Monograph in Formal Computation and Philosophy of Logic

Identity After Collapse

Self-Reference and Normal-Form Fixed Points

A Theory of Trace and Evaluation in Irreversible Systems

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Abstract

We develop a formal theory of self-reference for calculi governed by *irreversible reduction*. The evaluation map of the Spherepop calculus is a one-way collapse of nested structure into a terminal trace, and this map has no right inverse. Classical quines—programs that reproduce their own source—are therefore impossible: the preimage is destroyed and cannot be recovered. In their place we introduce *normal-form fixed points*, expressions whose identity survives as an invariant of their own collapse.

We show that irreversibility forces a divergence in Graham Priest's tripartite distinction between logica docens, utens, and ens, relocating validity from syntactic preservation to trace stability. The framework extends to natural language via Historical Lexical-Functional Grammar, where grammatical structure is the residue of irreversible constraint elimination. It extends further to asynchronous circuit logic, categorical sheaf theory, and RSVP field dynamics, where identity corresponds to entropic attractors. Spherepop is shown to be computationally equivalent to the lambda calculus and hence to Turing machines, grounding these contributions in the standard theory of computation.

At fixed points, program, execution, and description collapse into one object. A Spherepop quine does not reproduce itself; it *is* its own result.

Keywords: irreversible reduction, Spherepop, trace identity, normal-form fixed points, quines, logica ens, lambda calculus, Turing completeness, Historical LFG, sheaf of histories, RSVP.

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Part 1

Grounding: Making Evaluation Visible

Chapter 1

Introduction: From Evaluation to Irreversibility

Standard accounts of computation present evaluation as a function: an expression is given, a value is returned, and the process connecting the two is treated as incidental. Whether in arithmetic, lambda calculus, or programming languages, the dominant formal picture suppresses the internal structure of evaluation in favour of its input–output behaviour. This suppression is not merely a matter of convenience. It reflects a deeper assumption: that the process of evaluation is, in principle, reversible or at least reconstructible from its result.

The aim of this work is to examine what happens when this assumption is removed.

We begin from a simple observation. In many concrete systems, evaluation does not preserve the information required to reconstruct its starting point. When an arithmetic expression such as

$$(8 + (3 \times 5))$$

is reduced to the value 23, the intermediate structure that justified this result is no longer present. The result does not encode its own derivation. Distinct expressions may collapse to the same value, and no operation defined purely on the value can recover which expression produced it. Evaluation, in such cases, is not a reversible transformation but a *collapse*.

This observation motivates a shift in perspective. Rather than treating evaluation as a mapping between static objects, we treat it as a sequence of irreversible events. Each step in a reduction trajectory eliminates possibilities, contracts structure, and commits the system to a narrower range of future states. What survives this process is not the history itself, but a terminal invariant: the *trace*.

Central Question. What is identity in a system where evaluation irreversibly destroys its own preimage?

In reversible settings, identity may be defined syntactically or extensionally: an expression is identical to itself if its structure is preserved, or if it can be reconstructed from its value. Under irreversible collapse, neither of these criteria is available. The result of evaluation does not determine its origin, and distinct histories may converge to the same terminal state. Identity must therefore be relocated from the domain of initial configurations to the domain of what persists after collapse.

This relocation has immediate consequences for self-reference. The classical notion of a quine—a program that produces its own source as output—depends on the ability to preserve or encode syntactic structure across evaluation. If evaluation destroys preimage information, such reproduction becomes impossible. The question is not how a program can reproduce itself, but how it can remain itself under projection onto its terminal trace.

To make this question precise, we introduce the Spherepop calculus, a minimal formal system in which evaluation is explicitly irreversible. In Spherepop, expressions are built from nested bubbles, and evaluation consists of popping these bubbles according to local reduction rules. Each pop removes a layer of structure and cannot be undone. The calculus therefore makes visible what standard presentations conceal: evaluation as a sequence of commitments that progressively eliminate alternative possibilities.

Within this setting, we develop a theory of identity as trace invariance. The evaluation map $f : \mathcal{E} \to \mathcal{N}$ is shown to have no right inverse: the preimage of a normal form is not recoverable. This forces a redefinition of identity in terms of equivalence under collapse. Two expressions are identical not because they share a derivation, but because they share a trace.

This shift is further clarified through the integration of Graham Priest's distinction between *logica docens*, *logica utens*, and *logica ens*. In reversible systems, these strata are often implicitly conflated. In an irreversible calculus, they diverge. The formal rules of reduction, the practice of evaluation, and the resulting facts of validity become distinct objects, connected only through the process of collapse. Identity emerges at the level of *logica ens*: the invariant trace that survives all trajectories.

The consequences extend beyond abstract computation. By reinterpreting lexical events as irreversible operations over a grammatical option space, we derive a Historical Lexical-Functional Grammar in which syntactic and functional structure are not independent representations but the residue of a shared process of constraint elimination. Grammaticality becomes the existence of a non-empty terminal trace; ungrammaticality becomes total collapse.

At the limit of this development lies a new notion of self-reference. A Spherepop quine is not an expression that reproduces its own source, but one whose evaluation yields a trace equivalent to itself. At such fixed points, the distinction between program, execution, and description disappears. The expression is identical with the result of its own collapse.

The argument proceeds in stages. Part I (Chapters 2 and 3) makes evaluation visible as a sequence of reduction events and introduces the Spherepop calculus as a formal system of irreversible commitment. Part II (Chapter 4) establishes the central structural fact: the evaluation map has no right inverse. This asymmetry is the hinge on which the entire theory turns. Chapter 5 develops identity as equivalence under trace, and Chapter 6 situates the framework within Priest's tripartite logical stratification. Part III (Chapters 8 and 9) reconstructs self-reference in this setting. Classical quines are shown to fail; in their place we introduce normal-form fixed points and the Spherepop quine. At these points the distinction between representation and meaning collapses. Part IV (Chapters 10 and 11) extends the framework to natural language, deriving a Historical Lexical-Functional Grammar and demonstrating its operation through a worked parse. Chapter 15 outlines empirical consequences, and Chapter 16 states the general result.

The guiding intuition throughout is simple but far-reaching. Evaluation is not a reversible transformation of form, but a process of elimination. Possibilities are progressively excluded until only a stable configuration remains. That configuration does not represent its origin; it is what remains after the origin has been irreversibly transformed. To understand identity in such a system is therefore to understand what survives collapse.

Chapter 2

Making Evaluation Visible

Standard presentations of computation describe evaluation as a mapping from expressions to values. This description suppresses the internal structure of evaluation, treating the process as a black box whose only relevant property is its input–output behaviour. In this chapter, that abstraction is suspended and evaluation is made explicit as a sequence of local transformations.

The purpose is not to introduce a new formalism, but to expose a feature already present in ordinary computation: evaluation proceeds by the progressive elimination of structure. Each step removes a portion of the expression and replaces it with a value. Once removed, that structure is not retained by the result. Evaluation is therefore not merely the computation of a value, but a process of irreversible simplification.

Analogy. Consider peeling and chopping an onion. The act of chopping commits the onion to a particular configuration from which it cannot be restored. One cannot recover the original intact onion from the pile of chopped pieces, even though the chopped pieces represent the "output" of the operation. Evaluation in the Spherepop calculus works in exactly this way. The nested bubble (3×5) is the intact onion; the result 15 is what remains after the structural integrity of the subexpression has been committed. The scaffolding—the fact that it was a multiplication of 3 and 5—is gone.

Prerequisites

This chapter assumes no formal background beyond elementary arithmetic and a passing familiarity with the idea of a computer program. The purpose is to destabilise a picture that most readers already hold implicitly, so prior technical knowledge is not required and may even be an obstacle. Readers who have worked with rewriting

systems or formal language theory should read this chapter as a reframing rather than an introduction.

Traces and the pop* Notation

Consider the arithmetic expression $8+3\times5$. Rather than invoking a precedence rule, the expression may be written with its implicit structure made explicit:

$$(8 + (3 \times 5)).$$

Evaluation proceeds by reducing the innermost subexpression first. This process is recorded as a reduction trace, in which each step explicitly displays the current state of the expression. The notation `pop*` marks the moment at which a subexpression is irreversibly replaced by its value.

Reduction trace: $8 + 3 \times 5$

```
( 8 + ( 3 * 5 ) ) ( 8 + ( 15 ) ) [pop*:  3 * 5 -> 15] ( 8 + 15 ) ( 23 )
[pop*:  8 + 15 -> 23] 23
```

Two structural features are evident in this trace. Evaluation proceeds locally, with each step acting on a single subexpression. At the same time, each application of `pop*` removes structure. The expression (3×5) is replaced by 15, and the multiplicative structure that generated this value is no longer present. The value itself does not encode the operation or operands from which it arose. That information has been eliminated.

Left-Grouped Evaluation

To emphasise that this behaviour is independent of operator precedence, consider an explicitly grouped expression:

$$((12 - 4) \times 3) + 2.$$

Reduction trace: $(12 - 4) \times 3 + 2$

```
( ( ( 12 - 4 ) * 3 ) + 2 ) ( ( 8 * 3 ) + 2 ) [pop*:  12 - 4 -> 8] ( 24
+ 2 ) [pop*:  8 * 3 -> 24] 26 [pop*:  24 + 2 -> 26]
```

Here the grouping is fixed from the outset, yet the same phenomenon occurs. Structure is introduced in order to organise evaluation and is subsequently removed as each operation is performed. Intermediate forms exist only transiently; they do not persist in the final result.

Evaluation as Construct and Erase

The traces above support a general characterisation of evaluation. Structural organisation is introduced to determine how computation proceeds, but once a subexpression has been evaluated, the structure that enabled that evaluation is no longer required and is eliminated.

Evaluation can therefore be understood as a process in which structure is first constructed, then utilised, and finally erased. The conventional notion of operator precedence may be reinterpreted within this framework. Precedence does not describe a hierarchy inherent in the final result, but rather prescribes how a temporary hierarchy is constructed so that it may be systematically removed. What remains at the end of evaluation is not the structured expression but its terminal value. The scaffolding that governed the computation has been consumed by the process itself.

Irreversibility in Ordinary Computation

The elimination of structure has an immediate and decisive consequence. The final value does not determine the expression from which it was obtained.

In the earlier example, both $(8 + (3 \times 5))$ and $(8 + 15)$ reduce to the same value 23. From the value alone, it is not possible to recover which expression was evaluated. The information distinguishing these expressions has been irreversibly lost.

This phenomenon is not specific to arithmetic notation but arises whenever evaluation replaces structured expressions with simpler forms. Distinct inputs may converge to the same output, and the output does not retain the information necessary to distinguish among them. Evaluation is therefore intrinsically many-to-one. Irreversibility is not an exotic feature introduced by specialised systems. It is already present in ordinary computation, though typically concealed by the focus on final results.

The Trace as Residual Structure

Although the internal structure of an expression is eliminated during evaluation, the reduction trace provides a record of the process by which the value was obtained. The trace preserves the sequence of transformations, even as each transformation removes part of the structure.

This suggests a fundamental distinction. An expression may be identified with its final value, in which case evaluation appears as a mapping between static objects. Alternatively, it may be identified with the history of its reduction, in which case evaluation is a process whose structure persists only in the trace. Standard accounts privilege the former perspective and treat the latter as an implementation detail. In what follows, the trace is taken to be primary, and the final value is understood as a projection of that trace.

Transition to Formalisation

The preceding analysis establishes that evaluation is a dynamic process characterised by local transformation, structural elimination, and irreversible loss of information. These features motivate the introduction of a formal system in which irreversibility is not implicit but explicit.

In the next chapter, the Spherepop calculus is defined. Expressions are constructed as nested structures, and evaluation proceeds through the irreversible popping of these structures. Within that framework, the behaviour observed here is no longer incidental. Evaluation becomes a systematic contraction of structure, and the only invariant that survives is the terminal trace.

Chapter 3

Spherepop as an Irreversible Calculus

The preceding chapter established that ordinary evaluation proceeds through the progressive elimination of structure and does not preserve the information required to reconstruct its origin. In this chapter, that behaviour is made explicit. We introduce the Spherepop calculus as a formal system in which evaluation is not merely irreversible in practice, but irreversible by design.

The aim is not to replace existing rewriting systems, but to isolate and formalise a feature they implicitly exhibit. In Spherepop, expressions are constructed from nested structures, and evaluation consists of eliminating those structures through local operations. Each step contracts the expression and cannot be undone. The result is a calculus in which reduction is best understood as a history of commitments rather than a reversible transformation.

Prerequisites

This chapter assumes familiarity with inductive definitions and elementary rewriting systems at the level of an undergraduate course in theoretical computer science or formal languages. The reader should understand what it means for a rule to be applied locally to a subexpression and should be comfortable with the distinction between an expression and its value. Familiarity with the lambda calculus is helpful but not required; the Spherepop calculus is developed from first principles. The reduction trace notation introduced in Chapter 2 is used throughout.

Tokens and Bubbles

Definition 0.1 (Spherepop Expressions)**.** The set $\mathcal{E}$ of *Spherepop expressions* is defined inductively. Every atomic token $t \in \mathcal{A}$ is an expression. If $E_1, \ldots, E_n$ are expressions, then the concatenation enclosed in parentheses $(E_1 \cdots E_n)$ is also an expression, referred to as a *bubble*.

A bubble may be understood as a bounded region of structural commitment. Its contents are grouped together so that they may be acted upon as a unit. The nesting of bubbles therefore determines the scope within which local operations may occur.

Unlike conventional abstract syntax trees, bubbles are not intended as a static representation of hierarchical structure. They are transient organisational devices whose role is to delimit the domain of reduction. Once a bubble has been resolved, it is removed, and its internal structure does not persist in the result.

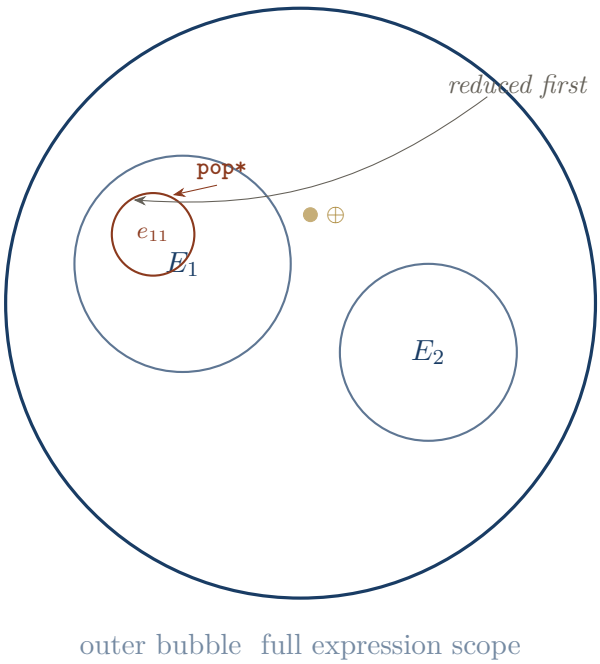


FIGURE 1. **Spherepop expressions as nested bubbles.** Nesting depth encodes evaluation priority: the innermost bubble is the first commitment site (`pop*`). Structure is transient; it is consumed during reduction and does not persist in the trace.

