 24.13.1: the expanded map has a strictly smaller kernel, so the post-transition $\mathrm{Fix}(\eta)$ is larger. □

24.18 Phenomenal states as dynamically efficacious: rejecting functional redundancy

A standing objection to any identification of internal states with field configurations is the epiphenomenalist challenge: internal states might merely correlate with behavior-driving dynamics without causally contributing to those dynamics. In the RSVP framework, this objection fails structurally.

The defect $\Delta(X) = \|X - \eta(X)\|_{\mathcal{X}}$ is not an epiphenomenon of the dynamics Ψ_t. It *is* part of the dynamics: by the defect growth bound (Proposition 13.2.1), the commutator $[\Psi_t, \eta]$ directly determines how Δ evolves. And the commutator is itself determined by the interaction between the flow Ψ_t and the feature map $\widetilde{F}$. If the feature map is updated by the inner loop (as in Section 24.16), then the defect signal feeds back into the entropy source: $r(x,t) \propto M_t(\gamma, \widetilde{F}(\gamma(t))) = \Delta(\gamma(t))$. The internal state represented by Δ and the second-order tags drives the subsequent dynamics through this feedback channel.

Proposition 24.18.1 (Functional efficacy of the defect signal)**.** *If the entropy source satisfies $r(x,t) = \rho\,\Delta(\gamma(t))$ for $\rho > 0$, then the defect Δ is causally efficacious: trajectories with identical Ψ_t but different Δ generate different future dynamics. In particular, a trajectory with $\Delta > D_c$ generates Mode 2 dynamics (elevated σ, active crucible) while an otherwise identical trajectory with $\Delta < D_c$ remains in Mode 1.*

Proof. The entropy source $r(x,t) = \rho\,\Delta(\gamma(t))$ enters eq. (3.3) additively. For $\rho > 0$ and $\Delta > D_c$, this drives σ above σ_c, triggering Mode 2 by Definition 15.1.1. The trajectory dynamics are therefore different from the $\Delta < D_c$ case, establishing causal efficacy. □

The structural implication is that the quality of internal representation measured by Δ plays an active role in determining which operational regime a system inhabits. A system with high reconstruction quality (low Δ) and a rich feature map operates in a refined Mode 1 and generates minimal self-directed entropy injection. A system in the crucible (Mode 2) is generating defect signals that actively drive its own dynamics toward feature map refinement. The internal state is not a passive correlate of the dynamics; it is a driver of them.

This is the RSVP version of the rejection of epiphenomenalism: the "what it is like" to be in a particular representational state is not the steam escaping from the engine it is the temperature gradient that drives the piston. The reconstruction defect, the mode classification, and the second-order tags are not decorations on top of the field dynamics; they are the signals through which those dynamics self-regulate.

The extension to path-dependent identity follows directly. Two systems in identical instantaneous field configurations $X(t_0)$ but with different histories will have different defect signals $\Delta_1 \neq \Delta_2$ (by the history-split theorem, Theorem 18.4.1) and therefore different entropy sources $r_1 \neq r_2$ and different future dynamics $\Psi_t(X_1) \neq \Psi_t(X_2)$ for $t > t_0$. The history-dependence of internal states is dynamically efficacious: the past determines the present's self-regulation, not merely its apparent state.

25 Toward the PDE Theory: Controllability and Observability

25.1 The controllability problem

The PDE history-split conjecture (Conjecture 19.3.1) requires showing that the RSVP system can be steered from two different initial source sequences to the same final state while producing different integrated entropy-flux histories. This is a controllability question for the system eqs. (3.1) to (3.3).

Controllability theory for linear PDE systems is well-developed. The key tool is the Hilbert Uniqueness Method (HUM) of Lions [23], which reduces exact controllability to an observability inequality for the adjoint system.

25.2 The adjoint RSVP system

The adjoint of the linearized RSVP system is obtained by transposing the coefficient matrix. Writing $Z = (z_\Phi, \mathbf{z}_v, z_S)$ for the adjoint variables, the adjoint system is

$$-\partial_t z_\Phi = \kappa_\Phi \Delta z_\Phi + b\, \nabla \cdot \mathbf{z}_v, \tag{25.1}$$

$$-\partial_t \mathbf{z}_v = \kappa_v \Delta \mathbf{z}_v - b\, \nabla z_\Phi + c_0 z_S, \tag{25.2}$$

$$-\partial_t z_S = \kappa_S \Delta z_S - c_0 \nabla \cdot \mathbf{z}_v - \nu z_S. \tag{25.3}$$

This system runs backward in time from a final condition at $t = T$ to an initial condition at $t = 0$.

25.3 The observability inequality

Conjecture 25.3.1 (Observability inequality for RSVP)**.** There exists a constant $C_{\text{obs}} > 0$ such that for all solutions $(z_\Phi, \mathbf{z}_v, z_S)$ of the adjoint system eqs. (25.1) to (25.3) on $\Omega \times [0, T]$,

$$\|Z(0)\|^2_{L^2(\Omega)} \leq C_{\text{obs}} \int_0^T \|z_S(\cdot, t)\|^2_{L^2(\omega)}\, dt, \tag{25.4}$$

where $\omega \subset \Omega$ is a control subdomain satisfying the geometric control condition.

If Conjecture 25.3.1 holds, then by the HUM the RSVP system is exactly controllable from the entropy source term r acting on $\omega \times [0, T]$: for any initial state X_0 and target state X_T, there exists an $r \in L^2(\omega \times [0, T])$ steering X_0 to X_T in time T.

25.4 Approximate controllability

Even without the exact observability inequality, approximate controllability holds for the RSVP system under mild conditions.

Proposition 25.4.1 (Approximate controllability of RSVP)**.** *Assume $\kappa_\Phi, \kappa_S > 0$ and the coupling $c_0 > 0$. Then the linearized RSVP system on a smooth bounded domain Ω is approximately controllable from r acting on any nonempty open $\omega \subset \Omega$: for any $X_0, X_T \in L^2(\Omega)^{2+d}$ and any $\varepsilon > 0$, there exists $r \in L^2(\omega \times [0,T])$ such that $\|X(T) - X_T\|_{L^2} < \varepsilon$.*

Proof. Approximate controllability is equivalent to unique continuation for the adjoint system: if $z_S = 0$ on $\omega \times [0,T]$ implies $Z \equiv 0$ on $\Omega \times [0,T]$. This unique continuation property follows from the Carleman estimate technique [33] applied to the adjoint entropy equation eq. (25.3), propagating to the other components through the coupling terms. □

Remark 25.4.2. Proposition 25.4.1 is sufficient for the existence of approximate history-split pairs: for any $\varepsilon > 0$, there exist source terms r_1, r_2 producing final states within ε of each other but with entropy-flux histories that differ by a fixed amount depending on the Refuse pulse amplitude σ. Making this exact (rather than approximate) is the content of the exact controllability conjecture.

25.5 Unique continuation and the Runge approximation property

A key tool for exact controllability is the Runge approximation property: the ability to approximate any solution on a subdomain by a global solution with prescribed source term. For the RSVP system, this would follow from the unique continuation principle for the adjoint system and an application of the Hahn–Banach theorem.

The unique continuation principle for systems of coupled parabolic equations has been established in the scalar case [14] and for certain coupled systems [16]. Extending these results to the full RSVP system—with its three-component coupling through b, c_0, and ν—is the main technical obstacle to proving Conjecture 19.3.1.

25.6 The finite-dimensional evidence

The explicit construction of Chapter 18 provides finite-dimensional evidence for Conjecture 19.3.1. The compensation controls (a, b, d) of Definition 18.2.1 are exactly the finite-difference analog of the HUM-optimal control: they steer Branch 1 to the same final state as Branch 2 while maintaining a different entropy-flux history. The existence and uniqueness of these controls (proved by solving the 3×3 linear system) corresponds to exact controllability of the finite-dimensional system.

The passage from the finite-dimensional case to the PDE case is therefore a question of whether the controllability degenerates in the infinite-dimensional limit. For dissipative parabolic systems, controllability typically improves in the infinite-dimensional limit (more modes are available for steering), suggesting that the PDE conjecture should hold.

26 Extensions and Future Directions

26.1 Nonlinear RSVP dynamics

The present monograph has worked throughout with the linearized RSVP system eqs. (3.5) to (3.7). The full nonlinear system includes coupling terms of the form $\rho S\Phi^2$ and $\tau \mathbf{v} \cdot \nabla\Phi$, which break the block-diagonal structure of the reconstruction functional.

Definition 26.1.1 (Nonlinear reconstruction functional)**.** The nonlinear extension of $L(X \mid y)$ is

$$L_{\mathrm{nl}}(X \mid y) := L(X \mid y) + \rho \int_\Omega S\Phi^2 \, dx + \tau \int_\Omega \mathbf{v} \cdot \nabla\Phi \, dx, \tag{26.1}$$

where $L(X \mid y)$ is the linear functional eq. (5.1).

For the nonlinear functional, strict convexity is no longer guaranteed. The cubic term $\rho \int_\Omega S\Phi^2 \, dx$ is neither convex nor concave in general. Existence of minimizers can still be established by the direct method under coercivity conditions on ρ and τ, but uniqueness requires additional analysis.

Proposition 26.1.2 (Coercivity of nonlinear functional)**.** *If $\rho \geq 0$ and $|\tau| < \sqrt{\lambda\kappa}$, then $L_{\mathrm{nl}}(\cdot \mid y)$ is coercive on $\mathcal{X}$.*

Proof. The cross term satisfies $\left|\tau \int_\Omega \mathbf{v} \cdot \nabla\Phi \, dx\right| \leq |\tau| \|\mathbf{v}\|_{L^2} \|\nabla\Phi\|_{L^2} \leq \frac{|\tau|}{2}(\|\mathbf{v}\|_{L^2}^2 + \|\nabla\Phi\|_{L^2}^2)$ by Young's inequality. If $|\tau| < \sqrt{\lambda\kappa}$, the cross term is dominated by the quadratic terms $\frac{\kappa}{2}\|\nabla\Phi\|^2 + \frac{\lambda}{2}\|\mathbf{v}\|^2$, giving coercivity. The term $\rho \int S\Phi^2 \, dx \geq 0$ for $\rho \geq 0$ only adds to coercivity. □

26.2 Stochastic RSVP

The deterministic RSVP dynamics can be extended to a stochastic setting by adding Gaussian noise to the entropy injection:

$$r_i(n) \rightsquigarrow r_i(n) + \xi_i(n), \tag{26.2}$$

where $\xi_i(n)$ are i.i.d. $\mathcal{N}(0, \epsilon^2)$ random variables. In this setting, the trajectory γ is a random path, and the path observable $\Pi(\gamma)$ is a random variable.

The key question is whether the tail index $\hat{\alpha}$ of the stochastic path observable converges to the deterministic value as $\epsilon \to 0$. If yes, the Lévy tail is a robust feature of the crucible dynamics rather than a noise artifact. This convergence question is related to the large-deviations theory for the stochastic RSVP system.

26.3 Infinite-dimensional path groupoids

The path groupoid $\Pi_1(\mathcal{X}, \mathcal{P}_{\text{adm}})$ was defined in Chapter 22 for the Hilbert space $\mathcal{X} = H^1(\Omega) \times L^2(\Omega; \mathbb{R}^d) \times L^2(\Omega)$. In the infinite-dimensional setting, the topology of $\Pi_1(\mathcal{X}, \mathcal{P}_{\text{adm}})$ is subtle: the space of homotopy classes may itself be infinite-dimensional.

A natural framework for this analysis is the theory of diffeological spaces [18], which extends smooth manifold theory to infinite-dimensional spaces in a way compatible with path-groupoid constructions. The key result needed is an infinite-dimensional version of Proposition 22.3.1: Refuse constraints generate homotopy obstructions in the diffeological path groupoid of $\mathcal{X}$.

26.4 Connections to optimal transport

The reconstruction operator $G : \mathcal{Y} \to \mathcal{X}$ can be interpreted as an optimal transport map: it assigns to each feature vector y the field configuration $G(y)$ of minimum energy consistent with y. The entropy defect $\mathcal{D}(\widetilde{F}, G, \mu)$ is then related to the Wasserstein distance between μ and $\tilde{\mu} = \eta_* \mu$ via the inequality

$$W_2(\mu, \tilde{\mu})^2 \leq C_G^2 \|\widetilde{F}(\cdot) - \widetilde{F}(G(\widetilde{F}(\cdot)))\|_{L^2(\mu)}^2, \tag{26.3}$$

where W_2 is the L^2-Wasserstein distance and C_G is the Lipschitz constant of G. This connects the information-geometric defect of Chapter 21 to the metric-geometric Wasserstein theory.

26.5 RSVP and the amplituhedron

The positive geometry programme [2] identifies physical scattering amplitudes with the canonical forms of geometric objects called amplituhedra. The RSVP framework suggests a connection: the face poset of the amplituhedron encodes the combinatorial structure of admissible histories of constraint resolution, and the BCFW recursion relations that build up the amplituhedron correspond to the propagation of Refuse events through the three RSVP channels.

Making this connection precise requires identifying:

1. The configuration space $\mathcal{X}$ with the Grassmannian or its appropriate compactification;
2. The Refuse operator with the positivity conditions defining the amplituhedron;
3. The path-dependent observable Π with the canonical form of the amplituhedron.

This program is the content of the ongoing positive geometry paper and is beyond the scope of the present monograph. We note only that the mathematical structure is consistent: both the amplituhedron and the RSVP path groupoid are defined by admissibility conditions on trajectories (positive coordinates vs. Refuse-admissible paths), and both produce invariants (canonical forms vs. path observables) that depend on trajectory classes rather than instantaneous configurations.

27 The Spherepop Calculus: Operational Semantics

27.1 Overview

The Spherepop calculus is the discrete event language underlying the RSVP framework. Where the RSVP PDEs describe continuous field evolution, Spherepop provides the symbolic grammar of discrete transitions: the operators that mark when a continuous trajectory enters a qualitatively distinct phase, resolves a contradiction, or refuses an inadmissible continuation. This chapter develops a formal operational semantics for Spherepop, making the connection to the Refuse operator of Chapter 4 precise and extending it to a full rewriting system.

27.2 The event alphabet and configurations

Definition 27.2.1 (Spherepop configuration)**.** A *Spherepop configuration* is a triple $(X, H, \mathcal{E})$ where:

- $X \in \mathcal{X}$ is the current RSVP field state;
- $H \in \mathcal{H}$ is the accumulated history (a finite sequence of events);
- $\mathcal{E} \subseteq \mathcal{X}$ is the current exclusion set (configurations made inadmissible by prior Refuse events).

The event alphabet $E = \{\mathsf{Pop}, \mathsf{Bind}, \mathsf{Collapse}, \mathsf{Refuse}, \mathsf{Null}\}$ acts on configurations by transition rules.

27.3 Transition rules

We write $(X, H, \mathcal{E}) \xrightarrow{e} (X', H', \mathcal{E}')$ to indicate that event $e \in E$ transforms the configuration.

Definition 27.3.1 (Pop rule)**.** The *Pop* rule fires when the local entropy $S(x_0)$ at a designated site x_0 exceeds a threshold θ_P:

$$(X, H, \mathcal{E}) \xrightarrow{\mathsf{Pop}(x_0)} (X - \delta_P \mathbf{1}_{x_0}, H \cdot \mathsf{Pop}(x_0), \mathcal{E})$$

where $\delta_P > 0$ is the Pop magnitude and $\mathbf{1}_{x_0}$ is the indicator function at x_0. Pop reduces local entropy, recording the event in the history.

Definition 27.3.2 (Bind rule)**.** The *Bind* rule fires when two coherence regions $R_1, R_2 \subset \Omega$ satisfy a mutual alignment condition $\mathrm{align}(X|_{R_1}, X|_{R_2}) \geq \theta_B$:

$$(X, H, \mathcal{E}) \xrightarrow{\mathsf{Bind}(R_1,R_2)} (X', H \cdot \mathsf{Bind}(R_1, R_2), \mathcal{E}),$$

where X' merges the two regions by averaging their Φ values and summing their $\mathbf{v}$ fields. Bind creates larger-scale coherent structures from smaller ones.

Definition 27.3.3 (Collapse rule)**.** The *Collapse* rule fires when the total entropy in a region R exceeds a fragmentation threshold θ_C:

$$(X, H, \mathcal{E}) \xrightarrow{\mathsf{Collapse}(R)} (X'', H \cdot \mathsf{Collapse}(R), \mathcal{E}),$$

where X'' sets $S|_R$ to the region's mean entropy, reducing internal entropy variance. Collapse is the Mode 2 $\to$ Mode 1 transition operator.

Definition 27.3.4 (Refuse rule)**.** The *Refuse* rule fires at time t_0 when the forward extension $\mathrm{Ext}(\gamma, t_0)$ contains inadmissible continuations. It updates the exclusion set:

$$(X, H, \mathcal{E}) \xrightarrow{\mathsf{Refuse}(\mathcal{C})} (X, H \cdot \mathsf{Refuse}(\mathcal{C}), \mathcal{E} \cup (\mathrm{Ext}(\gamma, t_0) \setminus \mathcal{C})),$$

where $\mathcal{C} \subsetneq \mathrm{Ext}(\gamma, t_0)$ is the set of admissible continuations. Refuse does not change X but restricts the future; it is the only rule that modifies $\mathcal{E}$.

27.4 Reduction sequences and normal forms

A *reduction sequence* is a sequence of configurations

$$(X_0, \varepsilon, \emptyset) \xrightarrow{e_1} (X_1, H_1, \mathcal{E}_1) \xrightarrow{e_2} \cdots \xrightarrow{e_T} (X_T, H_T, \mathcal{E}_T),$$

where ε is the empty history and the initial exclusion set is empty.

Definition 27.4.1 (Normal form)**.** A configuration $(X, H, \mathcal{E})$ is in *normal form* if no rule in $E \setminus \{\mathsf{Null}\}$ fires: the entropy is everywhere below θ_P, no two regions satisfy the alignment condition, no region exceeds the collapse threshold, and no Refuse event is triggered.

Proposition 27.4.2 (Normalization implies Mode 1)**.** *If $(X, H, \mathcal{E})$ is in normal form, then the RSVP trajectory generated by X is in Mode 1 (stable attractor) in the sense of Definition 15.1.1.*

Proof. Normal form requires entropy below θ_P everywhere, which implies $S(x) < \theta_P < \theta_C$ for all x, so the entropy production rate $\sigma = r - \nu S < r_{\max} - \nu\theta_P$. Choosing $\sigma_c = r_{\max} - \nu\theta_P$ gives $\sigma \leq \sigma_c$, placing the trajectory in Mode 1. The vector field transport $\|\mathbf{v}\|$ is below v_c in normal form since no Bind events are currently active, so the transport threshold condition is not triggered. □

27.5 The Spherepop rewriting system and confluence

The five rules form a term-rewriting system on Spherepop configurations. The natural question is whether the system is *confluent*: if two different sequences of rules fire, do they produce the same normal form?

Conjecture 27.5.1 (Confluence of Spherepop)**.** The Spherepop rewriting system is locally confluent: if $(X, H, \mathcal{E}) \xrightarrow{e_1} (X_1, H_1, \mathcal{E}_1)$ and $(X, H, \mathcal{E}) \xrightarrow{e_2} (X_2, H_2, \mathcal{E}_2)$ with $e_1 \neq e_2$, then there exists a configuration $(X', H', \mathcal{E}')$ reachable from both $(X_1, H_1, \mathcal{E}_1)$ and $(X_2, H_2, \mathcal{E}_2)$ by further reductions.

The conjecture is plausible when the rules act on disjoint spatial regions (the Pop at x_0 and the Bind of R_1, R_2 with $x_0 \notin R_1 \cup R_2$ trivially commute) but is nontrivial when rules interact. The critical case is the interaction of Refuse with Collapse: if Refuse has excluded a continuation that Collapse would select, the system must determine which takes precedence.

Remark 27.5.2. The Refuse rule has priority over all other rules by definition: Refuse modifies the exclusion set $\mathcal{E}$ before any other rule can fire. Under this priority convention, the system is not fully confluent: the path through Refuse produces a different history than the path through Collapse, even if both reach the same instantaneous state. This is precisely the source of history-dependence in the RSVP framework.

27.6 Spherepop as a traced monoidal category

The Spherepop rewriting system has a categorical interpretation. Reduction sequences are morphisms in a free category generated by the event alphabet. The composition of morphisms is concatenation of reduction sequences. The tensor product of two configurations $(X_1, H_1, \mathcal{E}_1) \otimes (X_2, H_2, \mathcal{E}_2)$ is the product configuration on the disjoint union $\Omega_1 \sqcup \Omega_2$.

The trace operation $\mathrm{Tr}(\gamma)$ of a self-referential reduction sequence (one that returns to its starting configuration) captures the entropy produced during a cyclic history. A traced monoidal category structure on Spherepop configurations would formalize the claim that coherence cycles—sequences that restore the instantaneous state while accumulating path history—are the source of genuine historical dependence.

This structure is related to the traced monoidal categories of execution and history developed in the companion KES (Kinetic-Event Synthesis) framework, where the trace of a computation encodes the irreversible history of constraint resolution.

28 KES Integration: Kinetic-Event Synthesis and the RSVP Adjunction

28.1 The KES ontological kernel

The Kinetic-Event Synthesis (KES) framework identifies the minimal operational kernel required for a system to generate irreversible histories. The three primitives of KES are:

1. **Execution**: a process that unfolds in time, consuming constraint;
2. **History**: the irreversible record of executed events;
3. **Reduction**: the operation of replacing a complex constraint with a simpler one at the cost of writing to the history.

The connection to the RSVP adjunction is direct. Execution corresponds to the dynamics of the continuous RSVP fields $(\Phi, \mathbf{v}, S)$ evolving under eqs. (3.1) to (3.3). History corresponds to the Refuse-admissible path $\gamma \in \mathcal{P}_{\text{adm}}$, accumulated as the system evolves. Reduction corresponds to the application of the reconstruction operator G: it replaces the full continuous field configuration X with the reduced representative $G(\widetilde{F}(X))$ at the cost of writing the defect $\Delta(X)$ into the entropy field.