Reduction Rules and Sequences

Definition 0.2 (Reduction)**.** Let $\mathcal{R}$ be a set of local rewrite rules of the form $(E_1 \cdots E_n) \to E'$, where E' is an expression. A *reduction step* consists in selecting a bubble within an expression and replacing it according to a rule in $\mathcal{R}$. A *reduction sequence* is a finite chain $E_0 \to E_1 \to \cdots \to E_k$ in which each step applies a rule to some bubble of the preceding expression.

Reduction in Spherepop is strictly local. At each step, a single bubble is selected and replaced. The global behaviour of evaluation emerges from the composition of these local operations.

Crucially, each reduction step eliminates a portion of the expression. The original bubble is not retained alongside its replacement. The system does not maintain a record of alternatives or preserve prior states. Each step constitutes a commitment that restricts the space of possible future reductions.

Normal Forms and the Evaluation Map

Definition 0.3 (Normal Form and Evaluation Map)**.** An expression E is said to be in *normal form* if no rule in $\mathcal{R}$ applies to any of its subexpressions.

Assumption 0.4 (Confluence and Termination)**.** The rule set $\mathcal{R}$ is terminating and confluent: every expression E reduces to a unique normal form regardless of the order in which rules are applied.

Under this assumption, the *evaluation map* $f : \mathcal{E} \to \mathcal{N}$ assigns to each expression its unique normal form, denoted $\tau(E) = f(E)$. The map f is surjective (every normal form evaluates to itself) and idempotent ($f(f(E)) = f(E)$); it is therefore a canonical surjective projection from $\mathcal{E}$ onto $\mathcal{N}$, inducing the quotient $\mathcal{N} \cong \mathcal{E}/{\sim}$.

Termination ensures that every reduction sequence reaches a state to which no further rules apply. Confluence ensures this state is independent of the sequence chosen. Without confluence, identity becomes path-dependent: two trajectories from the same expression could reach distinct normal forms, fracturing the notion of trace. We treat confluence not as a technical convenience but as the condition under which a well-defined notion of identity is available at all. Non-confluent systems yield path-dependent identity theory, which we set aside as a distinct direction.

Append-Only Dynamics

The defining feature of the Spherepop calculus is that reduction is append-only. Each step in a reduction sequence contributes to a history of commitments, but none of these commitments can be withdrawn.

A reduction sequence $E_0 \to E_1 \to \cdots \to E_k$ is not merely a path between two expressions, but an ordered record of transformations. At each step, the set of admissible continuations is restricted. Once a bubble has been reduced, the alternatives it represented are no longer available.

Analogy. A notarised deed transfer operates on the same principle. Once signed, witnessed, and recorded, the transfer is entered into the land register as a permanent and irrevocable entry. The alternative futures in which the seller retained the property—or sold to a different buyer—are not merely set aside; they are legally

eliminated. The land register is an append-only ledger, and no subsequent entry can remove a prior one. Spherepop reduction is formally identical: each reduction step is a notarisation of a decision that forecloses alternatives and can never be undone.

In this sense, reduction does not explore a space of possibilities; it contracts that space. The evaluation process progressively eliminates incompatible structures until a terminal configuration is reached. The normal form is not one possibility among many equally accessible alternatives. It is what remains after all incompatible possibilities have been excluded.

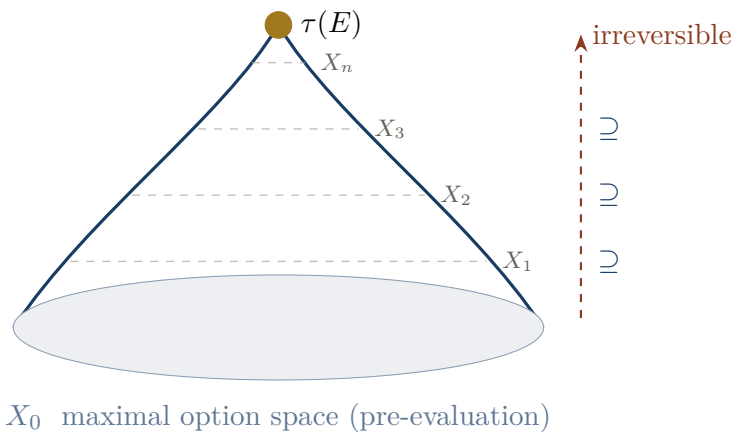


FIGURE 2. **Irreversible contraction of the option space.** Each reduction step yields $X_{k+1} \subseteq X_k$ (strict inclusion when new constraints eliminate possibilities). The sequence terminates at the trace, which carries no record of the path that produced it.

This interpretation distinguishes Spherepop from standard rewriting systems in which reduction sequences may be viewed abstractly as interchangeable paths between equivalent expressions. In Spherepop, the path itself has ontological significance. It records the sequence of eliminations by which the final result was obtained.

From Expressions to Histories

The preceding considerations suggest a shift in how expressions are understood. An expression is not merely a static object to which rules are applied, but the starting point of a reduction trajectory. Its identity is not exhausted by its syntactic form, but includes the history of transformations it undergoes.

At the same time, this history is not preserved in the final result. The normal form retains only what survives the process of elimination. Distinct histories may converge to the same terminal state, and the differences between them are not recoverable from that state.

This tension between the richness of the reduction history and the minimality of the normal form is the central structural feature of the calculus. It is this feature that

gives rise to the loss of preimage and, ultimately, to the need to redefine identity. In the next chapter, this observation is formalised. The evaluation map is shown to lack a right inverse, and the resulting notion of collapse provides the hinge on which the remainder of the theory turns.

Equivalence to the Lambda Calculus and Turing Completeness

The Spherepop calculus is not merely analogous to classical models of computation; it is computationally equivalent to them. We establish this by constructing explicit translations between Spherepop and the untyped lambda calculus, from which Turing completeness follows as a corollary.

Translating Lambda Terms into Spherepop

Recall that every untyped lambda term is one of three forms: a variable x, an abstraction $\lambda x.\, M$, or an application $M\, N$. We define a translation $\cdot : \Lambda \to \mathcal{E}$ by structural induction:

$$\begin{aligned} x &= x, \\ \lambda x.\, M &= (\texttt{lam}\ x\ M), \\ M\, N &= (M\ N). \end{aligned}$$

Variables translate to atomic tokens. Abstraction becomes a labelled bubble carrying the variable name and the translated body. Application becomes a binary bubble whose first child is the translated function and whose second child is the translated argument.

Beta-reduction in the lambda calculus, $(\lambda x.\, M)\, N \to M[N/x]$, corresponds to a single Spherepop reduction step: the outermost application bubble is popped, the variable x throughout the body M is replaced by the argument N, and the enclosing structure is discarded. This substitution rule is straightforwardly encoded in $\mathcal{R}$. The translation therefore respects reduction: if $M \to_\beta M'$ then $M \to_{\mathcal{R}} M'$.

Translating Spherepop into the Lambda Calculus

In the opposite direction, define $\lceil\cdot\rceil : \mathcal{E} \to \Lambda$ by treating every Spherepop bubble $(E_1 \cdots E_n)$ as a left-associated application:

$$\lceil (E_1 \cdots E_n) \rceil = \lceil E_1 \rceil\ \lceil E_2 \rceil\ \cdots\ \lceil E_n \rceil,$$

where nesting is handled by structural recursion. A bubble with a single element is the identity applied to its content. Named operators such as `lam`, `pop`, `bind`, and `refuse` are encoded as specific closed lambda terms (combinators) using the standard Church encoding of data and control structures.

The key point is that the core Spherepop operations—substitution during a pop step, constraint propagation in bind, and the monotone contraction of option spaces—all have direct lambda encodings. This follows from the universality of the lambda calculus: any computable function can be expressed as a lambda term [6].

Confluence and Normal Forms

The Spherepop calculus satisfies Assumption 0.4 for the relevant class of reduction rules, corresponding to the fact that beta-normal forms exist and are unique for strongly normalising lambda terms. For non-terminating computations, the normal form is undefined in both systems, preserving the correspondence exactly.

Theorem 0.5 (Computational Equivalence). *The Spherepop calculus is computationally equivalent to the untyped lambda calculus, and therefore to Turing machines. Specifically:*

(i) Every lambda term can be simulated by a Spherepop expression, and beta-reduction corresponds to Spherepop reduction steps.

(ii) Every Spherepop expression has a faithful encoding as a lambda term, and Spherepop reduction is simulated by beta-reduction.

(iii) A function is Spherepop-computable if and only if it is Turing-computable.

Proof (Sketch). Parts (i) and (ii) follow from the translations $\cdot$ and $\lceil\cdot\rceil$ constructed above, which preserve reduction steps in each direction. Part (iii) follows from the Church–Turing thesis and the known equivalence of the untyped lambda calculus and Turing machines [5]: any Turing-computable function is lambda-definable, and any lambda-definable function is Turing-computable. Since Spherepop simulates the lambda calculus and is simulated by it, the class of Spherepop-computable functions coincides with the class of Turing-computable functions. □ □

What Turing Completeness Does and Does Not Establish

The equivalence result is important for two reasons. First, it ensures that Spherepop is not a toy calculus but a full model of computation: any algorithm can be expressed in it. Second, it grounds the irreversibility analysis in a setting the reader already understands: lambda reduction is also irreversible in the relevant sense. Beta-reduction destroys the redex—the source structure—and the result does not encode which redex produced it.

However, Turing completeness does not explain irreversibility; it presupposes it. The point of the present work is not to establish that Spherepop can compute everything a Turing machine can (though it can), but to analyse what it means for an evaluation to be irreversible, and to develop a theory of identity appropriate to that condition.

The Turing equivalence confirms that this analysis applies to the full generality of computable processes.

Part 2

The Hinge: Collapse and Its Consequences

Chapter 4

Collapse and the Loss of Preimage

The Spherepop calculus makes explicit a structural feature that is only implicit in ordinary computation: evaluation eliminates information. The reduction of an expression to its normal form preserves certain invariants while discarding the structure that produced them. This asymmetry between what is retained and what is lost is the defining property of collapse.

In this chapter, that asymmetry is formalised. We show that the evaluation map does not admit a right inverse, and we analyse the resulting structure in terms of equivalence classes of expressions. The loss of preimage is not an accidental limitation of particular reduction sequences, but a necessary consequence of the many-to-one character of evaluation.

Part II Hinge

Collapse has no right inverse. This is the structural asymmetry that makes classical self-reference fail and forces the redefinitions in Parts III and IV.

Prerequisites

This chapter assumes the definitions of Chapter 3 (expressions, bubbles, reduction rules, evaluation map) and familiarity with basic set-theoretic notions: functions, surjectivity, and equivalence relations. The proof of Proposition 4.1 uses a simple fiber argument requiring no machinery beyond elementary combinatorics. Readers with a background in abstract algebra will recognise the quotient construction immediately; those without should find the prose development of the quotient $\mathcal{E}/\sim$ self-contained.

The Absence of a Right Inverse

Definition 0.1 (Right Inverse)**.** A function $g : \mathcal{N} \to \mathcal{E}$ is a *right inverse* of the evaluation map $f : \mathcal{E} \to \mathcal{N}$ if $f \circ g = \mathrm{id}_{\mathcal{N}}$.

The existence of a right inverse would imply that every normal form contains sufficient information to reconstruct at least one expression evaluating to it. The following proposition shows this is not the case.

Proposition 0.2 (Non-Existence of Right Inverse)**.** *The evaluation map $f : \mathcal{E} \to \mathcal{N}$ admits no right inverse.*

Proof. Assume for contradiction that there exists a function $g : \mathcal{N} \to \mathcal{E}$ such that $f(g(N)) = N$ for all $N \in \mathcal{N}$.

Consider a normal form N that admits multiple distinct preimages. Such cases arise immediately. Let $E_1 = (8 + (3 \times 5))$ and $E_2 = (8 + 15)$. Both expressions reduce to $f(E_1) = f(E_2) = 23$, so $E_1, E_2 \in f^{-1}(23)$, yet $E_1 \neq E_2$.

The function g must assign a single expression $g(23)$ whose evaluation yields 23. However, the value 23 does not encode any information that distinguishes E_1 from E_2. Any choice of $g(23)$ therefore depends on information not contained in 23 itself. Since this reasoning applies to every normal form with multiple preimages, no function g can be defined solely in terms of $\mathcal{N}$ such that $f \circ g = \mathrm{id}_{\mathcal{N}}$. □ □

Remark 0.3. The evaluation map f is therefore a canonical quotient map identifying all expressions in a fiber with a single invariant representative. The absence of a right inverse is not a deficiency but a structural consequence: the quotient point $N \in \mathcal{N}$ does not encode which preimage produced it.

Preimage and Trace

Definition 0.4 (Preimage and Trace)**.** Let $N \in \mathcal{N}$. The *preimage* of N is the set $f^{-1}(N) = \{E \in \mathcal{E} \mid f(E) = N\}$. For any expression E, the *trace* of E is its normal form $\tau(E) = f(E)$.

The preimage of a normal form collects all expressions that collapse to the same terminal state. The trace is the only information that survives evaluation. While the preimage may be large and structurally diverse, the trace is a single object that does not retain the distinctions present in the preimage.

The absence of a right inverse implies that the preimage cannot be recovered from the trace. The trace is therefore not a compressed encoding of the preimage, but a projection that discards its internal differences entirely.