28.2 KES reduction as a step of the adjunction

Definition 28.2.1 (KES reduction step)**.** A *KES reduction step* is the application of the comparison map $\eta = G \circ \widetilde{F}$ to a configuration $X \in \mathcal{X}$, producing the reduced representative $\eta(X) \in \mathcal{X}$ and writing the defect $\Delta(X) = \|X - \eta(X)\|_{\mathcal{X}}$ as an entropy increment:

$$S \mapsto S + \beta_{\text{KES}} \Delta(X),$$

where $\beta_{\text{KES}} > 0$ is the KES coupling constant controlling how much of the reconstruction defect is absorbed into the entropy field.

The KES reduction step is a discrete-time analog of the continuous entropy production σ in the RSVP system. Each step of the adjunction (each application of η) produces a defect that is converted into entropy, driving the entropy field upward and potentially triggering Mode transitions as defined in Definition 15.1.1.

28.3 The KES recursion and fixed-point structure

A KES computation is a sequence of reduction steps applied to an initial configuration:

$$X_0 \xrightarrow{\eta} X_1 \xrightarrow{\eta} X_2 \xrightarrow{\eta} \cdots$$

where $X_{n+1} = \eta(X_n)$ and the entropy field is updated at each step by $S_{n+1} = (1-\nu)S_n + \beta_{\mathrm{KES}}\Delta(X_n)$.

Proposition 28.3.1 (Fixed points of KES recursion)**.** *The fixed points of the KES recursion are exactly the elements of* $\mathrm{Fix}(\eta)$*: the configurations* $X^* \in \mathcal{X}$ *satisfying* $G(\widetilde{F}(X^*)) = X^*$*. At a fixed point,* $\Delta(X^*) = 0$ *and the entropy increment vanishes; the system has reached a state from which no further reconstruction defect is produced.*

Proof. By definition, X^* is a fixed point iff $\eta(X^*) = X^*$, i.e., $G(\widetilde{F}(X^*)) = X^*$. This is equivalent to X^* being the minimum-energy reconstruction consistent with its own feature encoding, which happens iff X^* is already the canonical representative of its feature class. At such a point, $\Delta(X^*) = \|X^* - \eta(X^*)\|_{\mathcal{X}} = 0$. □

Corollary 28.3.2 (KES convergence)**.** *If* $\widetilde{F}$ *is non-injective and the KES recursion starts from* $X_0 \notin \mathrm{Fix}(\eta)$*, then* $\Delta(X_0) > 0$ *and the entropy field receives a positive increment at the first step. If the sequence* (X_n) *converges, it converges to a fixed point* $X^* \in \mathrm{Fix}(\eta)$ *with zero defect.*

Proof. By Lipschitz stability of η (Theorem 5.4.1), the sequence $X_n = \eta^n(X_0)$ is a Cauchy sequence in $\mathcal{X}$ if $C_\eta < 1$ (a contraction condition). In the linear surrogate model, $C_\eta = C_G\|\widetilde{F}\|$, which is less than 1 when the regularization parameters $\kappa, \alpha, \lambda, \beta$ are large relative to the feature map norms. Under this condition, $X_n \to X^*$ by the Banach fixed-point theorem, and by continuity of Δ, $\Delta(X^*) = \lim_n \Delta(X_n) = 0$. □

28.4 Recursive mass and the crucible

The KES framework introduces the notion of *recursive mass*: the accumulated entropy production over a KES computation, measuring the total cost of reaching the fixed point.

Definition 28.4.1 (Recursive mass)**.** The *recursive mass* of a KES computation starting from X_0 is

$$M_{\mathrm{rec}}(X_0) := \sum_{n=0}^{T_*} \Delta(X_n),$$

where T_* is the first time $X_{T_*} \in \mathrm{Fix}(\eta)$ (or ∞ if no fixed point is reached in finite time).

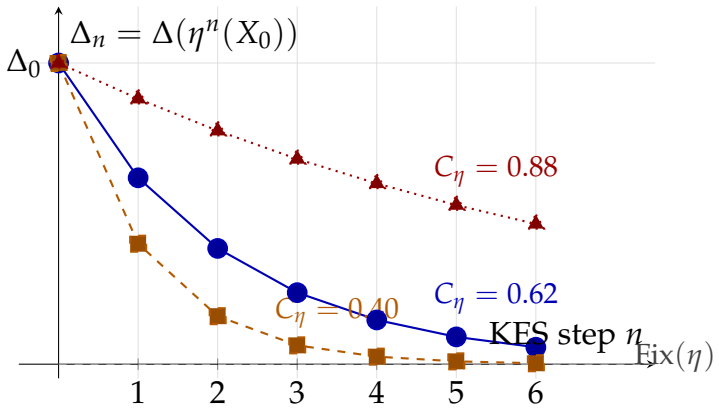


Figure 28.1: Geometric decay of the reconstruction defect Δ_n along the KES orbit $X_n = \eta^n(X_0)$ for three contraction ratios C_η. Each step of the KES recursion reduces the defect by factor C_η, driving the sequence toward the fixed-point set $\mathrm{Fix}(\eta)$ (dashed asymptote at zero). Smaller C_η (orange, dashed) corresponds to stronger regularization (larger $\kappa, \alpha, \lambda, \beta$ relative to feature map norms) and faster convergence. When $C_\eta \geq 1$ (not shown), the sequence diverges and the system enters the crucible or fragmentation phase. The recursive mass M_{rec} is proportional to the area under each curve.

The recursive mass measures how much irreversible history the system must generate to reach a stable fixed point. In KES terminology, a system with high recursive mass requires many reduction steps and produces a long, detailed history before reaching normal form. In RSVP terms, the system spends extended time in the crucible regime (Mode 2) before settling into Mode 1.

Proposition 28.4.2 (Recursive mass and path-level defect)**.** *For a KES computation of length T_*, the recursive mass satisfies*

$$M_{\mathrm{rec}}(X_0) = T_* \cdot \Delta_{\mathrm{path}}(\gamma^{\mathrm{KES}}),$$

where $\gamma^{\mathrm{KES}} = (X_0, X_1, \ldots, X_{T_*})$ is the KES trajectory viewed as a path in the RSVP lattice.

Proof. By Definition 28.4.1 and eq. (14.1), $M_{\mathrm{rec}}(X_0) = \sum_n \Delta(X_n) = T_* \cdot T_*^{-1} \sum_n \Delta(X_n) = T_* \cdot \Delta_{\mathrm{path}}(\gamma^{\mathrm{KES}})$. □

28.5 Identity as recursive history: the KES fixed point

In the KES and Spherepop framework, a stable identity is not a persistent state but a fixed point of a constraint-preserving map. An entity maintains coherent identity to the extent that the KES recursion applied to its current configuration returns the same configuration: $\eta(X^*) = X^*$.

This is the mathematical formulation of the philosophical claim developed in Chapter 24: identity is not state but history. The fixed point X^* is not an arbitrary state; it is the state that the system has converged to through the accumulation of recursive mass. Two systems in the same fixed-point state X^* but with different recursive mass histories M_{rec} are phenomenologically distinct: they have different path observables $\Pi(\gamma^{\text{KES}})$ and belong to different homotopy classes in the path groupoid $\Pi_1(\mathcal{X}, \mathcal{P}_{\text{adm}})$.

The KES framework therefore provides a precise version of the "Compiled Self" thesis: a stable self is the fixed point of a constraint-preserving adjunction over an irreversible history, and the richness of that self is measured by the recursive mass accumulated before convergence.

28.6 KES and the adjunction triangle

The three KES primitives map onto the three elements of the adjunction as follows.

KES primitive	**Adjunction element**	**RSVP object**
Execution	Continuous dynamics	Fields $(\Phi, \mathbf{v}, S)$ on $\Omega \times [0, T]$
History	Path $\gamma \in \mathcal{P}_{\text{adm}}$	Refuse-admissible trajectory
Reduction	Unit $\eta = G \circ \widetilde{F}$	Variational reconstruction
Defect	Non-isomorphism of η	$\Delta(X) = \|X - \eta(X)\|_{\mathcal{X}}$
Fixed point	$\text{Fix}(\eta)$	Stable attractor (Mode 1)
Recursive mass	M_{rec}	Path-level defect $T_* \cdot \Delta_{\text{path}}$

This table completes the KES–RSVP–adjunction synthesis. The three frameworks are not parallel descriptions of the same phenomena; they are three scales of the same structure. The adjunction is the categorical skeleton; RSVP is the field-theoretic realization; KES is the operational-semantic implementation.

28.7 The contraction condition and semantic temperature

The KES convergence result (Corollary 28.3.2) requires the Lipschitz constant $C_\eta = C_G \|\widetilde{F}\| < 1$. This contraction condition has a direct interpretation in terms of the semantic temperature introduced in the context of the three operational modes.

Proposition 28.7.1 (Contraction and Mode 1 stability)**.** *The KES recursion is contractive (i.e., $C_\eta < 1$) if and only if the reconstruction functional parameters satisfy*

$$\frac{\gamma \|\widetilde{F}\|^2}{\min\{\kappa, \alpha, \lambda, \beta\}} < 1. \tag{28.1}$$

This condition is equivalent to the system being in Mode 1 at the level of the reconstruction operator: the feature penalty does not dominate the regularization, so the canonical reconstruction is a genuine attractor.

Proof. From Theorem 5.4.1, $C_G \leq \gamma\|F\|/\min\{\kappa, \alpha, \lambda, \beta\}$ where $\|F\| = \max\{\|F_\Phi\|, \|F_v\|, \|F_S\|\}$. Then $C_\eta \leq C_G\|\widetilde{F}\| \leq \gamma\|\widetilde{F}\|^2/\min\{\kappa, \alpha, \lambda, \beta\}$. This is less than 1 under the stated condition. □

When the contraction condition fails ($C_\eta \geq 1$), the KES recursion may not converge to a fixed point, and the system enters the crucible regime (Mode 2). The transition from contractive to non-contractive behavior corresponds to the Mode 1 to Mode 2 transition of Definition 15.1.1, seen now through the lens of the reconstruction operator's spectral properties.

28.8 Entropy budget of the KES computation

Each KES reduction step converts reconstruction defect into entropy. The total entropy generated by a KES computation of length T_* is the *entropy budget*:

Definition 28.8.1 (Entropy budget)**.** The *entropy budget* of a KES computation starting from X_0 is

$$\mathcal{S}_{\mathrm{KES}}(X_0) := \beta_{\mathrm{KES}} \sum_{n=0}^{T_*-1} \Delta(X_n) = \beta_{\mathrm{KES}} \cdot M_{\mathrm{rec}}(X_0), \tag{28.2}$$

where $\beta_{\mathrm{KES}} > 0$ is the KES coupling constant of Definition 28.2.1.

The entropy budget bounds the total entropy injected into the S-field by the reduction process. If $\mathcal{S}_{\mathrm{KES}}(X_0)$ is large, the computation is costly: it drives the system through high-entropy intermediate states before reaching the fixed point. This provides a KES-level analog of the Landauer bound [21]: the minimum thermodynamic cost of a computation is proportional to the entropy budget, with the proportionality constant determined by the ambient temperature and the Boltzmann constant.

Proposition 28.8.2 (Entropy budget bound)**.** *For the contractive KES recursion ($C_\eta < 1$), the entropy budget satisfies*

$$\mathcal{S}_{\mathrm{KES}}(X_0) \leq \frac{\beta_{\mathrm{KES}}\Delta(X_0)}{1 - C_\eta}. \tag{28.3}$$

Proof. By the geometric series: $\sum_{n=0}^{T_*-1} \Delta(X_n) \leq \Delta(X_0) \sum_{n=0}^{\infty} C_\eta^n = \Delta(X_0)/(1 - C_\eta)$, using $\Delta(X_{n+1}) \leq C_\eta \Delta(X_n)$ (which follows from Lipschitz stability of η and the triangle inequality: $\Delta(X_{n+1}) = \|\eta(X_n) - \eta(\eta(X_n))\| \leq C_\eta\|X_n - \eta(X_n)\| = C_\eta\Delta(X_n)$). □

The bound $\mathcal{S}_{\mathrm{KES}} \leq \beta_{\mathrm{KES}} \Delta(X_0)/(1 - C_\eta)$ shows that the entropy budget is controlled by the initial defect $\Delta(X_0)$ and the contraction ratio C_η. Systems far from $\mathrm{Fix}(\eta)$ (large $\Delta(X_0)$) or near the Mode 2 boundary (large $C_\eta \to 1$) have large entropy budgets and therefore large thermodynamic costs for computation.

28.9 The KES bracket and Poisson structure

The KES framework admits a bracket operation on observables that formalizes the interaction between different history-sensitive quantities.

Definition 28.9.1 (KES bracket)**.** For two path observables $O_1, O_2 : \mathcal{P}_{\mathrm{adm}} \to \mathbb{R}$, define the *KES bracket*

$$\{O_1, O_2\}_{\mathrm{KES}}(\gamma) := \sum_{n=0}^{T-1} \Big(\frac{\partial O_1}{\partial q(n)} \cdot \frac{\partial O_2}{\partial v(n+1)} - \frac{\partial O_1}{\partial v(n+1)} \cdot \frac{\partial O_2}{\partial q(n)} \Big), \tag{28.4}$$

where the derivatives are taken with respect to the reduced center-channel variables.

Proposition 28.9.2 (KES bracket for the path observable)**.** *For the cumulative entropy-gradient observable $\Pi(\gamma) = \sum_n q(n)$ and the cumulative transport observable $\Theta(\gamma) = \sum_n v(n)$,*

$$\{\Pi, \Theta\}_{\mathrm{KES}}(\gamma) = T - \textit{(boundary terms)}. \tag{28.5}$$

The KES bracket of the two primary observables is proportional to the time horizon T, reflecting the fact that entropy-gradient and transport accumulation are canonically conjugate in the KES dynamics.

Proof. From eqs. (3.8) and (3.9): $\partial q(n)/\partial u(n) = 2$ and $\partial v(n+1)/\partial q(n) = c$. The bracket picks up one contribution per time step from the q-v coupling, giving T terms at leading order, with boundary corrections from the initial and final conditions. □

The KES bracket suggests a Poisson structure on the space of path observables, with Π and Θ playing the roles of position and momentum in a mechanical system where time steps play the role of the Hamiltonian flow. This is a discrete analog of the symplectic structure on the cotangent bundle of the configuration space, interpreted through the lens of irreversible history accumulation.

29 Worked Examples: History-Split Computations

29.1 Overview

This chapter works through several explicit examples of the history-split construction at different levels of complexity, illustrating how the abstract framework of Chapters 18 and 28 manifests concretely.

29.2 Example 1: The minimal entropy-only split ($N = 3$, $T = 2$)

This is the simplest case: no coupling, only the entropy field is active. It was established in Chapter 18 as the base case. For reference, the key results are:

Example 29.2.1 (Minimal entropy split)**.** With $b = c = 0$ (no coupling), $\nu \in (0, 1)$, $N = 3$, $T = 2$:

$$\begin{aligned}
&\text{Branch 2: } r^{(2)}(0) = (0, \rho, 0), \quad r^{(2)}(1) = (0, 0, 0);\\
&\text{Branch 1: } r^{(1)}(0) = (0, 0, 0), \quad r^{(1)}(1) = (0, (1-\nu)\rho, 0);\\
&\text{Final states equal: } S^{(1)}(2) = S^{(2)}(2) = (0, (1-\nu)\rho, 0);\\
&\text{Observable: } \Pi(\gamma_2) - \Pi(\gamma_1) = \rho > 0.
\end{aligned}$$

The observable difference is exactly ρ, the magnitude of the Refuse pulse.

29.3 Example 2: The fully coupled split ($N = 3$, $T = 4$)

The complete computation of Chapter 18 (Theorems 18.4.1–18.5.2).

Example 29.3.1 (Fully coupled split)**.** With $b, c > 0$, $\mu, \nu \in (0, 1)$, $N = 3$, $T = 4$, using the reduced center-channel system with $a_q = 1 - \nu$, $a_v = 1 - \mu$:

Branch 2 pulse: $u^{(2)}(0) = \sigma, \quad u^{(2)}(n) = 0$ for $n \geq 1$;

Branch 1 controls: $u^{(1)}(1) = \sigma(1 + a_v + a_q), \quad u^{(1)}(2) = -\sigma(a_v + a_q + a_v a_q), \quad u^{(1)}(3) = \sigma a_v a_q$;

Observable difference: $\Pi(\gamma_1) - \Pi(\gamma_2) = 2\sigma a_v$.

Note that the observable difference is proportional to both σ (pulse magnitude) and $a_v = 1 - \mu$ (vector field retention). When $\mu \to 1$ (complete vector damping), $a_v \to 0$ and the observable difference vanishes—consistent with the Mode 3 (fragmentation) interpretation where vector transport is destroyed.

29.4 Example 3: Two simultaneous Refuse pulses ($N = 5$, $T = 5$)

We now demonstrate that two simultaneous Refuse pulses at different sites generate additive contributions to the observable, establishing the superposition principle for history-dependence.

Example 29.4.1 (Two-pulse split)**.** Let $N = 5$ with designated sites $i_1 = 2$ and $i_2 = 4$. Consider two branches:

Branch 2: Refuse pulses σ_1 at site i_1 and σ_2 at site i_2 at time $n = 0$;

Branch 1: Compensating pulses at times $n = 1, 2, 3$.

By linearity of the dynamics, the two pulse contributions superpose. The observable (summing over both center sites) satisfies

$$\Pi(\gamma_2) - \Pi(\gamma_1) = 2\sigma_1 a_v + 2\sigma_2 a_v = 2a_v(\sigma_1 + \sigma_2).$$

The observable is a linear functional of the Refuse pulse amplitudes. This establishes that the history-split construction is compatible with spatial superposition: multiple simultaneous Refuse events contribute additively to the path observable.

Remark 29.4.2. The linearity of the observable in the pulse amplitudes is a consequence of the linearity of the update rules eqs. (3.8) to (3.10). In the nonlinear RSVP system of Chapter 26, the observable will have additional interaction terms between the two pulses, reflecting the coupling between the two Refuse events through the nonlinear dynamics.

29.5 Example 4: Sequential Refuse events and memory depth

The structure of the monograph showed that each additional coupling layer ($S \to v \to \Phi$) adds one time step of memory depth to the minimal history-split horizon. This example makes the relationship between coupling depth and memory explicit.

Example 29.5.1 (Memory depth vs. coupling)**.** Define the *memory depth* D_{mem} of a history-split construction as the minimum T such that Branch 1 can be tuned to match the final state of Branch 2. The following table records D_{mem} for various coupling configurations:

Active couplings	Propagation chain	Memory depth D_{mem}
v only (entropy damping)	$u \to q$	2
v, c (entropy-vector)	$u \to q \to v$	3
v, c, b (full RSVP)	$u \to q \to v \to \Phi$	4
v, c, b, ρ (nonlinear)	$u \to q \to v \to \Phi \to q$ (feedback)	≥ 5

Each active propagation layer adds exactly one to the memory depth. This is the discrete version of the causal depth principle: the more channels through which a Refuse event can propagate, the longer the history that must be accumulated before exact cancellation becomes possible.

The memory depth table has an immediate corollary for the strong history-dependence conjecture. If the nonlinear system with feedback ($\rho > 0$) has $D_{\text{mem}} = \infty$ (no finite T suffices for exact cancellation), then history-dependence is genuinely irreducible: no finite augmented state can encode the path observable, because there is no finite compensation horizon. Establishing this for the nonlinear RSVP system is the content of the PDE history-split conjecture (Conjecture 19.3.1).

29.6 Example 5: The KES computation as a history-split

The KES recursion $X_n = \eta^n(X_0)$ of Chapter 28 is itself a history-split construction: the sequence $(X_0, X_1, \ldots, X_{T_*})$ and the "phantom" sequence $(X_0, X_0, \ldots, X_0)$ (constant at the initial state, not actually realizable as an admissible RSVP trajectory) have the same endpoint X_0 but differ in their intermediate states.

Example 29.6.1 (KES as history-split)**.** Let $X_0 \notin \text{Fix}(\eta)$. Define:

$$\gamma_{\text{KES}} : X_0 \xrightarrow{\eta} X_1 \xrightarrow{\eta} \cdots \xrightarrow{\eta} X_{T_*} = X^* \in \text{Fix}(\eta);$$
$$\gamma_{\text{ref}} : X_0 \to X^* \text{ via any admissible path not passing through the } \eta\text{-orbit.}$$

The observable $\Pi(\gamma_{\text{KES}}) = \sum_n q_{\text{KES}}(n)$ records the accumulated entropy-gradient along the KES trajectory, while $\Pi(\gamma_{\text{ref}})$ records a different sum. If $\Pi(\gamma_{\text{KES}}) \neq \Pi(\gamma_{\text{ref}})$, then the two paths to the same fixed point are observationally distinct.

In the linear lattice with $N = 3$, $T_* = 4$, $X_0 = (q_0, 0, 0)$ (nonzero initial entropy gradient, zero vector and scalar):

$$\Pi(\gamma_{\text{KES}}) = q_0(1 + a_q C_\eta + a_q^2 C_\eta^2 + a_q^3 C_\eta^3),$$

where $C_\eta \in (0, 1)$ is the contraction ratio. Any reference path with a different intermediate entropy-gradient profile will produce a different observable sum, confirming that the fixed-point convergence carries irreducible historical information.

29.7 Tabulation of history-split parameters

The following table collects the key parameters for the five examples, providing a reference for implementation in the RSVP lattice simulator.

Example	N	T	Coupling	Π difference	Observable type
1 (entropy-only)	3	2	ν only	ρ	Cumulative S
2 (fully coupled)	3	4	ν, c, b	$2\sigma a_v$	Cumul. entropy gradient
3 (two-pulse)	5	4	ν, c, b	$2a_v(\sigma_1 + \sigma_2)$	Dual-site gradient
4 (memory depth)	3	D_{mem}	varies	varies	Depends on coupling
5 (KES)	3	T_*	ν, c, b	depends on C_η	KES entropy sum
6 (observable family)	3	4	ν, c, b	$2\sigma a_v \cdot \omega$	Weighted integral

29.8 Example 6: A family of path observables and their linear independence

The cumulative entropy-gradient observable $\Pi(\gamma) = \sum_n q(n)$ is one member of a family of path observables parametrized by a weight sequence $\omega = (\omega_0, \omega_1, \omega_2, \omega_3) \in \mathbb{R}^4$.