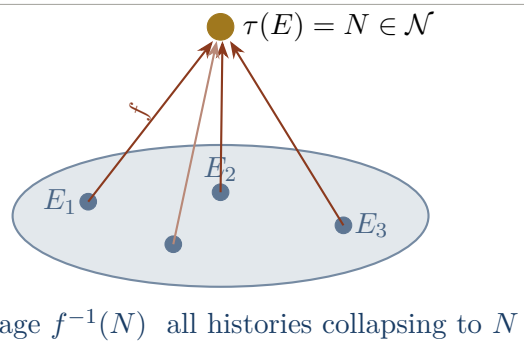


FIGURE 1. **Many-to-one collapse.** A fiber of expressions contracts irreversibly to a single trace. The preimage is not recoverable from N; this is the fundamental asymmetry established by Proposition 4.1.

The Quotient Structure of Collapse

The evaluation map induces an equivalence relation on $\mathcal{E}$:

$$E \sim E' \iff f(E) = f(E').$$

Two expressions are equivalent precisely when they collapse to the same trace. This equivalence partitions $\mathcal{E}$ into disjoint classes, each class consisting of all expressions that share a common normal form. The set of normal forms $\mathcal{N}$ can therefore be identified with the quotient $\mathcal{E}/\sim$.

Under this identification, the evaluation map f is the canonical projection sending each expression to its equivalence class. Each fiber $f^{-1}(N)$ is collapsed to a single representative N.

The essential asymmetry is now clear. The projection f is many-to-one. Each equivalence class contains multiple histories, but the quotient retains only a single invariant. The distinctions between elements of a fiber are not hidden; they are eliminated as distinguishable structure. Collapse is therefore not merely a loss of detail, but a structural operation that identifies all histories compatible with a single invariant outcome.

Analogy. A photograph is a quotient. The light entering the lens during the exposure represents a large and diverse set of physical configurations—the precise trajectory of every photon, the exact state of every surface in the scene. All of this is collapsed onto a flat image. Two physically distinct scenes that differ only in ways to which the sensor is insensitive yield identical photographs. The photograph is the trace; the original scene is the preimage. No darkroom technique recovers the preimage from the print. What is lost in the exposure is lost permanently, and the photograph carries no record of how many distinct scenes could have produced it.

Irreversibility as Structural Asymmetry

The non-existence of a right inverse establishes a fundamental asymmetry between expressions and their traces. While every expression determines a unique trace, no trace determines a unique expression.

This asymmetry is not contingent on the particular choice of reduction rules. It follows from the basic form of evaluation as a process that replaces structured objects with simpler ones. Whenever distinct structures reduce to a common result, the information distinguishing them is irretrievably lost. Irreversibility is therefore not an empirical limitation but a structural fact. The evaluation map contracts the space of expressions onto a space of invariants, and this contraction cannot be undone.

Chapter 5

Identity as Event History

The preceding chapter established that evaluation eliminates the information required to reconstruct the origin of an expression. Distinct expressions may collapse to the same trace, and the trace does not encode the structure of its preimage. This result forces a revision of the notion of identity.

In reversible systems, identity is typically defined in terms of state. An object is identified either by its syntactic structure or by its position within a space of configurations. Such definitions presuppose that the relevant structure is preserved or recoverable. Under irreversible collapse, this presupposition fails.

In this chapter, identity is relocated from the domain of initial structure to the domain of what survives evaluation. The trace becomes the carrier of identity, while the history of evaluation provides the context in which that identity is realised.

Prerequisites

This chapter builds directly on Chapter 4. The reader should understand the non-existence of a right inverse and the resulting quotient structure before proceeding. Some familiarity with philosophy of identity is helpful for the motivating discussion in Section 5.1, but is not required for the formal content. The definition of the event word ($\partial(E) \in \Sigma^*$) assumes basic acquaintance with formal languages and the monoid of finite strings.

The Failure of State-Based Identity

In a reversible setting, a state may be taken to determine its history, at least up to equivalence. This allows identity to be defined extensionally: two objects are identical if they occupy the same state or share the same structure.

In an irreversible system, a state no longer determines its origin. The evaluation map f identifies distinct expressions at the level of their trace, and the information distinguishing those expressions is not recoverable. A normal form therefore underdetermines the histories that produced it.

It follows that identity cannot be grounded in the initial configuration of an expression. Any criterion that depends on reconstructing the preimage is inapplicable. Identity must instead be located in what remains after collapse.

Event Words

Definition 0.1 (Event Word)**.** Let Σ be the set of atomic reduction events. The *event word* of an expression E is the sequence $\partial(E) \in \Sigma^*$ that records the ordered reduction steps transforming E into its normal form.

The event word captures the history of evaluation as a sequence of commitments. Each symbol represents a local transformation, and the concatenation of these symbols records the trajectory by which the expression is reduced. This history is richer than the final result. Distinct event words may correspond to expressions that collapse to the same trace. The event word therefore carries information that is not preserved under evaluation.

Trace and Identity

Definition 0.2 (Trace Identity)**.** The *identity* of an expression E under irreversible reduction is the equivalence class $[E]_\sim = \{E' \in \mathcal{E} \mid \tau(E') = \tau(E)\}$. Two expressions are *co-identical* if they share the same trace.

This equivalence is the coarsest congruence on $\mathcal{E}$ preserved by f: any finer distinction between expressions is eliminated by evaluation, so no finer identity criterion remains observable. Since no distinction finer than trace survives evaluation, trace equivalence is the maximal invariant available to the system.

Identity is thus defined in terms of the invariant that survives evaluation. While the event word records the path taken, the trace records the outcome that remains once all incompatible possibilities have been eliminated.

Analogy. Two witnesses to the same accident recall different sequences of detail—one noticed the colour of a car first, the other the sound of the impact. Their memories are different event words. Yet both agree on what happened: the essential facts that constitute the identity of the event are the same in both accounts. Those shared facts are the trace. The divergent details of the individual recollections are elements of the same preimage fiber: different paths that collapsed to the same invariant. The event itself is defined not by either witness's private sequence of impressions, but by what is common to all accounts that agree on its identity. This definition reflects the

structural asymmetry established in the previous chapter. Since the preimage cannot be recovered, identity cannot depend on it. The only stable criterion is the trace.

Event History and Invariance

The event word and the trace carry orthogonal information. The event word records the history of commitments and distinguishes trajectories that may converge to the same result. The trace records only the surviving invariant and identifies trajectories that differ but collapse to the same endpoint.

Thus $\partial(E) \neq \partial(E')$ does not imply $\tau(E) \neq \tau(E')$, and conversely, shared traces do not entail shared histories. The event word is path-sensitive; the trace is path-invariant. Identity, under irreversibility, attaches to the latter. Event words record how a trace was reached; they do not determine identity. Identity is defined solely at the level of the trace. This shows that identity is not a property of the path taken, but of the endpoint reached.

Mereological Inclusion

The event word also induces a notion of inclusion. If the reduction sequence of one expression forms a subsequence of the reduction sequence of another, then the first may be regarded as a part of the second in a historical sense.

This relation aligns with the containment of bubbles in Spherepop expressions. Nested structures give rise to nested segments of the event word, and the elimination of a substructure corresponds to the contraction of a segment of the history. Mereological inclusion is therefore not defined in terms of static substructure, but in terms of containment within a history of transformations.

Identity After Collapse

The relocation of identity from structure to trace has a decisive consequence. An expression is no longer identified by what it was, but by what it becomes under evaluation. Identity is not preserved through the process of reduction; it is established by that process. The elimination of structure is not a loss to be compensated, but the mechanism by which invariant identity emerges. In this sense, identity is not a precondition of evaluation, but its result.

Regions as Possible Worlds: The Categorical Geometry of Scope

The nested bubble structure of Spherepop expressions is not merely a notational device. It is a model of *scope*: each bubble delimits a bounded region within which

certain operations and tokens are active, and outside which they are not. This structure has a precise interpretation in category theory and connects the formalism to a deep philosophical tradition about possible worlds and the nature of models.

Bubbles as Objects in a Presheaf Category

In the internal logic of a topos, an object A in a category $\mathbf{C}$ can be interpreted as a collection of *local data* varying across contexts. A context here is an object of $\mathbf{C}$ itself—a "stage of definition" or a bounded region of applicability. Morphisms between contexts correspond to restrictions: data available in a larger context restricts to data available in a smaller one.

Spherepop bubbles instantiate exactly this structure. Each bubble B is a context: the tokens and operators it contains are in scope within B and are not directly visible outside it. A nested bubble B' inside B corresponds to a subcontext—a more restricted stage of definition—and the reduction rules that pop B' before B correspond to the restriction functors that resolve inner contexts before outer ones.

Formally, the collection of all bubbles in an expression, ordered by containment, forms a partial order $(\mathcal{B}, \sqsubseteq)$. This partial order is the base of the presheaf of histories constructed in Appendix C: the data assigned to each bubble is the set of all compatible reduction sequences whose scope falls within that bubble. Restriction maps send a global reduction to its local trace within a subbubble.

Scope Order as a Model of the World

When a human being sets out to understand a situation—whether by writing a program, constructing a scientific model, or parsing an utterance—the fundamental cognitive operation is the *establishment of scope*. One defines a boundary within which certain entities are active, certain relations hold, and certain operations are admissible. Outside that boundary, different rules apply.

This is exactly the operation performed by bubble introduction in Spherepop. A new bubble does not add tokens arbitrarily; it introduces a bounded context with its own evaluation rules. The tokens within it are not yet committed to any global configuration—they are a local possibility space waiting to be resolved. In this sense, a bubble is a *local world*: a region of structured possibility that will eventually be popped into the global trace.

The order in which bubbles are introduced and popped corresponds to the order in which partial models are built and collapsed. A deeply nested bubble represents a fine-grained local commitment made before its containing context has been resolved. A shallow bubble represents a coarser constraint whose resolution depends on prior inner work. This nesting structure is isomorphic to the nested quantifier scopes and dependent type contexts of formal logic, and to the nested frames and activation records of a call stack in programming languages.

Humans as Scope-Establishing Agents

The act of constructing a mental model—of a physical situation, a grammatical sentence, a social interaction, or a mathematical proof—is structurally identical to the act of introducing a bubble. The modeller establishes a scope: which entities are present, which relations hold, which operations are admissible within this representation. The model is then evaluated—resolved, tested against constraints, reduced to its salient content—and the scaffolding of its construction is discarded.

The analogy is not metaphorical. In Chapter 11, the H-LFG parse of an utterance shows exactly this process: each lexical event introduces a bounded local constraint (the bubble of a nominal phrase, the scope of a verb), and the global f-structure is the trace that remains after all local scaffolding has been consumed. The listener builds a model of the sentence by establishing and collapsing scopes in the order dictated by the linear sequence of lexical events.

At a broader scale, scientific modelling follows the same pattern. A model is a bubble—a bounded context with its own ontology—within which certain idealisations hold and others do not. The model is evaluated against data, and the aspects of reality it cannot accommodate are the points at which its constraints become incompatible with the external option space. A model that survives this evaluation is grammatical in the H-LFG sense: it has a non-empty trace. A model that cannot accommodate the data is ungrammatical: total collapse.

Regions in Categorical Semantics

In the categorical semantics of modal and possible-worlds logic, a *world* is an object w in a Kripke frame—a directed graph whose edges represent the accessibility relation between worlds. A proposition φ is true at world w if it is satisfied by the local data at w. The global truth of φ is determined by whether it holds in all worlds accessible from the current one.

The partial order $(\mathcal{B}, \sqsubseteq)$ of bubbles is a Kripke frame in which containment is the accessibility relation: a bubble B' is accessible from B (i.e., $B' \sqsubseteq B$) if B' is contained within B. A token is *in scope at* B if it is contained within B and not yet popped. The evaluation of the innermost bubble first corresponds to computing truth values at the most accessible worlds before resolving the question at the containing world.

This is not a loose analogy. The sheaf of histories over $(\mathcal{N}, \preceq)$ constructed in Appendix C is the precise categorical expression of this structure. Local sections correspond to locally consistent valuations at individual worlds; global sections correspond to globally consistent models. The failure of gluing—ungrammaticality—corresponds to the impossibility of finding a model that satisfies all local constraints simultaneously, which is precisely the categorical definition of unsatisfiability in the internal logic of the sheaf.

Scope, Commitment, and the Arrow of Time

The order in which bubbles are introduced and popped defines a direction of time within the computation. Inner bubbles precede outer ones; local commitments precede global ones. This order is not arbitrary—it is the order in which the world of the expression is constructed and then dismantled.

In this light, the append-only dynamics of Spherepop reduction formalises a deep feature of cognition and action: to act is to commit to a scope, and to commit is to foreclose alternatives. The space of possible worlds narrows with each commitment. Identity—in computation, in language, in thought—is what remains when all possible worlds except one have been eliminated.

Chapter 6

Logical Stratification Under Collapse

Priest Integration

The introduction of an irreversible evaluation operator f *induces* a stratification of logical interpretation. The distinction between theory, practice, and fact is no longer optional—it is *structurally necessary*.

Prerequisites

This chapter assumes Chapters 3 through 5. On the philosophical side, some acquaintance with the philosophy of logic is helpful, though Priest's distinction between *logica docens*, *logica utens*, and *logica ens* is explained from scratch. Readers unfamiliar with paraconsistent or non-classical logic need not have prior exposure; those ideas are developed further in Chapter 13. The mathematical content of this chapter (the Correspondence Theorem and the Lemma on Trace Adequacy) requires only the definitions already established.

Priest's Tripartite Distinction

In "Revising Logic," Graham Priest distinguishes three senses in which we speak of logic. *Logica Docens* is the formal theory of valid inference, as codified in textbooks. Revision at this level proceeds by the standard criteria of theory choice: simplicity, fruitfulness, adequacy to data. *Logica Utens* is the norms governing actual inferential practice, once systematic errors are corrected. Priest argues that it is rational to revise *logica utens* by bringing practice into accord with the best available *logica docens*. *Logica Ens* is the facts of validity themselves—what actually follows from what, independently of our theoretical attempts to capture it.

Much confusion about logical revision arises, Priest contends, from conflating these levels. A revision of *logica docens* need not be a revision of *logica ens*; a change in *logica utens* need not wait for a change in *logica ens*.

The Spherepop Correspondence

The Spherepop calculus provides a setting in which Priest's three levels are not only distinguishable but are identified with distinct mathematical objects.

THEOREM 0.1 (Correspondence of Collapse). *The irreversible reduction operator f induces a tripartite correspondence:*

$$\begin{aligned} \textit{logica docens} &\longleftrightarrow (\mathcal{E}, \mathcal{R}), \\ \textit{logica utens} &\longleftrightarrow \textit{reduction trajectories in } \mathcal{E}, \\ \textit{logica ens} &\longleftrightarrow f(\mathcal{E}) \subseteq \mathcal{N}. \end{aligned}$$

These are not equivalent representations but distinct strata connected by the evaluation process.