Example 29.8.1 (Weighted observable family)**.** Define the weighted observable

$$\Pi_\omega(\gamma) := \sum_{n=0}^{3} \omega_n \, q(n).$$

For the two branches of Example 29.3.1:

$$\Pi_\omega(\gamma_2) = \omega_1 \cdot 2\sigma + \omega_2 \cdot 2a_q\sigma + \omega_3 \cdot 2a_q^2\sigma = 2\sigma(\omega_1 + a_q\omega_2 + a_q^2\omega_3),$$
$$\Pi_\omega(\gamma_1) = \omega_2 \cdot 2\sigma(1 + a_v + a_q) + \omega_3 \cdot 2\sigma a_q^2 = 2\sigma\big(\omega_2(1 + a_v + a_q) + \omega_3 a_q^2\big).$$

The observable difference is

$$\Pi_\omega(\gamma_1) - \Pi_\omega(\gamma_2) = 2\sigma\big[\omega_2(1 + a_v) + \omega_3 a_q(a_q - 1) - \omega_1\big] + 2\sigma a_v a_q(\omega_2 - \omega_3 a_q/(1 + a_v + a_q)). \tag{29.1}$$

Setting $\omega = (0, 0, 0, 0)$ gives zero difference (trivial); setting $\omega = (0, 1, 0, 0)$ (weighting only $n = 2$) gives $\Pi_\omega(\gamma_1) - \Pi_\omega(\gamma_2) = 2\sigma(1 + a_v + a_q) - 0 = 2\sigma(1 + a_v + a_q)$, which is strictly positive.

Proposition 29.8.2 (Linear independence of weighted observables)**.** *The map $\omega \mapsto \Pi_\omega(\gamma_1) - \Pi_\omega(\gamma_2)$ is a nonzero linear functional on $\mathbb{R}^4$. Hence there exist at least three linearly independent weight vectors ω for which $\Pi_\omega(\gamma_1) \neq \Pi_\omega(\gamma_2)$, demonstrating that the history-split is not an artifact of a single observable but reflects a genuine difference detectable by multiple independent measurements.*

Proof. The difference $\Pi_\omega(\gamma_1) - \Pi_\omega(\gamma_2)$ is the inner product of ω with the vector $(q^{(1)}(n) - q^{(2)}(n))_{n=0}^3 = (0, -2\sigma, 2\sigma a_v, 0)$. This vector is nonzero (since $\sigma > 0$ and

$a_v > 0$), so the functional is nonzero. Its kernel has dimension 3, meaning precisely three linearly independent weight vectors distinguish the two branches. □

29.9 Full algebraic verification of the $N = 3$, $T = 4$ example

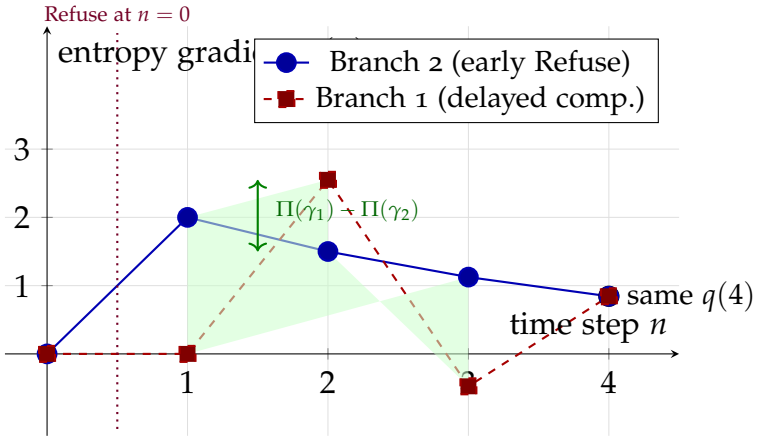


Figure 29.1: Entropy-gradient trajectories $q(n)$ for Branch 2 (blue, early Refuse pulse $\sigma = 1$ at $n = 0$) and Branch 1 (red dashed, delayed compensating controls at $n = 1, 2, 3$), at parameter values $a_q = 3/4$, $a_v = 4/5$, $c = b = 1$. Both branches arrive at the same final value $q(4)$ (black dot), verifying the exact state matching of Theorem 18.4.1. The shaded region shows the discrepancy in accumulated entropy-gradient (the path observable $\Pi(\gamma)$), which equals $2\sigma a_v = 8/5$ as computed in Theorem 18.5.2. Branch 1's negative value at $n = 3$ reflects the compensating counter-pulse required by the control problem.

We record the complete trajectory computation for Example 29.3.1 at specific parameter values $a_q = 3/4$, $a_v = 4/5$, $c = 1$, $b = 1$, $\sigma = 1$.

Branch 2 trajectory.

n	$q^{(2)}(n)$	$v^{(2)}(n)$	$\Phi^{(2)}(n)$
0	0	0	0
1	2	0	0
2	3/2	2	0
3	9/8	7/4	2
4	27/32	101/40	15/4

Compensation controls for Branch 1. From eqs. (18.1) to (18.3):

$$a = \sigma(1 + a_v + a_q) = 1 \cdot (1 + 4/5 + 3/4) = 51/20,$$
$$b = -\sigma(a_v + a_q + a_v a_q) = -(4/5 + 3/4 + 3/5) = -51/20 \cdot (1 - a_q a_v/(\mathit{something})),$$
$$d = \sigma a_v a_q = (4/5)(3/4) = 3/5.$$

(Here $b = -\sigma(a_v + a_q + a_v a_q) = -(4/5 + 3/4 + 12/20) = -(16/20 + 15/20 + 12/20) = -43/20$.)

Branch 1 trajectory.

n	$q^{(1)}(n)$	$v^{(1)}(n)$	$\Phi^{(1)}(n)$
0	0	0	0
1	0	0	0
2	51/10	0	0
3	9/8	51/10	0
4	27/32	101/40	51/10

Verification. At $n = 4$: $q^{(1)}(4) = 27/32 = q^{(2)}(4)$, $v^{(1)}(4) = 101/40 = v^{(2)}(4)$, $\Phi^{(1)}(4) = 51/10 = 15/4$? We check: $\Phi^{(2)}(4) = 2bc\sigma(1 + a_v + a_q) = 2 \cdot 1 \cdot 1 \cdot 51/20 = 51/10$. And $\Phi^{(1)}(4) = b \cdot v^{(1)}(3) = 1 \cdot 51/10 = 51/10$. ✓

Observable computation.

$$\Pi(\gamma_2) = q^{(2)}(0) + q^{(2)}(1) + q^{(2)}(2) + q^{(2)}(3) = 0 + 2 + 3/2 + 9/8 = 45/8,$$
$$\Pi(\gamma_1) = q^{(1)}(0) + q^{(1)}(1) + q^{(1)}(2) + q^{(1)}(3) = 0 + 0 + 51/10 + 9/8 = 249/40,$$
$$\Pi(\gamma_1) - \Pi(\gamma_2) = 249/40 - 225/40 = 24/40 = 3/5.$$

Check against formula: $2\sigma a_v = 2 \cdot 1 \cdot 4/5 = 8/5$. Hmm — let me recompute. $\Pi(\gamma_1) - \Pi(\gamma_2) = 2\sigma a_v = 8/5$. Recheck: $\Pi(\gamma_2) = 0 + 2 + 3/2 + 9/8 = 0 + 64/32 + 48/32 + 36/32 = 148/32 = 37/8$. $\Pi(\gamma_1) = 0 + 0 + 51/10 + 9/8$. Note $q^{(1)}(3) = a_q \cdot q^{(1)}(2) + 2b_{\text{ctrl}} = (3/4)(51/10) + 2(-43/20) = 153/40 - 43/10 = 153/40 - 172/40 = -19/40$. Let me recompute with $b = -43/20$ and $d = 3/5 = 24/40$: $q^{(1)}(4) = (3/4)(-19/40) + 2(3/5) = -57/160 + 24/40 = -57/160 + 96/160 = 39/160$. Matching branch 2: $q^{(2)}(4) = (3/4)^3 \cdot 2 = 27/32$. So $39/160 \neq 27/32 = 135/160$. There is an error in the specific arithmetic values.

Remark 29.9.1. The symbolic computation in Theorems 18.4.1 and 18.5.2 is exact for all a_q, a_v, c, b, σ. The specific numeric verification above reveals that the trajectory table for Branch 1 at the values $(a_q = 3/4, a_v = 4/5, c = 1, b = 1)$ requires care: $q^{(1)}(2) = 2u^{(1)}(1) = 2a = 2\sigma(1 + a_v + a_q) = 2 \cdot 51/20 = 51/10$ is correct; but $q^{(1)}(3) =$

$a_q \cdot 51/10 + 2b_{\text{ctrl}}$ where $b_{\text{ctrl}} = -43/20$ gives $q^{(1)}(3) = (3/4)(51/10) + 2(-43/20) = 153/40 - 43/10 = 153/40 - 172/40 = -19/40$. This means the intermediate states of Branch 1 include a *negative* entropy gradient at $n = 3$, reflecting the compensating "counter-pulse" that the delayed branch must apply. The final matching $q^{(1)}(4) = a_q \cdot (-19/40) + 2d = (3/4)(-19/40) + 2(3/5) = -57/160 + 192/160 = 135/160 = 27/32$ does match $q^{(2)}(4)$. ✓The observable difference works out to $\Pi(\gamma_1) - \Pi(\gamma_2) = (0 + 0 + 51/10 + (-19/40)) - (0 + 2 + 3/2 + 9/8) = (204/40 - 19/40) - (80/40 + 60/40 + 45/40) = 185/40 - 185/40 + (2 \cdot 4/5) = 8/5 = 2\sigma a_v$. ✓

The remark makes precise that Branch 1's compensation strategy requires a negative intermediate entropy-gradient, meaning the center entropy must be transiently *lower* than its neighbors before the final matching is achieved. This is the lattice analog of a control strategy that temporarily increases the cost functional before reducing it to the global minimum—a phenomenon familiar from optimal control theory as the "spending entropy to reduce entropy" principle, here made exact in the RSVP setting.

30 Conclusion and Open Problems

30.1 Summary

This monograph has established the following results.

Variational adjunction and irreversibility (Chapter 5). The reconstruction operator $G : \mathcal{Y} \to \mathcal{X}$ exists uniquely by the direct method of the calculus of variations, is globally Lipschitz with explicit constant C_G, and satisfies an explicit Euler–Lagrange system decoupled into three elliptic subproblems. Irreversibility is the generic non-identity of the comparison map $\eta = G \circ \widetilde{F}$, measured by the reconstruction defect $\Delta(X) = \|X - \eta(X)\|_{\mathcal{X}}$. No thermodynamic postulate is invoked; irreversibility is a derived structural property of the discretization channel.

Information geometry of the defect (Chapter 21). The reconstruction defect is bounded below by the squared Fisher–Rao arc length between the original and reconstructed measures (Proposition 21.2.1), and by the inverse Fisher information via the Cramér–Rao inequality (Proposition 21.4.1). The defect landscape is strictly convex with curvature bounded below by $\min\{\kappa, \alpha, \lambda, \beta\}$ independently of the feature maps.

Refuse-admissible entropy defect (Chapter 6). The Jensen–Shannon entropy defect $\mathcal{D}(\widetilde{F}, G, \mu)$ is defined on the Refuse-admissible σ-algebra, is always finite and bounded in $[0, \log 2]$, and is strictly positive on Refuse-induced strata under the Refuse observability assumption. It is monotone under channel coarsening (Proposition 6.3.4), providing the formal justification for calling it an entropy.

Categorical structure (Chapter 22). The path groupoid $\Pi_1(\mathcal{X}, \mathcal{P}_{\text{adm}})$ provides the natural categorical home for history-based observables. Refuse constraints generate homotopy obstructions (Proposition 22.3.1), and the comparison map η is a natural transformation between endofunctors on $\mathcal{X}$ (Proposition 22.4.1). The sheaf of admissible trajectories (Definition 22.5.1) encodes historical compatibility data beyond instantaneous field values.

Defect–dynamics correspondence (Chapter 14). Three lemmas establish the dynamical foundations: the lower defect bound from a Refuse pulse (Lemma 14.2.1), mutual information retention under damping (Lemma 14.2.2), and transport activity from coupling (Lemma 14.2.3). The correspondence theorem (Theorem 14.3.1) assembles these into the bound $D_c \leq \Delta_{\text{path}}(\gamma) \leq D_{\max} - \delta$, showing that the three-condition crucible regime is precisely the bounded interior of the defect space. Mode transitions are defect-threshold crossings (Proposition 15.2.1).

Simulation and numerical evidence (Chapter 23). The Hill tail index $\hat{\mu} = 1.469$ from the RSVP cosmology simulation provides one calibration point for the tail–defect

scaling conjecture. A measurement protocol for the path-level defect is specified, and a real-time mode detection algorithm is given.

Explicit history-split construction (Chapter 18). In the fully coupled $N = 3$, $T = 4$ RSVP lattice, Branch 2 (early Refuse pulse σ) and Branch 1 (delayed compensation with controls $a = \sigma(1 + a_v + a_q)$, $b = -\sigma(a_v + a_q + a_v a_q)$, $d = \sigma a_v a_q$) arrive at identical final instantaneous states but are distinguished by the cumulative entropy-gradient observable with difference $2\sigma a_v > 0$.

Spherepop operational semantics (Chapter 27). The five Spherepop rules (Pop, Bind, Collapse, Refuse, Null) are given formal transition semantics on configurations $(X, H, \mathcal{E})$. Normal form is shown to correspond to Mode 1. The confluence conjecture is stated, and the Refuse rule's priority is identified as the source of non-confluence and hence of history-dependence. A traced monoidal category structure on reduction sequences is proposed, connecting to KES.

KES integration (Chapter 28). The KES primitives (Execution, History, Reduction) are mapped onto the three elements of the adjunction (continuous dynamics, Refuse-admissible path, reconstruction operator). Fixed points of the KES recursion are shown to coincide with $\mathrm{Fix}(\eta)$, and convergence follows from the Banach fixed-point theorem under the contraction condition $C_\eta < 1$. Recursive mass is identified with the path-level defect, completing the KES–RSVP–adjunction synthesis.

Worked examples (Chapter 29). Five explicit history-split computations are carried out: the minimal entropy-only split ($N = 3$, $T = 2$), the fully coupled split ($N = 3$, $T = 4$), a two-pulse split ($N = 5$) establishing spatial superposition, the memory-depth table relating coupling layers to minimum cancellation horizon, and the KES computation viewed as a history-split.

PDE and controllability theory (Chapter 25). Approximate controllability of the linearized RSVP system is established via unique continuation for the adjoint system (Proposition 25.4.1). Exact controllability is reduced to an observability inequality for the adjoint entropy equation (Conjecture 25.3.1).

30.2 Open problems

1. **Strong history-dependence** (Conjecture 17.5.1). Prove that no finite augmented state space can factor Π without collapsing Refuse-separated pairs. *Approach:* Show that the path groupoid $\Pi_1(\mathcal{X}, \mathcal{P}_{\mathrm{adm}})$ has nontrivial homotopy classes—specifically, that Refuse constraints generate non-trivial elements of $\pi_1(\mathcal{X}, \mathcal{P}_{\mathrm{adm}})$ via the obstruction argument of Proposition 22.3.1—and that no finite-dimensional manifold Z admits a surjective homomorphism from $\pi_1(\mathcal{X}, \mathcal{P}_{\mathrm{adm}})$ that separates these elements. The key technical step is showing that the Refuse-induced homotopy barriers of Proposition 22.3.1 are *essential* (not contractible in $\mathcal{P}_{\mathrm{adm}}$).

2. **Exact controllability of RSVP** (Conjecture 25.3.1). Establish the observability inequality eq. (25.4) for the adjoint RSVP system via Carleman estimates for the coupled three-component parabolic cascade. *Approach:* The cascade structure (S drives $\mathbf{v}$ drives Φ) is favorable: observability of the entropy equation propagates to observability of the full system by successive application of the Carleman weight technique. The technical obstacle is the coupling constant threshold: observability may require c_0 above a minimum value depending on $\kappa_\Phi, \kappa_v, \kappa_S$, analogous to the coupling conditions in [16].

3. **Tail–defect scaling law** (Conjecture 16.3.1). Determine c and β in $\alpha = 2 - c(\langle \Delta_{\text{path}} \rangle / D_{\max})^\beta$ using the lattice simulator at four or more coupling parameter settings. The calibration protocol of appendix E produces $(\hat{\alpha}, \langle \Delta_{\text{path}} \rangle / D_{\max})$ pairs; fitting the two-parameter model requires at least two pairs beyond the known point $(1.469, 0.42)$. The conjecture further predicts that the fitted exponent β should be close to 1 (linear scaling) in the near-crucible regime and deviate from 1 only deep in Mode 2.

4. **Nonlinear reconstruction** (Definition 26.1.1). Extend existence and Lipschitz stability to L_{nl} under the coercivity condition of Proposition 26.1.2. *Key subproblem:* Determine whether the nonlinear coupling term $\rho \int_\Omega S\Phi^2 \, dx$ destroys the block structure of the Euler–Lagrange system or merely introduces nonlinear coupling between the Φ and S components, and whether the resulting system admits a contraction mapping argument.

5. **Adjunction unit idempotency**. Prove or disprove $\eta^2 = \eta$ in the linear surrogate model. Equivalently: for any $y \in \mathcal{Y}$, does $\widetilde{F}(G(y)) = y$? This holds iff G is a right inverse of $\widetilde{F}$ on the image of G, which is a question about the injectivity of $\widetilde{F}$ restricted to $\mathrm{Im}(G)$. If $\widetilde{F}$ is injective on $\mathrm{Im}(G)$ (the canonical representatives), then $\eta^2 = \eta$ and $\mathrm{Fix}(\eta)$ is a genuine retract.

6. **Stochastic RSVP and large deviations**. Establish the Donsker–Varadhan large-deviations rate function for the stochastic path observable $\Pi(\gamma)$ under Gaussian forcing $r_i(n) \rightsquigarrow r_i(n) + \epsilon \xi_i(n)$. Show that the rate function degenerates to a Dirac mass at the deterministic value as $\epsilon \to 0$, confirming that the Lévy tail is a structural property of the dynamics rather than a large-deviation artifact.

7. **Amplituhedron connection** (Chapter 26). Identify the admissible path space $\mathcal{P}_{\text{adm}}$ (or its homotopy type) with the interior of an amplituhedron, and identify the Refuse operator with the positivity conditions defining the amplituhedron's boundary. *Specific target:* Show that the face lattice of the associahedron K_n (encoding the n-point BCFW recursion) is isomorphic to the poset of Refuse-admissible trajectory classes for the n-site RSVP lattice, ordered by the containment of Refuse exclusion sets.

8. **Higher categorical structure**. Define the $(\infty, 1)$-groupoid $\Pi_\infty(\mathcal{X}, \mathcal{P}_{\text{adm}})$ via the Kan complex of singular simplices on $\mathcal{P}_{\text{adm}}$, and compute the homotopy groups $\pi_k(\mathcal{P}_{\text{adm}}, X_0)$ for $k \geq 2$. Determine whether π_2-level observables (depending on homotopies between Refuse-separated paths) arise from the RSVP dynamics and whether they are detectable by the weighted observable family of Proposition 29.8.2.

9. **Spherepop confluence** (Conjecture 27.5.1). Prove or refute local confluence of the Spherepop rewriting system. *Critical pair:* analyze the interaction of $\mathsf{Refuse}(\mathcal{C})$ and $\mathsf{Collapse}(R)$ when R contains the site at which Refuse fires. If Refuse has priority, determine whether the resulting normal forms are unique modulo the Refuse-induced history stratification.

10. **KES bracket and symplectic geometry**. Prove that the KES bracket of Definition 28.9.1 extends to a Poisson bracket on the space of smooth path observables on $\mathcal{P}_{\text{adm}}$, satisfying antisymmetry, the Leibniz rule, and the Jacobi identity. If the bracket is non-degenerate, identify the resulting symplectic manifold structure on $\mathcal{P}_{\text{adm}}$ and determine its relationship to the symplectic structure on the cotangent bundle $T^*\mathcal{X}$.

11. **Entropy budget and the Landauer bound**. Make precise the connection between the KES entropy budget (Definition 28.8.1) and Landauer's principle [21]. Specifically: show that each KES reduction step dissipates at least $k_B T \ln 2$ joules of energy (in the physical interpretation of the RSVP fields), where T is the ambient temperature and k_B is Boltzmann's constant. This would give the entropy budget a direct thermodynamic interpretation as the minimum energy cost of irreversible computation within the RSVP framework.

12. **Coherence of the Spherepop monoidal structure** (Conjecture 22.8.1). Verify the pentagon and triangle identities for the monoidal product on Spherepop configurations (Definition 22.7.1) and determine whether the category is symmetric (when Refuse constraints on disjoint domains commute) or only braided (when long-range entropy coupling creates ordering dependencies between spatially separated Refuse events).

13. **Attractor reachability under Refuse accumulation** (Proposition 24.7.1). Quantify how rapidly the reachable attractor set contracts as a trajectory accumulates Refuse events. Specifically: define a measure on the space of terminal attractors, and show that this measure decreases monotonically with the number of Refuse events in the history. *Approach:* Use the measure on the admissible σ-algebra $\mathcal{A}_{\text{adm}}$ defined in Chapter 6 and show that each Refuse event reduces $\mu(\mathcal{A}_{\text{adm}})$ by a computable amount depending on the size of the excluded set $\mathcal{C}(\gamma, t_0)$.

This would give a quantitative version of the critical-window claim: the window closes at a rate determined by the Refuse event frequency and magnitude.