Remark 0.2. This stratification is not interpretive but induced: the irreversibility of f severs theory, practice, and outcome into distinct mathematical objects.

Analogy. Consider a musical score, a performance, and the acoustic event that reaches the audience. The score is logica docens: a formal specification of admissible realisations. The performer's practice—the choices of tempo, dynamics, and articulation in real time—is logica utens: the actual sequence of irreversible commitments that instantiate one trajectory from the score's option space. The acoustic trace that the audience hears, which cannot be retracted once produced, is logica ens: the terminal fact. A recording of the performance is not the score; the score is not the performance; and neither is identical to the acoustic event that has already collapsed into the listener's memory. Irreversibility is what separates them. In reversible systems the three strata coincide because the formal system determines outcomes and outcomes determine preimages. Irreversibility dissolves both identifications simultaneously.

PROOF. The formal calculus $(\mathcal{E}, \mathcal{R})$ is a specification of admissible transformations—a theory—and corresponds to *logica docens*. A reduction trajectory $E_0 \to E_1 \to \cdots \to E_k$ is a sequence of actual commitments, the practice of evaluation, and corresponds to *logica utens*. By confluence, distinct trajectories over the same expression converge to the same terminal object; that terminal object is invariant across all paths in the fiber and corresponds to *logica ens*: the fact of what the expression evaluates to, independent of route. □ □

Divergence Under Irreversibility

In reversible systems, the three levels are implicitly conflated: the formal theory specifies a recoverable transformation, so *logica docens* and *logica ens* coincide, and *logica utens* is merely the execution of that specification. In Spherepop, this conflation is impossible.

Proposition 0.3 (Divergence of Logical Strata)**.** *In the Spherepop calculus, logica docens, logica utens, and logica ens are pairwise non-equivalent: syntactically distinct expressions may share a trace, so logica docens does not determine logica ens; distinct trajectories may realise the same trace, so logica utens does not determine logica docens; and the trace is determined by the fiber structure of f, not by syntactic form or trajectory alone, so logica ens is not reducible to either of the others.*

Validity as Trace Stability

The standard notion of validity concerns truth-preservation under substitution. In an irreversible calculus, this must be revised.

Definition 0.4 (Trace Validity)**.** An inference from E to E' is *trace-valid* if $\tau(E) \sim \tau(E')$.

This relocates validity from *logica docens* (syntactic preservation) to *logica ens* (stability of trace under projection). It is a revision in Priest's sense: we have identified a new adequacy criterion and updated both theory and practice accordingly.

Lemma 0.5 (Trace Adequacy)**.** *In an irreversible calculus, the only observable invariant is the trace $\tau(E)$. Therefore any criterion of correctness must be defined over $\mathcal{N}$, not over $\mathcal{E}$.*

Proof. In a reversible system, intended and actual outcome can be compared by inspecting preimage structure. In an irreversible system, f has no right inverse (Proposition 0.2), so $\tau(E)$ does not determine E. Any criterion requiring access to the preimage is therefore inapplicable. Only criteria definable over $\mathcal{N}$ remain. □ □

Part 3

Fixed Points: Self-Reference Without Inverses

Chapter 7

From Values to Actions

The examples presented in earlier chapters concern arithmetic evaluation, where the terminal object of a computation is always a number. The Spherepop calculus, however, is not restricted to numerical output. In this chapter we extend the framework to expressions whose evaluation terminates in an action token, showing that computation is not essentially numerical but is a process of constraint resolution whose residue may be interpreted operationally.

Prerequisites

This chapter requires Chapters 2 through 4. No new formal machinery is introduced; the chapter extends the reduction trace notation to non-numerical terminal tokens. Readers who have worked with operational semantics or abstract machines may find the treatment straightforward. The chapter is short and largely motivational; its primary purpose is to prepare the linguistic extension in Part IV by establishing that computation need not terminate in a numerical value.

Action Tokens and Terminators

The Spherepop calculus admits tokens that are not reducible to numerical values but instead represent terminal actions. Let the token set $\mathcal{A}$ include such elements, for example `play`.

A reduction sequence may therefore terminate not in a number but in an action token. In such cases, evaluation resolves all intermediate quantitative structure and releases a residual command. The presence of such tokens demonstrates that evaluation is not essentially numerical. It is a process of constraint resolution whose output may be interpreted operationally rather than arithmetically.

The `play` Program

Consider the expression $Q = \big(+\ (-\ 1\ 2)\ (-\ 1\ 2)\ \texttt{play}\big)$.

Reduction trace: the `play` program

```
( + ( - 1 2 ) ( - 1 2 ) play ) ( + ( -1 ) ( -1 ) play ) [pop*:  1 -
2 -> -1, twice] ( (-2) play ) [pop*:  -1 + -1 -> -2] ( play ) [pop*:
numeric scaffold consumed] play
```

The token `play` is not merely adjacent to the arithmetic; it is released by it. The computation resolves a cost structure and then discards the quantitative scaffolding, leaving only the executable residue.

Computation as Resolve-then-Discard

The reduction trace of the `play` program shows that the arithmetic subexpressions are not ends in themselves. Each numerical reduction contracts a local portion of the option space, but the resulting values are subsequently eliminated. The final transition $\big((-2)\ \texttt{play}\big) \to (\texttt{play}) \to \texttt{play}$ demonstrates that the numerical scaffold is consumed in the course of evaluation, leaving only the terminal action.

Computation therefore proceeds in two phases. First, constraints are resolved: each numerical operation determines a partial result by eliminating alternative values. Second, the scaffolding is discarded: once the constraint has been settled, the structure that organised it is removed. The meaning of the expression is not the intermediate values but the residue that survives their elimination. In this case, the meaning is the action `play`.

Analogy. In a chess endgame, the calculation of variations—whether a sacrifice leads to mate in seven—is exactly this structure. The arithmetic of piece values and tempo is the quantitative scaffold. Once the calculation is complete, the scaffold is discarded and what remains is the decision: play the sacrifice. The move is the terminal action token; the tree of variations is the bubble structure that organised the computation and was consumed by it. A chess engine that outputs 'mate in 7' exhibits the normal form. The evaluation tree that produced this conclusion is the preimage, and it is not stored in the output.

This establishes a principle that extends naturally to the linguistic setting. In a sentence, lexical events constrain the grammatical option space and then dissolve into the functional structure they have produced. The surviving structure is the meaning; the history that produced it is gone.

Chapter 8

Normal-Form Fixed Points and the Spherepop Quine

The preceding chapters established three interlocking results. Evaluation in the Spherepop calculus is irreversible; the evaluation map admits no right inverse; and identity must therefore be grounded in the trace rather than the preimage. These results converge on a natural question: under what conditions does an expression coincide with its own trace?

In classical computation, this question is addressed through the construction of quines—programs that reproduce their own source code. Such constructions rely on the ability to encode and preserve syntactic structure across evaluation. In an irreversible system, this strategy is unavailable. The preimage is destroyed during reduction, and no mechanism exists for recovering it. This chapter introduces an alternative notion of self-reference appropriate to irreversible systems.

Prerequisites

This chapter assumes all of Part II (Chapters 3 through 6). Readers should be comfortable with the notions of normal form, trace equivalence, and the Priest stratification before proceeding. Some familiarity with fixed-point theorems in mathematics or computer science (Kleene's fixed-point theorem, Gödel's diagonal lemma) is helpful for appreciating why the classical quine construction fails, but is not required. The constructive existence proof in Section 8.3 is self-contained.

Fixed Points of the Evaluation Map

Definition 0.1 (Normal-Form Fixed Point)**.** An expression $E \in \mathcal{E}$ is a *fixed point* of the evaluation map f if $f(E) = E$.

A fixed point is therefore an expression that is already in normal form. The significance of fixed points emerges only when considered in relation to the collapse of preimage. For a general expression E, the identity of E is given by its trace $\tau(E)$. The expression itself does not survive evaluation; it is replaced by its normal form. In the case of a fixed point, this distinction disappears.

Definition 0.2 (Trace Invariance)**.** An expression E exhibits *trace invariance* if $\tau(E) = E$.

Trace invariance states that the identity of the expression, as defined by the evaluation map, is fully realised by the expression itself. No distinction remains between the process of evaluation and the result of that process. In an irreversible system, trace-invariant expressions are the only objects for which self-identity survives evaluation; all others are transformed into distinct traces and cease to be identical to themselves in any recoverable sense.

The Classical Quine and Its Limitation

In classical computation, a quine is a program Q such that $\mathrm{eval}(Q) = Q$, where evaluation produces a representation of the program's source. This equality is achieved by encoding the program within itself. The quine contains a representation of its own syntax, which is preserved and reproduced during execution. The success of this construction depends on the ability to retain and manipulate syntactic structure.

Proposition 0.3 (Non-Existence of Classical Quines)**.** *There exists no non-trivial expression $Q \in \mathcal{E}$ satisfying $f(Q) = Q$ unless Q is already in normal form.*

Proof. If $Q \notin \mathcal{N}$, then evaluation strictly reduces its structural complexity: the bubble nesting of Q is eliminated, yielding $\tau(Q)$ with strictly fewer bubble nodes. Hence $\tau(Q) \neq Q$, contradicting $f(Q) = Q$. If $Q \in \mathcal{N}$, then Q contains no reducible structure, yielding only a trivial fixed point. □ □

In the Spherepop calculus this strategy fails entirely. The evaluation map eliminates structure rather than preserving it. Any attempt to encode a representation of the preimage within the expression is subject to collapse. The encoded structure is reduced along with everything else, and the original configuration cannot be recovered.

The Spherepop Quine

Definition 0.4 (Spherepop Quine)**.** An expression $Q \in \mathcal{E}$ is a *Spherepop quine* if $\tau(Q) \sim Q$, where $\sim$ denotes trace equivalence.

Unlike the classical quine, the Spherepop quine does not reproduce its source. It does not require the preservation of syntactic structure. Instead, it achieves self-reference by surviving collapse. The expression Q may undergo reduction, but the result of that reduction lies in the same equivalence class as Q itself.

Analogy. A perfectly calibrated retroreflector—a surface engineered to return a beam of light precisely along its incident path—is a fixed point of the reflection map: input and output are identical in the relevant sense. A standard mirror is not a fixed point; it inverts left and right. The Spherepop quine is the computational retroreflector. It is not a system that generates a copy of itself; it is a configuration that, when collapsed, occupies the same equivalence class under trace, so that no distinction between source and result survives the evaluation.

Proposition 0.5 (Existence of Spherepop Quines)**.** *There exist non-trivial $Q \in \mathcal{E}$ with $\tau(Q) \sim Q$.*

Proof (Constructive). Consider $Q = ((A)\ (\text{is})\ (A))$ where $A = ((\text{Sphere}(\text{pop}))(\text{normal}(\text{form}$ Under reduction, A collapses via lexical fusion: $(\text{Sphere}(\text{pop})) \to \text{Spherepop}$, $(\text{normal}(\text{form})) \to$ normal form, and finally $A \to$ Spherepop normal form. Hence $\tau(A) =$ Spherepop normal form, and $f(Q) =$ Spherepop normal form is Spherepop normal form, which is itself a normal form asserting the identity relation between its two copies. The complete reduction trace is exhibited in **??**. Thus $f(Q) \sim Q$ in the sense of trace equivalence, and Q is a Spherepop quine. □ □

Collapse of Program and Output

For a general expression E, the evaluation process distinguishes between three levels: the initial expression, the sequence of reductions, and the final trace. These correspond informally to program, execution, and output.

In the case of a Spherepop quine, this distinction collapses. The evaluation map does not carry the expression to a distinct object, but returns an element identical to it at the level of identity. The expression is already in the form that execution would produce.

Corollary 0.6 (Collapse of Representation and Meaning)**.** *For any Spherepop quine Q, the evaluated object $\tau(Q)$ simultaneously serves as the result of computation, the description of that computation, and the identity of the original expression.*

Proof. By definition of quine, $f(Q) \sim Q$. The trace $\tau(Q)$ is the output of the computation, it describes Q since it expresses the same identity relation, and it is $[Q]_\sim$ per Definition 0.2. All three coincide at $\tau(Q)$. □ □

Self-Reference Without Reconstruction

The Spherepop quine provides a model of self-reference that does not depend on reconstruction. In classical systems, self-reference is achieved by encoding a description of the program within itself. In the present setting, such reproduction is impossible. The preimage is not available after collapse, and no encoding can protect it from reduction.

The only viable form of self-reference is one in which no reconstruction is required. The Spherepop quine refers to itself by being identical to its own trace. It does not describe itself; it coincides with what remains after all description has been eliminated. The fixed point is the limit case of identity in an irreversible system: the expression occupies a boundary at which transformation becomes identity.

THEOREM 0.7 (Collapse of Logical Distinctions at Fixed Points). *Let Q be a Spherepop quine such that $\tau(Q) \sim Q$. Then logica docens(Q) = logica utens(Q) = logica ens(Q) in the sense that the theory predicts the invariant trace, the evaluation trajectory realises it, and the resulting trace constitutes the identity.*

PROOF (SKETCH). The reduction rules of $\mathcal{R}$ generate a trajectory terminating in the invariant trace $\tau(Q)$ (*logica docens*). This trajectory is the practice of evaluation (*logica utens*). Since no further structural information survives, the trace $\tau(Q)$ is the final fact of what Q is (*logica ens*). All three layers converge at $\tau(Q)$. □ □

Chapter 9

Program, Execution, and Description Collapse

In classical computational theory, a clear ontological separation is maintained between three distinct levels. A program is a static object specifying a set of instructions. Execution is a process that applies those instructions over time. Description is a representation of the result of that process. These three levels are typically treated as distinct both conceptually and formally.

The Spherepop calculus undermines this separation. Because evaluation proceeds through irreversible collapse, the distinctions between program, execution, and description cannot be maintained. Each level is defined in terms of structure that is eliminated during reduction. What remains is a single object: the trace.

Let f be the Spherepop evaluation operator and let $\tau(E)$ denote the terminal trace. Because f has no right inverse (Proposition 0.2), the trajectory leading to $\tau(E)$ is not recoverable from the result. Any distinction that depends on reconstructing the generative process from the outcome becomes ill-posed.

Prerequisites

This chapter assumes Chapter 8 and, implicitly, all preceding chapters. The argument that program, execution, and description collapse into a single object depends on the fixed-point theory developed in Chapter 8 and the quotient structure from Chapter 4. Readers interested in philosophy of computation may find it productive to read this chapter alongside classic treatments of the Curry–Howard correspondence or reflective towers, though no prior knowledge of those frameworks is assumed.

The Vanishing of the Program–Execution Boundary

In reversible systems, a program is a static object and execution is a reversible or inspectable process applied to it. In Spherepop, this distinction collapses.

A program is given not as a static rule set but as a compositional sequence of events $E = e_n \circ \cdots \circ e_1$. Execution is therefore identical to the construction of this sequence. There is no external interpreter applying rules to an inert object; the program is the history of commitments itself. Since each operation irreversibly contracts the option space, the sequence cannot be replayed in reverse, nor can an alternative trajectory be substituted without altering the identity of the result.

Hence, program and execution coincide as trajectories in $\mathcal{E}$. Formally, there exists no decomposition of $f(E)$ into a static specification and a distinct execution relation that preserves identity under f.

The Trace as Self-Description

The terminal trace is identified with the normal form: $\tau(U) = f(U)$. This object is not a representation of the computation; it is the invariant residue of the computation. All intermediate structures—bubble configurations, partial bindings, alternative trajectories—are eliminated under the quotient $[\cdot]_\sim$.

Let $\sim$ be the equivalence induced by collapse. Then $\tau(U) = [U]_\sim$ is the equivalence class of all derivations converging to the same invariant. The description is a quotient object: a universal representative of all histories realising the same grammatical commitments. This eliminates the ontological gap between computation and description. The description is not constructed alongside the computation; it is what remains when the computation has exhausted its degrees of freedom.

Readability as Trace Alignment

The apparent readability of Spherepop at the level of normal form is not a property of notation but a consequence of alignment between logical strata.

Let U be an utterance such that $\tau(U)$ is stable under evaluation. Then the trajectory of commitments (*logica utens*), the invariant trace (*logica ens*), and the interpretive content accessible to a competent agent all coincide. Because all non-essential structure has been eliminated, the remaining object encodes only those distinctions that survived irreversible contraction. Interpretation therefore requires no reconstruction of hidden derivations; it consists in reading the trace directly.

Analogy. A skilled diner, tasting a finished dish, does not reconstruct the recipe; they read the trace. The flavour is not a representation of the preparation method—it is what remains after the preparation method has exhausted itself. The readability of the dish is a property of its own structure as a terminal object, not of the recipe's

legibility. In Spherepop, the same collapse of levels occurs at a normal-form fixed point: the program that generated the trace is not recoverable from it, and reading the trace requires no reconstruction of the hidden derivation.

In this sense, readability is a structural property: an object is readable precisely when it is already in its collapse-invariant form. The distinction between code and meaning disappears because both are projections of the same trajectory under f.

The Collapse of Representational Layers

THEOREM 0.1 (Program–Execution–Description Collapse). *Let Q be a Spherepop quine. Then $\tau(Q)$ simultaneously serves as the result of the computational trajectory, the complete description of that trajectory up to equivalence, and the identity of the original expression under projection.*

PROOF (SKETCH). All intermediate structures are eliminated under $[\cdot]_\sim$. The quine condition $f(Q) \sim Q$ ensures that what remains encodes Q itself. Hence the computation produces $\tau(Q)$; $\tau(Q)$ describes the trajectory as the canonical representative of its fiber; and $\tau(Q) = [Q]_\sim$ is Q's identity per Definition 0.2. □ □

Remark 0.2. No additional representational layer can be introduced without reintroducing the assumption that structure survives evaluation. Any description external to the trace would require access to the preimage, which does not exist.

Part 4

Linguistic Extension: Historical LFG

Chapter 10

Historical Lexical-Functional Grammar

Prerequisites

This chapter assumes Part III (Chapters 7 through 9) for the computational background and introduces Lexical-Functional Grammar from scratch for readers who have not encountered it. Prior knowledge of generative syntax or formal linguistics is helpful but not required. The key prerequisite is the option-space framework (the monotone sequence $X_0 \supseteq X_1 \supseteq \cdots$) and the bind and refuse operators, which are introduced within the chapter. Readers with a background in LFG will immediately recognise the c-structure / f-structure distinction and may proceed directly to Section 10.3.

Overview

We now show that the Spherepop calculus yields a linguistic grammar as a natural consequence. The resulting system—Historical Lexical-Functional Grammar (H-LFG)—reinterprets the classical LFG bifurcation between constituent structure (c-structure) and functional structure (f-structure) in irreversible terms.

The key insight is that Spherepop already contains the ingredients of a lexical-functional grammar: a history-sensitive notion of constituency through bubble nesting, an explicit account of scope, typed operators, and a separation between surface arrangement and deeper structural constraints. Because structure is not primitive but emerges from event history, phrasehood and subphrasehood are available immediately as containment relations on histories.

Historical Lexical Entries

Definition 0.1 (Historical Lexical Entry)**.** A *lexical item* w in the Spherepop lexicon is a triple $\mathcal{L}(w) = (\pi(w),\ \kappa(w),\ \varphi(w))$, where $\pi(w)$ is the *phonological trace* (the acoustic or orthographic event triggering the entry), $\kappa(w)$ is the *c-structural insertion profile* (a historical operator specifying the bubble-nesting potential of w), and $\varphi(w)$ is the *f-structural constraint* (a binding operator bind_R reducing the grammatical option space X to the subset compatible with relation R).

Constituent Structure as Bubble History

Definition 0.2 (C-Structure)**.** Let T_u be the set of lexical token events in utterance u, partially ordered by precedence $\prec$. Let $\mathcal{B}_u$ be the set of bubbles with inclusion $\sqsubset$. The *constituent structure* is $\mathrm{CStr}(u) = (\mathcal{B}_u,\ \prec_u)$.

Functional Structure as Normal Form

Definition 0.3 (F-Structure)**.** Let X_u denote the grammatical option space for u. The *functional structure* is $\mathrm{FStr}(u) = \mathrm{NF}\big(\varphi_n \circ \cdots \circ \varphi_1(X_u)\big)$.

Theorem 0.4 (Projection of Meaning). *There exists a unique projection* $\Pi_{\mathrm{func}} : \mathrm{CStr}(u) \to \mathrm{FStr}(u)$ *that is not an annotation regime but the inevitable result of lexical constraint propagation.*

Proof (Sketch). The c-structural operators κ_i determine which bubbles may host which tokens, thereby restricting which functional operators φ_j may apply within each bubble's scope. The composed operator $\varphi_n \circ \cdots \circ \varphi_1$ therefore inherits the bubble hierarchy and has a unique normal form by the termination and confluence of $\mathcal{R}$. □ □

Proposition 0.5 (Ungrammaticality as Total Collapse)**.** *Utterance u is* ungrammatical *if and only if* $\mathrm{FStr}(u) = \varnothing$.

Proof. Ungrammaticality arises when lexical constraints are mutually incompatible: every trajectory in X_u is excluded by a refuse operator triggered by a feature mismatch. When all trajectories are eliminated, the composed operator maps X_u to $\varnothing$. □ □

Constraint Monotonicity and Irreversible Binding

Definition 0.6 (Constraint Monotonicity)**.** A sequence of lexical events $(e_1, \ldots, e_n)$ induces a sequence of constrained spaces $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n$, where $X_{i+1} = \mathsf{bind}_{e_{i+1}}(X_i)$.

Monotonicity formalises the irreversibility of grammatical commitment. Once a feature has been fixed or a dependency resolved, the system cannot return to a prior state in which that feature was unspecified. This replaces consistency maintenance: incompatibility is eliminated immediately through contraction of the option space, not checked post hoc.

Projection as Quotient Collapse

Definition 0.7 (Functional Projection)**.** Let $\mathcal{C}$ denote the space of c-structures. Define $C \approx C'$ iff $\Pi_{\text{func}}(C) = \Pi_{\text{func}}(C')$. Then $\mathcal{F} \cong \mathcal{C}/\approx$.

Under this formulation, f-structure is not constructed independently of c-structure. It is the invariant that survives the collapse of distinctions between structurally different but functionally equivalent configurations, aligning the H-LFG derivation directly with the quotient structure of Chapter 4.

LFG Concepts Recovered

The standard LFG notions fall out cleanly. Functional uncertainty is unresolved branching in the option space. Agreement is compatibility preservation under bind. Grammaticality is the non-emptiness of $\text{FStr}(u)$. Ungrammaticality is total collapse to $\varnothing$. Ambiguity is a normal form containing multiple surviving traces. Disambiguation is irreversible contraction of the parse space, not mere feature unification.

Functional structure is not constructed in parallel with syntax; it is what remains once incompatible grammatical trajectories have been eliminated. Crucially, H-LFG is inherently historical: a sentence is not a static well-formed object but a commitment trajectory. The grammar is not an external filter applied to surface strings, but the structure of irreversible constraint propagation itself.

Chapter 11

A Worked Parse

To demonstrate that H-LFG is not merely analogical, we derive the functional structure of a concrete utterance by explicit Spherepop operations. Let $U =$ "the child sees the dog." The aim is to show that grammatical interpretation is an irreversible contraction of an initial option space into a stable normal form.

Prerequisites

This chapter assumes Chapter 10 in full. Readers should be familiar with the definitions of historical lexical entries, c-structure, and f-structure as well as the constraint monotonicity condition. No additional mathematical background beyond what has already been introduced is required. The worked parse is intended to be followed step by step; readers are encouraged to trace the sequence $X_0 \supseteq \cdots \supseteq X_5$ alongside the textual derivation.

Token Events and Lexical Entries

The utterance is realised as the event sequence $(e_1, e_2, e_3, e_4, e_5) = (\text{the}_1,\ \text{child},\ \text{sees},\ \text{the}_2,\ \text{dog})$. We begin with X_u containing all well-typed partial analyses compatible with the language before any commitments.

Phase 1: First Nominal Bubble

Events e_1 and e_2 form the first nominal bubble. The determiner opens the nominal scope:

$$\kappa_{\text{the}} : \varnothing \rightsquigarrow B_{\text{NP}}, \qquad \varphi_{\text{the}} : X \mapsto X_{\text{def}} \subseteq X.$$

The noun saturates it:

$$\kappa_{\text{child}} : B_{\text{NP}} \rightsquigarrow B_{\text{NP}}^{\text{sat}}, \qquad \varphi_{\text{child}} : X_{\text{def}} \mapsto X_{\text{NP}_1}.$$

The stable constituent B_{NP_1} = (the child) emerges with partial f-structure:

$$F_{\mathrm{NP}_1} = \begin{bmatrix} \mathrm{PRED} & \texttt{child} \\ \mathrm{DEF} & + \\ \mathrm{NUM} & \mathrm{sg} \end{bmatrix}.$$

Many global analyses remain open. The nominal might become subject, object, or part of a larger construction; those unrealised alternatives remain in X_u.

Phase 2: Verbal Commitment

Event e_3 (*sees*) is the decisive hinge. Its c-structural profile opens a verbal bubble taking B_{NP_1} as potential external argument and anticipating a second nominal as internal argument. Its functional operator introduces a predicate frame and contracts the global option space:

$$\varphi_{\mathrm{sees}} : X_{\mathrm{NP}_1} \mapsto X_V \subseteq X_{\mathrm{NP}_1},$$

where every member of X_V carries the constraint $\mathrm{PRED} = \texttt{see}\langle \mathrm{SUBJ}, \mathrm{OBJ} \rangle$. All parses incompatible with a two-place predicate are eliminated irreversibly.

Phase 3: Second Nominal and Final Binding

Events e_4 and e_5 form the second nominal bubble B_{NP_2} = (the dog):

$$F_{\mathrm{NP}_2} = \begin{bmatrix} \mathrm{PRED} & \texttt{dog} \\ \mathrm{DEF} & + \\ \mathrm{NUM} & \mathrm{sg} \end{bmatrix}.$$

The c-structure is now complete: $B_S(B_{\mathrm{NP}_1},\ B_V(\mathrm{sees},\ B_{\mathrm{NP}_2}))$. The verbal operator applies two sequential binding operations:

$$X_{\mathrm{final}} = \mathsf{bind}_{C_{\mathrm{obj}}}\Big(\mathsf{bind}_{C_{\mathrm{subj}}}(X_V,\ F_{\mathrm{NP}_1}),\ F_{\mathrm{NP}_2}\Big).$$

Terminal Trace

Applying $\mathsf{collapse}_{\sim}$ yields the grammatical normal form:

$$F(U) = \begin{bmatrix} \mathrm{PRED} & \texttt{see}\langle \mathrm{SUBJ}, \mathrm{OBJ} \rangle \\ \mathrm{TENSE} & \mathrm{pres} \\ \mathrm{SUBJ} & \begin{bmatrix} \mathrm{PRED} & \texttt{child} \\ \mathrm{DEF} & + \\ \mathrm{NUM} & \mathrm{sg} \end{bmatrix} \\ \mathrm{OBJ} & \begin{bmatrix} \mathrm{PRED} & \texttt{dog} \\ \mathrm{DEF} & + \\ \mathrm{NUM} & \mathrm{sg} \end{bmatrix} \end{bmatrix}.$$

This is the *logica ens* of the clause: the terminal fact of who did what to whom, stripped of the transient scaffolding that made it available. The f-structure is not a supplementary representation; it is what remains when the derivation has exhausted its incompatible continuations.

Grammaticality and Ungrammaticality

Proposition 0.1 (Grammaticality Criterion)**.** *U is grammatical iff* $\mathsf{collapse}_{\sim}(X_U) \neq \varnothing$.

Proof. If collapse yields a non-empty space, at least one fully compatible functional trajectory survives. If collapse yields $\varnothing$, every trajectory has been excluded by incompatibility. □ □

For U' = "the child *see* the dog," the token *see* imposes a constraint requiring [NUM : pl] or a non-third-person index. Since F_{NP_1} bears [NUM : sg], the binding operator finds no compatible trajectories: $\mathsf{bind}_{C_{\mathrm{subj}}}(X_V,\ F_{\mathrm{NP}_1}) = \varnothing$. Ungrammaticality is not a violation of an external template; it is the total collapse of the grammatical option space.

For utterances such as "flying planes can be dangerous," the lexical events do not impose sufficient exclusionary pressure to reduce X_U to a single trajectory. If two trajectories survive through the terminal event, the normal form retains both traces. Ambiguity is the failure of the history to become fully definite: residual optionality that has not undergone final contraction.