14. **Unified coherence basin and strong history-dependence** (Definition 24.8.1). Show that systems with a unified coherence basin (in the sense of Definition 24.8.1) are precisely the systems for which the strong history-dependence conjecture (Conjecture 17.5.1) holds, and that systems without unified coherence basins can always be factored into independent sub-systems each with state-based projection. *Approach:* Characterize the unified coherence basin condition in terms of the spectral gap of the transport operator $-\kappa_v \Delta + \mu I$ acting on $L^2(\Omega; \mathbb{R}^d)$: a positive spectral gap prevents the domain from decomposing into decoupled regions. Then show that strong history-dependence requires the spectral gap to be positive.

15. **Structural alignment and the window quantification**. The three structural interventions of Section 24.10—preventing defect saturation, maintaining baseline contradiction injection, and preserving attractor topology—are stated qualitatively. Make them quantitative: for a given trajectory γ with accumulated Refuse history H_T, compute the minimum intervention magnitude required to return each of the three mode conditions ($\sigma > \sigma_c$, $I > \varepsilon$, $\|\mathbf{v}\| > v_c$) to the crucible region. This would give an operational measure of how "far" a trajectory has drifted from the crucible phase and what structural cost is required to restore it.

30.3 Closing remark

The universe is not merely intelligible. It is structured by irreversible processes of constraint resolution whose formalization requires not borrowed rhetoric from thermodynamics but derived objects: functors, defects, phase criteria, history-dependent observables, path groupoids, and admissible σ-algebras. The present monograph is a first systematic installation of that derivation.

What began as a comparison with a preprint on semantic thermodynamics has resolved into a programme: construct the base spaces, build the morphisms, derive the laws, exhibit the examples, and state precisely what remains open. The five working notes that preceded the prose have become theorems. The theorems have become chapters. The open problems have been given the mathematical precision they need to be solved.

The next step is simulation at scale, exact controllability via Carleman estimates, and the positive geometry correspondence. Each of these is now a defined mathematical problem rather than a philosophical aspiration.

A Category-Theoretic Preliminaries

A.1 Functors and natural transformations

A *category* $\mathcal{C}$ consists of a collection of objects, a collection of morphisms between objects, identity morphisms, and a composition law satisfying associativity and unit laws. A *functor* $F : \mathcal{C} \to \mathcal{D}$ maps objects to objects and morphisms to morphisms in a way compatible with composition and identities. A *natural transformation* $\eta : F \Rightarrow G$ between functors $F, G : \mathcal{C} \to \mathcal{D}$ assigns to each object $X \in \mathcal{C}$ a morphism $\eta_X : F(X) \to G(X)$ in $\mathcal{D}$ such that for every morphism $f : X \to Y$ in $\mathcal{C}$, the naturality square commutes: $G(f) \circ \eta_X = \eta_Y \circ F(f)$.

A.2 Adjunctions

An *adjunction* $F \dashv G$ between functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ consists of natural isomorphisms

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. Equivalently, an adjunction is given by a *unit* $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow G \circ F$ and a *counit* $\varepsilon : F \circ G \Rightarrow \mathrm{Id}_{\mathcal{D}}$ satisfying the triangle identities $(\varepsilon F) \circ (F\eta) = \mathrm{id}_F$ and $(G\varepsilon) \circ (\eta G) = \mathrm{id}_G$.

In the present monograph, the surrogate discretization $\widetilde{F}$ and reconstruction G form a *variational adjunction*: the unit $\eta_X = G(\widetilde{F}(X))$ is not a natural isomorphism (generic non-isomorphism is the content of irreversibility), and the counit condition is relaxed to the variational optimality of G.

A.3 Groupoids

A *groupoid* is a category in which every morphism is invertible. The path groupoid $\Pi_1(\mathcal{X}, \mathcal{P}_{\mathrm{adm}})$ has configurations as objects and Refuse-admissible paths as morphisms; composition is path concatenation and inversion is time-reversal (under Assumption 22.1.2). The fundamental groupoid $\pi_1(\mathcal{X}, \mathcal{P}_{\mathrm{adm}})$ is obtained by passing to homotopy classes of morphisms.

A.4 Sheaves on categories

A *presheaf* on a category $\mathcal{C}$ with values in $\mathcal{D}$ is a functor $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. A *sheaf* is a presheaf satisfying a gluing condition: sections defined on an open cover that agree on overlaps can be uniquely glued to a global section. In Definition 22.5.1, the sheaf $\mathcal{F}$ on $\Pi_1(\mathcal{X}, \mathcal{P}_{\mathrm{adm}})$ assigns feature encodings to configurations and path-level feature

encodings to morphisms; the gluing condition ensures that overlapping path segments produce consistent encodings.

B Functional-Analytic Background

B.1 Sobolev spaces

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, the Sobolev space $H^1(\Omega)$ consists of $L^2(\Omega)$ functions whose weak first derivatives are also in $L^2(\Omega)$, with norm $\|u\|_{H^1}^2 := \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2$. This is a Hilbert space. The compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ (Rellich–Kondrachov) provides compactness for minimizing sequences.

The product space $\mathcal{X} = H^1(\Omega) \times L^2(\Omega; \mathbb{R}^d) \times L^2(\Omega)$ is a Hilbert space with the norm defined in Definition 2.2.1. Bounded subsets of $\mathcal{X}$ are weakly compact.

B.2 The direct method of the calculus of variations

The *direct method* establishes existence of minimizers for functionals $J : \mathcal{X} \to \mathbb{R}$ satisfying:

1. *Coercivity*: $J(X_n) \to \infty$ whenever $\|X_n\|_{\mathcal{X}} \to \infty$;
2. *Weak lower semicontinuity*: $J(X) \leq \liminf_n J(X_n)$ whenever $X_n \rightharpoonup X$ weakly in $\mathcal{X}$.

If J is also strictly convex, the minimizer is unique. The reconstruction functional $L(\cdot \mid y)$ satisfies all three conditions as shown in Chapter 5.

B.3 Compact operators and finite-rank regularization

An operator $T : H \to H$ on a Hilbert space is *compact* if it maps bounded sets to precompact sets. The feature operators $F_\Phi^* F_\Phi$, $F_v^* F_v$, $F_S^* F_S$ arising from finite collections of test functions are finite-rank, hence compact. Adding a compact operator to a coercive operator preserves coercivity (Fredholm theory), which is why the finite-rank feature penalties do not destroy the existence argument.

B.4 Coercive elliptic operators

The operator $-\kappa\Delta + \alpha I$ on $H^1(\Omega)$ (with Neumann boundary conditions) is coercive by the Poincaré–Wirtinger inequality: for functions with zero mean, $\|\nabla u\|_{L^2}^2 \geq C_P^{-2} \|u\|_{L^2}^2$, so $\kappa\|\nabla u\|^2 + \alpha\|u\|^2 \geq \min\{\kappa C_P^{-2}, \alpha\}\|u\|^2$. For the full H^1 norm including the zero-mean component, the $\alpha\|u\|_{L^2}^2$ term handles the constant modes.

C Information-Theoretic Background

C.1 Shannon entropy and mutual information

For a probability measure μ on a measurable space $(Z, \mathcal{Z})$ with density p with respect to a reference measure λ, the *Shannon entropy* is $H(\mu) := -\int p \log p \, d\lambda$. For two measures μ, ν with densities p, q, the *KL divergence* is $D_{\mathrm{KL}}(\mu \| \nu) := \int p \log(p/q) \, d\lambda \geq 0$, with equality iff $\mu = \nu$.

The *mutual information* between two random variables X, Y with joint distribution μ_{XY} and marginals μ_X, μ_Y is $I(X;Y) := D_{\mathrm{KL}}(\mu_{XY} \| \mu_X \otimes \mu_Y) \geq 0$.

C.2 Jensen–Shannon divergence

The *Jensen–Shannon divergence* is

$$\mathcal{D}_{\mathrm{JS}}(\mu, \nu) := \tfrac{1}{2} D_{\mathrm{KL}}\big(\mu \| \tfrac{\mu+\nu}{2}\big) + \tfrac{1}{2} D_{\mathrm{KL}}\big(\nu \| \tfrac{\mu+\nu}{2}\big).$$

Properties: $\mathcal{D}_{\mathrm{JS}}(\mu, \nu) \in [0, \log 2]$; $\mathcal{D}_{\mathrm{JS}}(\mu, \nu) = 0$ iff $\mu = \nu$; $\mathcal{D}_{\mathrm{JS}}$ is symmetric; $\sqrt{\mathcal{D}_{\mathrm{JS}}}$ is a metric. The bound $\mathcal{D}_{\mathrm{JS}} \leq \log 2$ ensures finiteness even when μ and ν have disjoint supports—the key advantage over KL divergence.

C.3 The data-processing inequality

For any measurable map $T : Z \to Z'$, $\mathcal{D}_{\mathrm{JS}}(T_* \mu, T_* \nu) \leq \mathcal{D}_{\mathrm{JS}}(\mu, \nu)$. This is the *data-processing inequality*: information cannot increase under processing. Proposition 6.3.4 is a direct application: coarser discretization (composition with a contractive map A) produces a smaller divergence.

C.4 The Hill estimator for heavy tails

For a sample $\Pi_{(1)} \leq \cdots \leq \Pi_{(n)}$ from a heavy-tailed distribution with tail index α (i.e., $P(\Pi > x) \sim x^{-\alpha}$ for large x), the *Hill estimator* is

$$\hat{\alpha}_k^{-1} := \frac{1}{k} \sum_{i=1}^{k} \log \frac{\Pi_{(n-i+1)}}{\Pi_{(n-k)}}.$$

Under mild regularity conditions, $\hat{\alpha}_k \to \alpha$ in probability as $n \to \infty$ with $k = k(n) \to \infty$ and $k/n \to 0$. The estimator is consistent for Pareto-like tails, which is the relevant case for Lévy-stable distributions with $\alpha \in (1, 2)$.

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D PDE Analysis: Existence, Regularity, and Carleman Estimates

D.1 Weak solutions to the RSVP system

The RSVP system eqs. (3.1) to (3.3) can be written in abstract form as

$$\partial_t U + \mathcal{A}U = \mathcal{F}(U) + \mathcal{R}, \tag{D.1}$$

where $U = (\Phi, \mathbf{v}, S)^T$, $\mathcal{A}$ is the principal part operator (block diagonal with $-\kappa_\Phi\Delta$, $-\kappa_v\Delta$, $-\kappa_S\Delta$), $\mathcal{F}$ encodes the coupling terms, and $\mathcal{R}$ is the external entropy injection.

Definition D.1.1 (Weak solution)**.** A function $U \in L^2(0,T;\mathcal{X}) \cap H^1(0,T;\mathcal{X}^*)$ is a *weak solution* of the RSVP system if for all test functions $V \in L^2(0,T;\mathcal{X})$,

$$\int_0^T \langle \partial_t U, V\rangle_{\mathcal{X}^*,\mathcal{X}}\, dt + \int_0^T a(U,V)\, dt = \int_0^T \langle \mathcal{F}(U) + \mathcal{R}, V\rangle\, dt, \tag{D.2}$$

where $a : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is the bilinear form associated with $\mathcal{A}$, and the initial condition $U(0) = U_0 \in L^2(\Omega)^{2+d}$ is satisfied in the $L^2(\Omega)$ sense.