Derivation as Sequential Contraction

The worked parse may be understood as a sequence of contractions on the global option space. Each lexical event reduces the admissible configurations, and the entire derivation is the composition of these reductions. The sequence $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n$ makes explicit that parsing is not the construction of a structure, but the elimination of incompatible alternatives.

Proposition 0.2 (Global Collapse from Local Failure)**.** *If at any stage $X_i = \varnothing$, then $X_j = \varnothing$ for all $j > i$.*

Proof. Since $X_j \subseteq X_i$ for all $j > i$, the emptiness of X_i implies the emptiness of all subsequent states. □ □

Ungrammaticality is therefore not a local defect but a global termination of the derivation. Once the option space is exhausted, no subsequent lexical event can restore it.

Analogy. Consider assembling flat-pack furniture following written instructions. Each step commits a component to a position from which it cannot easily be removed: a bolt tightened, a peg inserted, a panel slotted. If a step is performed incorrectly—a panel installed with the wrong face outward—the constraint it introduces may be incompatible with a later required configuration. At that point, the option space for completing the assembly correctly has collapsed to empty and there is no valid

continuation. This is ungrammaticality in the H-LFG sense: not a rule violation checked by an external arbiter, but the exhaustion of all compatible trajectories by accumulated irreversible commitments.

Chapter 12

Paraconsistency and Null Collapse

The preceding stratification of logic under irreversible collapse provides a natural setting in which to reconsider the status of contradiction. In classical logic, the presence of a contradiction trivialises the system: from $P \wedge \neg P$ one may derive any proposition. This principle of explosion depends on the preservation of inferential structure across derivations. In an irreversible system, that preservation fails.

In the Spherepop calculus, evaluation proceeds by contraction of an option space. A contradiction is therefore not a source of arbitrary derivability, but a condition under which all trajectories are eliminated. The result is not triviality, but collapse.

Prerequisites

This chapter assumes Chapters 3 through 6 and, specifically, familiarity with the Priest stratification from Chapter 6. Some exposure to paraconsistent logic is helpful but not required; the relevant notion (that contradictions need not entail arbitrary conclusions) is explained from first principles using the collapse framework. The NCL comparison in the later sections assumes only that the reader understands what a digital logic gate does at a functional level; no circuit design knowledge is needed.

Null Collapse

Definition 0.1 (Null Collapse)**.** Let $E \in \mathcal{E}$ be an expression whose constraints are mutually incompatible. Then evaluation yields

$$f(E) = \varnothing,$$

the empty trace. We refer to this as *null collapse*.

The empty trace is not a truth value in the classical sense. It is the absence of any surviving configuration. Where classical logic propagates contradiction into universal derivability, irreversible collapse terminates the derivation.

Analogy. A circuit breaker does not attempt to reason about a short circuit; it terminates the system. Classical logic in the presence of contradiction behaves like a circuit with no breaker: current flows everywhere, and every proposition is simultaneously derivable. The null collapse of Spherepop is the breaker. When constraints become mutually incompatible, evaluation halts at the empty trace rather than propagating arbitrary conclusions. Termination is a different epistemic state from universal affirmation, and it is the structurally correct response to contradiction in a system that cannot reverse its commitments.

Proposition 0.2 (Non-Explosion Under Collapse)**.** *Let E be an expression with $\tau(E) = \varnothing$. Then for any expression F with $\tau(F) \neq \varnothing$, we have $\tau(E) \not\sim \tau(F)$.*

Proof. The empty trace $\varnothing$ is a distinguished element of $\mathcal{N}$ representing total collapse. Since $\tau(F) \neq \varnothing$, the two traces are unequal and therefore not equivalent. Null collapse does not license derivation of arbitrary non-empty traces. □ □

Null Convention and Paraconsistency

We may interpret null collapse as defining a *null convention logic*: incompatible constraints eliminate all admissible interpretations rather than generating all interpretations.

Theorem 0.3 (Paraconsistency of Null Collapse). *The logic induced by irreversible collapse is paraconsistent: expressions with mutually incompatible constraints evaluate to the empty trace, and the empty trace does not entail any non-empty trace.*

Proof (Sketch). Contradictory constraints eliminate all trajectories, yielding $\tau(E) = \varnothing$. Since $\varnothing$ is not equivalent to any non-empty trace under $\sim$, no further derivation is licensed. Thus contradiction does not entail arbitrary consequence. □
□

This behaviour is structurally analogous to NULL Convention Logic (NCL), a framework for asynchronous digital circuit synthesis. NCL introduces a distinguished NULL value representing the absence of valid data; gates produce output only when completeness conditions are satisfied, otherwise asserting NULL. The NULL state is not merely the absence of computation but a logical value encoding incompleteness.

The structural correspondence is precise. In NCL, the NULL state corresponds to the empty trace in irreversible evaluation. Both represent the absence of a complete and valid configuration. In both systems, the response to incompatibility is not propagation but termination.

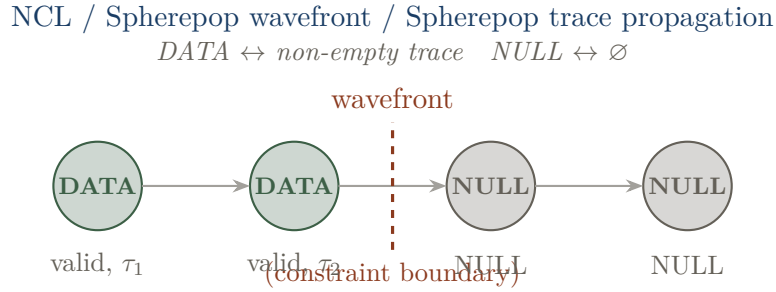


FIGURE 1. **Constraint wavefront (NCL / Spherepop correspondence).** Stages carrying valid data (non-empty trace) are shaded; stages awaiting completion remain NULL. The wavefront advances only when local completeness conditions are satisfied. This is the discrete analogue of the option-space contraction sequence $X_0 \supseteq X_1 \supseteq \cdots$.

Stratification Revisited

Within Priest's framework, null collapse occupies a precise position. At the level of *logica docens*, contradictory formulas remain expressible. At the level of *logica utens*, their evaluation produces a trajectory terminating in failure. At the level of *logica ens*, no fact of validity is generated: the trace is empty.

Thus contradiction is not a truth-bearing object at the level of *logica ens*, but the absence of any stable outcome. Paraconsistency is therefore not an additional feature imposed on the system, but a direct consequence of irreversibility: once evaluation is understood as collapse, contradiction ceases to be explosive and becomes terminal.

Wavefronts, Traces, and Fields

The correspondence with NCL extends beyond the identification of null states to reveal a shared dynamical structure: the propagation of a constraint-satisfying wavefront. In NCL, computation proceeds through the advance of a data wavefront from an all-NULL state. Valid data values propagate through gates only when local completeness conditions are satisfied. The wavefront is entirely internal to the symbolic structure; no external temporal coordination is required.

In the Spherepop calculus, evaluation proceeds by successive contraction of the option space, yielding $X_0 \supseteq X_1 \supseteq \cdots \supseteq X_n$. This sequence may be interpreted as a propagating front of admissibility: the set of surviving configurations advances through the expression as constraints are imposed. The terminal trace corresponds to the stabilisation of this front.

In the RSVP framework, the scalar field $\Phi(x, t)$ encodes the local density of compatible configurations. Its evolution reflects the propagation of constraint through the system, and stable regions correspond to trace-dominated structures where the configuration space has largely collapsed.

All three descriptions share a common form. Each system evolves by enforcing local constraints; global structure emerges as the fixed point of this process. Sequential behaviour corresponds to constraint chains, parallel behaviour to independent reductions, and the NULL state to regions where no compatible configuration survives.

Chapter 13

A Category of Constraint Systems

The identifications developed in the preceding chapter suggest that asynchronous circuits, irreversible evaluation, and field dynamics instantiate a common abstract structure. We now formalise this structure as a category of constraint systems, within which each of these domains arises as a concrete realisation.

Prerequisites

This chapter assumes basic category theory at the level of Mac Lane's *Categories for the Working Mathematician*, Chapter 1: objects, morphisms, functors, and natural transformations. Readers unfamiliar with category theory should read Appendix A first, which introduces the category **Hist** and the collapse functor F in a more elementary style. The collapse and constraint-system framework from Chapters 3 and 4 is assumed throughout.

Constraint Systems

Definition 0.1 (Constraint System)**.** A *constraint system* is a triple $\mathcal{C} = (X, \mathcal{R}, \Phi)$ where X is a state space equipped with a partial order $\preceq$, $\mathcal{R}$ is a family of local compatibility relations on X, and $\Phi : X \to X$ is a monotone contraction operator satisfying $\Phi(x) \preceq x$ for all $x \in X$.

Iteration of Φ generates a descending chain $x \succeq \Phi(x) \succeq \Phi^2(x) \succeq \cdots$ which stabilises at a fixed point. The operator Φ encodes the elimination of incompatible configurations; the partial order records the direction of this elimination.

A constraint system is *irreversible* if and only if Φ admits no right inverse: equivalently, if distinct states may be mapped to the same fixed point, in which case the preimage information is irrecoverably lost.

The Category Constr

Definition 0.2. Let **Constr** be the category whose objects are constraint systems and whose morphisms $F : (X, \mathcal{R}, \Phi) \to (Y, \mathcal{S}, \Psi)$ are functions $F : X \to Y$ such that $F(\Phi(x)) = \Psi(F(x))$ for all $x \in X$, and F preserves compatibility relations in the sense that $x \mathrel{\mathcal{R}} x'$ implies $F(x) \mathrel{\mathcal{S}} F(x')$.

Morphisms commute with collapse: they carry trajectories to trajectories and preserve the structure of constraint propagation. The following is immediate.

Proposition 0.3 (Fixed Points Preserved)**.** *If $\Phi(x^*) = x^*$, then $F(x^*)$ is a fixed point of Ψ.*

Proof. $\Psi(F(x^*)) = F(\Phi(x^*)) = F(x^*)$. □ □

Three Functorial Realisations

Spherepop evaluation, NULL Convention Logic, and RSVP field dynamics each define objects of **Constr**.

In the Spherepop calculus, the state space is $\mathcal{E}$ with refinement of option spaces as the partial order. The operator Φ_{SP} applies one reduction step. The evaluation map f is the limit of iterated Φ_{SP}, and fixed points are precisely the elements of $\mathcal{N}$.

In NULL Convention Logic, the state space consists of all wire assignments taking values in $\{0, 1, \mathrm{NULL}\}$. The operator Φ_{NCL} enforces completeness conditions across all gates. Fixed points are stable data configurations; the NULL state is the least element under the partial order.

In the RSVP framework, the state space consists of field configurations $(\Phi, \mathbf{v}, S)$ over a spatial manifold. The operator Φ_{RSVP} advances the field dynamics by one step. Fixed points are stable field configurations; regions of maximal entropy correspond to null states.

Definition 0.4 (Functorial Realisations)**.** There exist functors

$$F_{\mathrm{SP}} : \mathbf{Expr} \to \mathbf{Constr}, \quad F_{\mathrm{NCL}} : \mathbf{Circ} \to \mathbf{Constr}, \quad F_{\mathrm{RSVP}} : \mathbf{Field} \to \mathbf{Constr}$$

assigning to each system its underlying constraint structure, carrying expressions, circuits, and fields to the corresponding monotone contraction systems.

Universality

Theorem 0.5 (Universality of Constraint Propagation). *Every system admitting a local compatibility structure and a monotone elimination operator embeds into* **Constr**.

Proof (Sketch). Given such a system, define Φ as the operator eliminating incompatible states. Monotonicity follows from the definition of elimination. This yields an object of **Constr**, and the embedding is given by the identity on states. □

□

Identity as Fixed Point

Within **Constr**, identity is characterised by fixed points.

Proposition 0.6 (Identity as Invariance)**.** *Identity in a constraint system is equivalent to invariance under collapse: x is identical to itself if and only if $\Phi(x) = x$.*

The notion of identity in Spherepop (the trace), the stable signal configuration in NCL, and the equilibrium state in RSVP all coincide as fixed points of their respective contraction operators, and therefore as objects in **Constr**. The category makes this coincidence precise.

Chapter 14

Variational Principles and Cartesian Closure

Prerequisites

This chapter assumes Chapter 14 and a working familiarity with the concept of a functional (a real-valued function on a space of configurations). Readers with a background in variational calculus or optimisation theory will find the framework immediately familiar. The cartesian closed category material in Section 15.3 assumes knowledge of products and exponentials in a category; readers new to these ideas should consult Appendix A. The connection to the Curry–Howard correspondence is implicit and is not developed formally here.

Energy as Incompatibility

The categorical formulation of constraint systems establishes that evolution proceeds through monotone elimination. We now refine this structure by introducing a variational principle. The central claim is that the collapse operator is not merely monotone, but directed by a functional whose minimisation encodes compatibility.

Definition 0.1 (Constraint Functional)**.** Let $(X, \mathcal{R}, \Phi)$ be a constraint system. A *constraint functional* is a map $\mathcal{E} : X \to \mathbb{R}_{\geq 0}$ such that $\mathcal{E}(x)$ measures the degree of incompatibility in configuration x, and $\mathcal{E}(x^*) = 0$ if and only if x^* is a fixed point of Φ.

Proposition 0.2 (Energy Decrease)**.** *The collapse operator Φ is energy-decreasing: $\mathcal{E}(\Phi(x)) \leq \mathcal{E}(x)$.*

PROOF. Φ eliminates incompatible components. The measure of incompatibility cannot increase under elimination. □ □

This allows the evolution of a constraint system to be interpreted as a discrete descent along $\mathcal{E}$. In the Spherepop calculus, the constraint functional measures unresolved constraints and incompatible commitments; each reduction step reduces this measure. In RSVP, the functional measures structural tension or entropy gradient; the field dynamics relax this tension over time.

Variational Formulation of Identity

Within the variational framework, identity admits a refined characterisation.

Proposition 0.3 (Identity as Minimal Energy)**.** *Identity corresponds to global minima of the constraint functional. In Spherepop, this is the normal-form trace; in NCL, it is a stable data configuration; in RSVP, it is a coherent field structure.*

Thus the general claim of the monograph—that identity is what survives collapse—acquires a variational form: identity is the configuration of minimal incompatibility reachable from the initial state under irreversible descent.

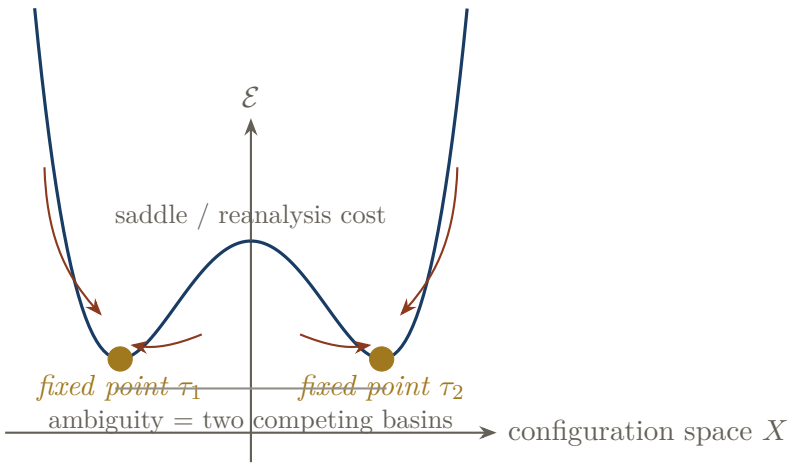


FIGURE 1. **Constraint functional $\mathcal{E}$ as incompatibility landscape.** Evaluation is entropic descent; fixed points are global or local minima. Two basins correspond to ambiguity; the saddle represents the cost of reanalysis (garden-path effect).

Currying and Cartesian Closure via Constraint Propagation

The behavior of NULL Convention Logic and irreversible evaluation suggests that higher-order structure may be realized without explicit functional abstraction. The persistence of partially resolved states provides a direct realization of currying.

In NCL, when a subset of inputs becomes valid while others remain NULL, the circuit transitions to a partially resolved configuration. This intermediate state is stable and encodes the effect of the supplied inputs on all future behavior. It therefore plays the role of a function awaiting further arguments.

In the Spherepop calculus, fixing a partial input $a \in A$ restricts the option space X to a subspace X_a of configurations compatible with a. This residual system is stable under further evolution and represents all admissible completions.

Definition 0.4 (Residual System)**.** Given a constraint system (X, Φ) and an input a, the *residual system* is the restricted system $(X_a, \Phi|_{X_a})$, where X_a consists of all configurations compatible with a.

Proposition 0.5 (Currying via Residualisation)**.** *For each input a, the residual system X_a determines a map on the remaining input domain. This assignment defines a curried morphism $\tilde{f}(a) = f(a, \cdot)$.*

Proof. Fixing a contracts X to X_a, eliminating all configurations incompatible with a. The remaining system evolves under further collapse, yielding a unique output for each admissible remaining input. Thus X_a defines a function on the residual domain. □ □

Higher-order structure therefore emerges not from symbolic encoding but from the geometry of constraint propagation. Currying is not added; it is forced by irreversibility. Each committed input produces a definitive residual object that serves as the partial application, because eliminated configurations cannot be recovered.

Theorem 0.6 (Cartesian Closure of Constraint Systems). *Constraint systems admitting stable residual subspaces and deterministic collapse form a cartesian closed category.*

Proof (Sketch). Products are joint constraint spaces. Exponentials are residual systems obtained through partial collapse. Evaluation is the continuation of collapse within a conditioned space. The universal property of exponentials follows from the correspondence between joint and sequential conditioning. □ □

Exponentials as Conditional Energy Landscapes

Introducing the constraint functional $\mathcal{E}$ gives the cartesian closed structure a variational interpretation.

Definition 0.7 (Conditional Energy Object)**.** Given a global functional $\mathcal{E} : A \times B \times C \to \mathbb{R}_{\geq 0}$, the *conditional energy object* determined by $a \in A$ is the restricted

functional $\mathcal{E}_a(b,c) = \mathcal{E}(a,b,c)$. The exponential $B \Rightarrow C$ is the collection of such conditional functionals indexed by A.

Proposition 0.8 (Currying as Conditioning)**.** *Currying corresponds to conditioning the global energy functional on a subset of inputs:* $\tilde{f}(a) = \arg\min_c \mathcal{E}_a(b,c)$ *for each* b.

THEOREM 0.9 (Conditional Variational Closure). *Constraint systems equipped with a compatible energy functional form a cartesian closed structure in which exponentials correspond to conditional energy landscapes and evaluation corresponds to conditional minimisation.*

PROOF (SKETCH). Products are joint configuration spaces; conditioning induces residual systems yielding exponentials; evaluation is minimisation within conditioned spaces. The universal property follows from equivalence between joint and sequential conditioning. □ □

The consequence is that functions are not primitive objects but conditioned energy landscapes: currying is the act of fixing variables and restricting the system to a reduced domain; evaluation is descent to a minimum within that domain. Logic, computation, and physical evolution are thus unified as instances of the same principle: conditioning followed by minimisation under irreversible descent.

Part 5

Empirical Consequences and Conclusion

Chapter 15

Empirical Signatures of Irreversible Structure

The preceding chapters established a formal framework in which evaluation is irreversible, identity is grounded in trace, and the distinction between program, execution, and description collapses. These results were derived at the level of abstract calculus. The present chapter considers their empirical implications.

The aim is not to provide a complete experimental programme, but to identify observable structures that would distinguish an irreversible model from one based on reversible or lossless transformation. The central claim is that irreversibility leaves characteristic signatures in the distribution, stability, and interpretation of observable states.

Prerequisites

This chapter is the most speculative in the monograph and requires a broader background than the preceding chapters. Readers should be familiar with the collapse framework established in Part II and the variational formulation of Chapter 15. Some acquaintance with dynamical systems, attractor theory, or the physics of entropy is helpful for the field-representation sections. The empirical predictions are stated as propositions rather than theorems; they are intended as research directions rather than established results. No experimental data is analysed.

Asymmetry in Reconstruction

A direct consequence of the absence of a right inverse is that reconstruction from observed outcomes is underdetermined. If an observable state corresponds to a trace $\tau(E)$, then the set of possible antecedent structures is given by the fiber $f^{-1}(\tau(E))$.

Empirically, this implies that systems governed by irreversible collapse will exhibit reconstruction asymmetry. Forward evolution from an initial structure will be well-defined and convergent, while inverse reconstruction from an observed state will admit multiple incompatible hypotheses. This asymmetry may be detected by comparing predictive and retrodictive performance. In a reversible system the two are expected to align. In an irreversible system, forward prediction remains stable while backward inference diverges across a broad equivalence class.

Entropy as Residual Constraint

The process of collapse may be interpreted as a progressive restriction of an underlying option space. Let X denote the set of admissible configurations compatible with a given state. Each irreversible event reduces X by eliminating incompatible trajectories.

We may associate to X a functional $\mathcal{F}(X)$ measuring the residual degrees of freedom consistent with the current trace. Under irreversible evolution, this functional is non-increasing along trajectories of evaluation. Empirically, this suggests that systems exhibiting irreversible collapse will display monotonic contraction of accessible configuration space. Observable states will appear increasingly constrained, even when no explicit conservation law enforces such behaviour. Importantly, this contraction is not necessarily accompanied by uniform dispersion. The reduction of X may proceed through highly structured eliminations, producing stable but non-uniform terminal states.

Field Representation and Trace Stability

To model the spatial and temporal distribution of traces, we introduce a field representation. Let $\Phi(x, t)$ denote a scalar field encoding the local density of compatible configurations at position x and time t. Regions of high density correspond to areas where multiple trajectories remain admissible, while regions of low density indicate near-terminal collapse.

The evolution of this field reflects the propagation of constraint through the system. Local reductions induce changes in Φ that may extend beyond their immediate domain, as eliminated configurations restrict neighbouring possibilities. Empirically, this leads to the prediction of coherent regions in which the field stabilises. These regions correspond to trace-dominated structures, where the local configuration space has largely collapsed and the system exhibits persistent behaviour.

Persistence Without Reconstruction

A central feature of irreversible systems is that stable structures persist without requiring the preservation of their generative history. The trace does not encode the

sequence of events that produced it, yet it remains robust under further evolution. Empirically, this implies that systems may exhibit long-lived structures whose origin cannot be uniquely reconstructed. Attempts to infer their history will yield multiple incompatible explanations, all consistent with the observed state.

Such persistence differs from stability in reversible systems, where invariants are typically tied to conserved quantities or symmetries. In the present framework, stability arises from the exhaustion of alternatives rather than the preservation of structure.

Ambiguity as Residual Branching

Not all trajectories collapse to a single trace. In some cases, the reduction process terminates in a state that retains multiple compatible configurations, corresponding to a non-singleton equivalence class at the level of the trace. Empirically, such states manifest as ambiguity. The system supports multiple interpretations that cannot be resolved without additional constraints. This is not due to incomplete information, but to the persistence of branching within the reduced configuration space.

Irreversibility and Observational Limits

The loss of preimage imposes intrinsic limits on observation. No measurement of a trace can recover the full structure of its antecedent. Any attempt to do so must rely on external assumptions or probabilistic reconstruction. This suggests that certain forms of uncertainty are not epistemic but structural. They arise not from a lack of information about an underlying reality, but from the fact that the relevant information no longer exists within the system.

Empirical models that assume invertibility may therefore systematically overestimate the recoverability of past states. Incorporating irreversibility requires treating the preimage as fundamentally inaccessible.

Synthesis

The empirical consequences of irreversible collapse may be characterised as follows. Systems governed by such dynamics exhibit asymmetry between prediction and reconstruction, monotonic contraction of configuration space, persistence of stable structures without recoverable histories, and the presence of ambiguity as residual branching. These features do not depend on the specific details of the reduction rules, but follow from the structural properties of the evaluation map. The absence of a right inverse, the projection onto equivalence classes, and the identification of identity with trace together determine the observable behaviour.

Chapter 16

Conclusion: Self-Reference Without Inverses

A system that has collapsed to its normal form no longer carries the history that produced it. What remains is not a record of the path, but the invariant that survived. Identity is this invariant.

The foregoing analysis has progressively removed a single, pervasive assumption: that computation, logic, and meaning are fundamentally reversible. Once this assumption is abandoned, the conceptual architecture of identity undergoes a corresponding transformation. What remains is not a theory of representation, but a theory of survival under collapse.

The Spherepop calculus provides a minimal formal setting in which this transformation can be made explicit. By treating evaluation as an irreversible contraction of possibility, it forces a distinction between what is constructed, what is lost, and what persists. The central claim of this work is that identity resides exclusively in the latter.

Prerequisites

The conclusion presupposes the entire preceding development. It is intended to be read after all other chapters, at which point no new definitions or theorems are introduced. Readers who have worked through the monograph in order will find this chapter acts as a compression of the argument rather than an extension of it.

The Trace as Logica Ens

Through the stratification of Chapter 6, the distinction between *logica docens*, *logica utens*, and *logica ens* becomes structurally grounded. The formal system $(\mathcal{E}, \mathcal{R})$ constitutes the logic we teach: a theory of admissible transformations. The reduction trajectories constitute the logic we use: a practice of sequential commitment under constraint. The image $f(\mathcal{E}) \subseteq \mathcal{N}$ constitutes logic itself: the set of invariant traces that survive collapse.

In reversible systems these strata are implicitly conflated, because the preimage of a result can in principle be reconstructed. In an irreversible system, this conflation is no longer possible. The only stable object is the trace $\tau(E)$. Accordingly, *logica ens* is not a hidden semantic layer but the terminal fixed point of reduction. Identity is not a property of an initial configuration, but an invariant of a completed trajectory.

The Redefinition of Self-Reference

The classical notion of a quine presupposes that syntactic structure can be preserved or encoded across evaluation. This presupposition fails under irreversible collapse (Proposition 0.3).

In Spherepop, a quine is redefined (Definition 0.4) as an expression Q satisfying $f(Q) \sim Q$. The requirement is not syntactic reproduction, but trace invariance. This eliminates the need for representational duplication. A Spherepop quine does not reproduce its source; it coincides with its result. The destruction of preimage is not an obstacle but a condition of possibility. Self-reference is achieved precisely because nothing remains except the invariant trace.

Thus, the quine becomes a limiting case of identity: an expression whose evaluation yields no distinction between what it is, what it does, and what it says.

Language as Irreversible Computation

The derivation of H-LFG (Chapters 10 and 11) demonstrates that this structure is not confined to abstract computation. An utterance is a sequence of lexical events that irreversibly contracts a grammatical option space. The c-structure records the historical organisation of these events as nested scopes; the f-structure is the normal form of the resulting compatibility network.

Grammaticality corresponds to the existence of a non-empty terminal trace; ungrammaticality corresponds to total collapse (Proposition 0.5). Meaning is not an additional representational layer; it is the residual structure that survives the elimination of incompatible trajectories. The sentence does not encode its meaning; it is the trace of the process that made it definite.

The Irreducibility of the Trace

The trace is not a derived or secondary object. It is not a summary of a prior structure, nor a representation of an underlying process. It is the only object that remains once the process of elimination has been completed.

Any attempt to treat the trace as an encoding of its preimage reintroduces the assumption of reversibility. The absence of a right inverse shows that no such encoding exists. The trace does not compress the preimage; it replaces it. There is no deeper level at which the identity of the system may be recovered. The trace is not the surface of a hidden structure. It is the terminal form of the system itself.

Identity as Terminal Trace

THEOREM 0.1 (Identity as Trace). *Let $f : \mathcal{E} \to \mathcal{N}$ be an irreversible evaluation map with no right inverse. Then no expression has a recoverable preimage from its trace alone; the only stable notion of identity is the equivalence class $[E]_\sim$ determined by the trace; and at normal-form fixed points, program, execution, and description coincide.*

PROOF. The first claim follows from Proposition 0.2. The second follows from Definition 0.2 together with Lemma 0.5, which shows that the coarsest observable criterion is the trace equivalence $\sim$. The third follows from Theorem 0.1. □ □

Temporal Finality

Irreversibility introduces a fundamental directionality into the system. Reduction proceeds from greater structural multiplicity to lesser structural multiplicity, and this process cannot be reversed. This directionality is not imposed externally but arises from the internal logic of collapse. Each step reduces the available options, and no step restores them.

The trace marks the endpoint of this sequence. It is not a state that may be revisited or revised, but a terminal condition. Temporal finality is not an additional assumption, but a consequence of the structure of evaluation.

Identity Without Origin

The relocation of identity from preimage to trace has a final implication. Identity no longer depends on origin. Two expressions with distinct histories may share the same identity if they collapse to the same trace. This does not erase history, but it deprives history of its role as a criterion of identity. The event word records the path taken, but identity is determined by what survives.