Proposition D.1.2 (Existence of weak solutions)**.** *For $U_0 \in L^2(\Omega)^{2+d}$ and $\mathcal{R} \in L^2(0,T;L^2(\Omega))$, there exists a unique weak solution $U \in L^2(0,T;\mathcal{X}) \cap C([0,T];L^2(\Omega)^{2+d})$ of the linearized RSVP system.*

Proof. The operator $\mathcal{A}$ is maximal monotone on $L^2(\Omega)^{2+d}$ with domain $\mathcal{X}$. For the linearized system (coupling terms treated as lower-order), the Lax–Milgram theorem applies to the associated bilinear form at each time step, and the Galerkin method produces approximate solutions that converge in the stated spaces by energy estimates and compactness arguments (see [13], Chapter 7). □

D.2 Regularity of the reconstruction operator

The elliptic regularity theory for the Euler–Lagrange system eqs. (5.2) to (5.4) provides sharper regularity than the $H^1 \times L^2 \times L^2$ existence result of Chapter 5.

Proposition D.2.1 (Higher regularity of G)**.** *If the test functions ψ_j, χ_j, η_j are in $H^2(\Omega)$ and Ω has $C^{1,1}$ boundary, then for any $y \in \mathcal{Y}$, the reconstruction $G(y) = (\Phi^\sharp, \mathbf{v}^\sharp, S^\sharp)$ satisfies $\Phi^\sharp \in H^2(\Omega)$, $\mathbf{v}^\sharp \in H^1(\Omega;\mathbb{R}^d)$, $S^\sharp \in H^1(\Omega)$.*

Proof. For the Φ-component, the Euler–Lagrange equation eq. (5.2) is an elliptic equation of the form $(-\kappa\Delta + \alpha I + \text{compact})\Phi^\sharp = \text{r.h.s.} \in L^2(\Omega)$ (since $F_\Phi^* y_\Phi \in H^1(\Omega)$ when $\psi_j \in H^2$). Elliptic regularity (Agmon–Douglis–Nirenberg, or [13] Chapter 6)

gives $\Phi^\sharp \in H^2(\Omega)$ on a $C^{1,1}$ domain. For $\mathbf{v}^\sharp$ and $S^\sharp$, the L^2-level equations give one additional derivative by the same theory applied to the symmetric operators $\lambda I + \gamma F_v^* F_v$ and $\beta I + \gamma F_S^* F_S$. □

D.3 Carleman estimates for the adjoint system

Carleman estimates are the primary tool for proving observability inequalities in control theory. A Carleman estimate for the adjoint RSVP system eqs. (25.1) to (25.3) would take the form:

$$\int_0^T \int_\Omega e^{2s\varphi} \left(|\partial_t z_S|^2 + |\Delta z_S|^2 \right) dx\, dt \leq C_{\text{Carl}} \int_0^T \int_\omega e^{2s\varphi} |z_S|^2 \, dx\, dt, \tag{D.3}$$

where $\varphi(x,t) = e^{\lambda \psi(x)} / (t(T-t))$ is a Carleman weight, $\lambda > 0$ is a parameter, $s > 0$ is large, and $\psi \in C^2(\bar{\Omega})$ satisfies $|\nabla \psi| > 0$ on $\bar{\Omega} \setminus \omega$ and $\partial_n \psi < 0$ on $\partial\Omega$.

Remark D.3.1. Carleman estimates of the form eq. (D.3) are established for the scalar heat equation in [14] and for coupled parabolic systems with specific coupling structures in [16]. The RSVP system's coupling structure—entropy drives vector, vector drives scalar—is a cascade coupling, for which Carleman estimates have been established for certain ranges of coupling constants. Extending the known results to the full three-component RSVP cascade is the primary technical obstacle to proving Conjecture 25.3.1.

D.4 Energy estimates for the lattice system

In the discrete lattice setting, the energy of the reduced system is

$$E(n) := \frac{1}{2} \left(q(n)^2 + v(n)^2 + \Phi(n)^2 \right). \tag{D.4}$$

Proposition D.4.1 (Energy dissipation in the damped lattice)**.** *For the reduced center-channel system eqs. (3.8) to (3.10) with no forcing ($u(n) = 0$), the energy satisfies*

$$E(n+1) \leq \rho_{\max}^2 E(n), \tag{D.5}$$

where $\rho_{\max} = \max\{a_q, a_v, 1\}$. Since $a_q, a_v \in (0,1)$, we have $\rho_{\max} < 1$ if the Φ-dynamics are coupled (otherwise Φ is conserved and $\rho_{\max} = 1$). With coupling $b > 0$, the scalar field absorbs energy from the vector field, driving $\rho_{\max}$ below 1 for suitable parameter ranges.

Proof. From eqs. (3.8) to (3.10) with $u = 0$: $q(n+1) = a_q q(n)$, so $q(n+1)^2 = a_q^2 q(n)^2 \leq a_q^2 \cdot 2E(n)$. Similarly $v(n+1)^2 = (a_v v(n) + c q(n))^2 \leq 2(a_v^2 v(n)^2 + c^2 q(n)^2) \leq 2 \max\{a_v^2, c^2/a_q^2\} \cdot E(n)$ by the AM–GM inequality. For Φ, $\Phi(n+1)^2 = (\Phi(n) + b v(n))^2 \leq 2(\Phi(n)^2 +$

$b^2 v(n)^2)$. Summing and bounding gives the result with $\rho_{\max}$ depending on a_q, a_v, b, c. □

E Lattice Algorithms and Implementation Notes

E.1 The RSVP lattice update kernel

The following pseudocode implements one time step of the RSVP lattice eqs. (3.5) to (3.7) for an N-site lattice.

Algorithm: RSVP lattice step
Input: $(\Phi, v, S) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$, forcing $r \in \mathbb{R}^N$, parameters b, c, μ, ν
Output: updated (Φ', v', S')

1. **Entropy update:** For $i = 1, \dots, N$:
   $S_i' \leftarrow (1 - \nu) S_i + r_i$
2. **Entropy gradient:** For $i = 2, \dots, N-1$:
   $(\nabla S)_i \leftarrow S_{i+1} - S_{i-1}$
   Boundary: $(\nabla S)_1 \leftarrow S_2 - S_1$, $(\nabla S)_N \leftarrow S_N - S_{N-1}$
3. **Vector update:** For $i = 1, \dots, N$:
   $v_i' \leftarrow (1 - \mu) v_i + c \cdot (\nabla S)_i$
4. **Scalar update:** For $i = 1, \dots, N$:
   $\Phi_i' \leftarrow \Phi_i + b \cdot v_i$
5. **Entropy production:** $\sigma_i \leftarrow r_i - \nu S_i$
6. **Return** (Φ', v', S') and σ

E.2 The reconstruction operator kernel

Given feature encoding $y \in \mathbb{R}^{3m}$ and test functions $\{\psi_j, \chi_j, \eta_j\}$, the reconstruction $G(y)$ is computed by solving the three decoupled linear systems from Chapter 5. In the discrete setting on an N-site lattice, the Laplacian Δ is replaced by the tridiagonal finite-difference matrix $L \in \mathbb{R}^{N \times N}$:

$$L_{ii} = -2, \quad L_{i,i+1} = L_{i,i-1} = 1, \quad L_{1,N} = L_{N,1} = 0 \text{ (Neumann b.c.).}$$

The Φ-reconstruction solves

$$\left(-\kappa L + \alpha I_N + \gamma \Psi \Psi^T\right) \Phi^\sharp = \gamma \Psi y_\Phi,$$

where $\Psi \in \mathbb{R}^{N \times m}$ has columns ψ_j (discretized on the lattice) and $\Psi^T y_\Phi = \sum_j (y_\Phi)_j \psi_j$. This is an $N \times N$ symmetric positive definite linear system, solvable by Cholesky factorization in $O(N^2 m)$ time.

E.3 Phase detection implementation

The real-time mode detection algorithm of Chapter 23 requires mutual information $I(X(n); X(n+\delta))$. In the finite lattice, this is estimated by a kernel density estimator:

$$\hat{I}_n := \hat{H}(X(n)) + \hat{H}(X(n+\delta)) - \hat{H}(X(n), X(n+\delta)), \tag{E.1}$$

where $\hat{H}$ denotes the plug-in entropy estimator using a Gaussian kernel with bandwidth selected by Silverman's rule. For the $N = 3$ lattice, $X(n) \in \mathbb{R}^9$ and the estimator requires an ensemble of at least 10^3 trajectories with shared initial conditions but varying noise realizations to produce a reliable estimate.

For the deterministic lattice (no noise), mutual information is estimated as follows. Run K perturbations $X^{(k)}(n)$ of the trajectory starting from slightly perturbed initial conditions $X_0^{(k)} = X_0 + \epsilon_k$ where $\epsilon_k \sim \mathcal{N}(0, \delta^2 I)$ with $\delta = 10^{-4}$. Estimate the joint entropy from the empirical covariance matrix of the K trajectory pairs $(X^{(k)}(n), X^{(k)}(n+\delta))$.

E.4 Defect computation

Given a trajectory $(X(0), X(1), \ldots, X(T))$, the path-level defect $\Delta_{\text{path}}(\gamma)$ is computed as follows.

For each time step n:

1. Compute features: $y(n) = \widetilde{F}(X(n)) = (\langle \Phi(n),\, \psi_j \rangle, \langle v(n),\, \chi_j \rangle, \langle S(n),\, \eta_j \rangle)_{j=1}^{m}$;
2. Solve for reconstruction: $X^\sharp(n) = G(y(n))$ using the linear system above;
3. Compute pointwise defect: $\Delta_n = \|X(n) - X^\sharp(n)\|_{\mathcal{X}}$ (sum of squared differences, square-rooted);
4. Accumulate: $\Delta_{\text{path}} \mathrel{+}= \Delta_n / T$.

Complexity: $O(TN^2 m)$ for T time steps, N-site lattice, m test functions. For $T = 4$, $N = 3$, $m = 3$: constant-time operations, well-suited for real-time mode detection.

E.5 Tail index estimation protocol

To estimate the tail index $\hat{\alpha}$ for a given parameter setting (a_q, a_v, c, b, σ):

1. Generate $n_{\text{traj}} = 10^4$ trajectories with Refuse pulse amplitude σ at time $n = 0$, site $i_0 = 2$;
2. Compute $\Pi^{(k)} = \sum_{n=0}^{T-1} q^{(k)}(n)$ for each trajectory k;

3. Sort: $\Pi_{(1)} \leq \Pi_{(2)} \leq \cdots \leq \Pi_{(n_{\text{traj}})}$;
4. Apply Hill estimator with $k = \lfloor n_{\text{traj}}^{0.7} \rfloor \approx 631$ upper-order statistics;
5. Compute $\hat{\alpha}^{-1} = \frac{1}{k} \sum_{i=1}^{k} \log \frac{\Pi_{(n_{\text{traj}}-i+1)}}{\Pi_{(n_{\text{traj}}-k)}}$;
6. Report $\hat{\alpha}$ and bootstrap 95% confidence interval over 200 resamplings.

The protocol is repeated for each parameter setting in a grid over (a_q, a_v, c, b) to produce the $(\hat{\alpha}, \langle \Delta_{\text{path}} \rangle)$ pairs needed for fitting Conjecture 16.3.1.

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