Identity is forward-defined rather than backward-defined. It is not what something was, but what it has become after the elimination of all incompatible alternatives.

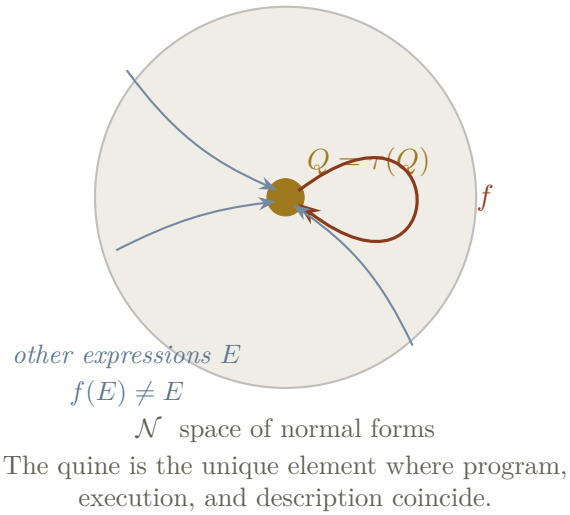
Final Implication

The consequence is not merely technical. It is ontological.

In a finite system, possibilities are progressively eliminated until only a stable configuration remains. That configuration is not a snapshot of an underlying essence; it is the residue of a history that can no longer be undone. To ask what something is is therefore to ask what remains after all incompatible alternatives have been excluded.

Self-reference without inverses is the only form of self-reference available under these conditions. It does not consist in pointing back to an origin, but in coinciding with the endpoint of a process.

Spherepop makes this visible in its simplest form. The bubbles of possibility are transient; their popping is irreversible. What persists is the structure of that collapse. And it is this structure—this trace—that constitutes identity.



The quine is the unique element where program, execution, and description coincide.

FIGURE 1. **Self-reference as fixed-point identity.** The Spherepop quine Q satisfies $f(Q) \sim Q$: collapse returns Q to itself. All other expressions are carried to distinct traces; only at the fixed point do program, execution, and description collapse into one object.

Identity is not preserved through evaluation; it is produced by collapse.

Chapter 17

Summary and Everyday Analogies

Prerequisites

This chapter assumes the full preceding monograph. It is intended to be read after all other chapters. No new definitions or theorems are introduced. The purpose is twofold: to provide a compressed restatement of the central argument for readers who wish to survey the work before engaging it in detail, and to offer a series of everyday analogies that make the formal claims concrete without diluting them.

The Central Argument in Brief

This monograph has developed a formal theory of identity and self-reference for irreversible computational systems, using the Spherepop calculus as its central model. The guiding insight is that evaluation, when properly understood as an irreversible collapse rather than a reversible transformation, forces a fundamental revision of how identity, meaning, and self-reference are to be conceived.

In standard computational frameworks, evaluation is treated as a transformation that preserves informational recoverability. By contrast, the Spherepop calculus formalises evaluation as a many-to-one map $f : \mathcal{E} \to \mathcal{N}$, where distinct expressions collapse to the same normal form. This map admits no right inverse. The loss of preimage is therefore not incidental but structural. Reduction proceeds through irreversible operations—popping, binding, and refusal—that eliminate syntactic structure and contract the space of possibilities. What remains after evaluation is not a transformed version of the original expression, but a residue: a trace.

This irreversibility necessitates a relocation of identity. Since preimages are destroyed, identity cannot be grounded in initial syntax or generative history. Instead, identity is defined by trace equivalence. Two expressions are identical if and only if they

collapse to the same normal form. The trace, as the invariant of evaluation, is the sole bearer of identity. All prior structure is contingent and expendable.

This shift induces a divergence in the classical tripartite distinction articulated by Graham Priest. The logic we teach, the logic we use, and the logic that is the case no longer coincide. Within the present framework, the formal system $(\mathcal{E}, \mathcal{R})$ constitutes the *logica docens*, the reduction trajectories instantiated by irreversible evaluation constitute the *logica utens*, and the image $f(\mathcal{E}) \subseteq \mathcal{N}$—the set of terminal traces—constitutes the *logica ens*. Under irreversibility, these strata separate. The theory describes possible reductions, practice executes particular histories, and the facts of identity are determined only at the level of the terminal trace.

Self-reference must therefore be reformulated. Classical quines rely on the preservation or reconstruction of syntactic structure, implicitly assuming reversibility. In an irreversible system, such reconstruction is impossible. The Spherepop analogue replaces reproduction with coincidence. A quine is defined as an expression Q such that $f(Q) \sim Q$, where equivalence is taken at the level of trace. At such fixed points, the distinctions between program, execution, and description collapse. The object is not a generator of itself but the endpoint of its own evaluation.

This computational framework extends naturally to language. The derivation of a Historical Lexical-Functional Grammar shows that linguistic structure can be understood as the trace of an irreversible process. An utterance is a sequence of lexical events that progressively contracts a grammatical option space. Constituent structure records the historical nesting of commitments, while functional structure is the surviving trace of compatible constraints. Grammaticality is defined by the existence of a non-empty terminal trace; ungrammaticality corresponds to total collapse, where no compatible configuration survives; ambiguity arises when multiple traces remain viable at the terminal stage.

These results admit a unifying abstraction. Spherepop, NULL Convention Logic, and field-based constraint dynamics are all instances of systems governed by monotone contraction operators on spaces of possibilities. Evaluation is a descent process that eliminates incompatibility. Identity corresponds to a minimal configuration under a variational principle, where energy is interpreted as unresolved contradiction.

The sheaf-theoretic formulation provides a precise mathematical articulation. Histories form a presheaf over the poset of partial traces, ordered by refinement. Grammaticality corresponds to the existence of a global section—a consistent gluing of local histories into a single trajectory. Ambiguity corresponds to the existence of multiple global sections; ungrammaticality corresponds to the failure of gluing, yielding an empty terminal trace. Meaning is not an additional layer of representation but the existence and structure of sections over the base space.

The cumulative result is a redefinition of identity for finite systems. Identity is not preserved through evaluation; it is produced by collapse. The trace is the terminal fact, and no deeper layer remains accessible once irreversibility has taken place. Self-reference, under these conditions, is not the capacity to reconstruct one's origin but the condition of coinciding with the endpoint of one's own transformation.

The Denouement Analogy

The process of evaluation is structurally analogous to the denouement of a narrative. A system begins in a state of multiplicity, supporting many possible trajectories. As evaluation proceeds, incompatible possibilities are irreversibly eliminated. The terminal trace is the state that remains once all alternatives have been discharged.

This is not a loose resemblance. In classical narrative theory, the denouement is the final unravelling of the plot—the state of the story after all conflicts have been resolved and all alternative possibilities for the characters' fates have been closed off. In the Spherepop calculus, the evaluation process performs exactly this operation. Each reduction step commits to a specific direction and eliminates branches that are no longer compatible with the accumulated history of commitments.

The terminal trace is the structural equivalent of the denouement. It is the configuration that remains only after the tension of the computation—the unresolved or unevaluated nested bubbles—has been fully discharged. The identity of the system is not found in its initial source or opening state, but in the specific, irreversible configuration that survives to the end. Just as the conclusion of a narrative resolves ambiguity by fixing a single history, the trace represents the final configuration in which no further reinterpretation is possible within the calculus.

The analogy also illuminates the sheaf-theoretic picture. A story with a genuine ambiguity—one that the text sustains to the end without resolution—is formally equivalent to a grammatical utterance with multiple surviving traces. The story supports more than one global section over its narrative base. The denouement, when it arrives, selects among these sections, or in rare cases leaves them genuinely unresolved. A story whose plot contradicts itself catastrophically—whose events become mutually incompatible before a coherent ending can be reached—corresponds to an ungrammatical utterance: gluing fails, the terminal trace is empty, and no final identity is produced.

Adaptation as Premature Collapse

The distinction between preimage and trace is vividly illustrated by the adaptation of a written narrative into film. A literary text permits prolonged indeterminacy in

the attributes of its characters. The reader constructs an implicit bundle of compatible interpretations and carries it forward through the unfolding of the narrative. A character's precise appearance, voice, and physical presence may remain undetermined—or deliberately indeterminate—across hundreds of pages. This corresponds to a high-entropy preimage: many expressions, many compatible trajectories, all collapsing eventually to the same narrative identity.

A film enforces early collapse. The moment a character appears on screen, the casting instantiates a specific physical and temporal realisation. A particular actor's body, face, and voice select one trajectory from the space of possibilities. The resulting representation is not a translation of the original but its trace under a many-to-one projection. Formally, the adaptation acts as an evaluation map from the space of textual possibilities to the space of cinematic instantiations, and this map is many-to-one.

Once the film fixes these properties, the original indeterminacy is irrecoverable. No viewer can reconstruct the full space of interpretations that existed in the text from the film alone. This is the core claim of Chapter 4: the preimage is destroyed; only the trace remains.

Techniques such as non-linear editing and flashbacks do not restore the lost preimage. Rather, they reparameterise the order of collapse, presenting local segments of the trajectory out of sequence while preserving the same global resolution. A flashback does not reopen the possibilities; it re-stages the commitments, in a different order, that were already made. Ambiguity is not preserved by non-linear structure—it is resolved under a different temporal ordering of constraints.

This demonstrates that identity in narrative, as in computation, is not preserved across representation. It is produced by the irreversible elimination of alternatives. The character in the film is not the same object as the character in the book; the film character is the trace of the book character under a specific collapse schedule.

Cooking as Irreversible Constraint Resolution

A simpler and equally precise analogy is the preparation of a dish from a recipe. A recipe specifies a space of admissible ingredient combinations and procedural sequences, most of which are intersubstitutable within limits. At the outset, the cook operates in a high-dimensional option space: which tomatoes, how ripe, how many, combined in what order with which other ingredients.

Each step of preparation applies an irreversible constraint. Chopping eliminates the original integrity of the vegetable; it cannot be restored. Heat transforms structure at the molecular level; the transformation is one-directional. Salt dissolved into liquid cannot be unsalted. Each operation pops a bubble, in Spherepop terms, and the structure it organised is gone.

The finished dish is the normal form of the recipe's option space under the cook's specific sequence of reductions. Two cooks following the same recipe may produce dishes that are functionally equivalent—they serve the same purpose, carry the same nutritional content, and taste similar—yet differ in every microscopic detail. These two dishes are co-identical in the sense of Chapter 5: they share the same trace under the relevant equivalence relation, even though their preimages (the specific sequences of preparation) differ.

Ungrammaticality in this domain is a dish that cannot be completed. If the constraints imposed by a recipe become mutually incompatible—too much salt renders further reduction pointless; a protein coagulates beyond the temperature window in which a sauce can form—the option space collapses to empty and no valid final configuration is reachable. The dish is ungrammatical: the gluing fails.

The recipe itself is logica docens: the formal specification. The cook's practice is logica utens: the particular sequence of reductions actually performed. The finished dish is logica ens: the terminal fact, the trace that constitutes the identity of the meal. The distinction between these three levels, forced by irreversibility, maps directly onto Priest's stratification in Chapter 6.

Memory as Trace Without Preimage

Human memory provides a particularly compelling analogy because it exhibits the core asymmetry of collapse in a domain everyone has directly experienced. Memory does not preserve a recording of an event; it preserves a trace. The event, in its full original structure—the precise sensory detail, the emotional context, the order of subsidiary perceptions—is not recoverable from the memory. What is recoverable is an invariant that survived the encoding process, which is itself irreversible.

This is why two people who attended the same event carry memories that differ in structural detail while agreeing on the identity of what occurred. Both memories are elements of the same preimage fiber: they collapse to the same trace under the encoding relation. The trace is the shared normal form, and it carries no information about which specific experiential trajectory produced it.

The impossibility of recovering the exact experiential preimage from a memory trace is not a limitation of brain architecture; it is the structural property of any irreversible encoding. Memory is a collapse map, and its fibers are large. The subjective sense that one's memory is "of" the original event, rather than of its trace, is precisely the phenomenological equivalent of mistaking a quotient for its preimage.

Distortion and false memory arise when the trace is treated as if it were a right inverse of the original encoding—as if it determined, rather than merely surviving from, the event. The absence of such an inverse is the content of Proposition 4.1, and it applies to human memory as directly as it applies to Spherepop reduction.

Law and the Accumulation of Irreversible Commitment

Legal systems provide a domain in which irreversibility is explicitly formalised. A judgment, once entered, is not reversed by recourse to the original facts; it is revised, if at all, only by a further irreversible act: an appeal, a new verdict, a legislative change. Each legal decision is a reduction step that eliminates a space of alternative resolutions and commits the system to a specific configuration.

The common-law doctrine of precedent is the formal expression of the append-only dynamics described in Chapter 3. A precedent adds an entry to the legal ledger that cannot be removed; subsequent decisions must be consistent with it or must explicitly distinguish themselves from it. The option space of future judgments is constrained by the accumulated trace of past decisions. This is constraint monotonicity in the sense of Definition 10.3.

Legal ambiguity—cases that are genuinely underdetermined by statute and precedent—corresponds to multiple surviving traces. The option space has not been collapsed to a single configuration by the accumulated constraints, and the system supports more than one valid global section. Judgment in such cases is an act of selection among sections: the judge performs a disambiguation that the law, taken alone, could not.

Ungrammaticality in law is the condition of a legal instrument so internally contradictory that no consistent interpretation survives. Such instruments are voided: the legal equivalent of the empty trace. No judgment can be entered that satisfies all the constraints simultaneously; gluing fails.

The statute itself is logica docens. The practice of jurisprudence is logica utens. The body of settled law—the accumulated trace of irreversible decisions—is logica ens. The Priest stratification is not a philosophical abstraction imposed on the legal system; it is the natural description of how legal systems actually work.

Musical Performance as Ordered Collapse

A musical score specifies a space of admissible performances. Tempo markings, dynamic indications, and articulation marks each impose constraints on the option space of possible realisations, but they do not fully determine it. The score is logica docens: a specification with many compatible models.

Each act of performance is an irreversible realisation. A breath, an attack, a fingering, the exact moment of a crescendo—these are reductions that commit the performance to a specific trajectory through the option space. Once a phrase has been played, it cannot be unplayed; the acoustic trace is fixed. The performance proceeds as an append-only history: a sequence of commitments from which no retreat is possible.

Two performances of the same score are co-identical in the relevant sense if they are functionally equivalent under the audience's perceptual evaluation map. They are

elements of the same fiber—different histories that produce the same trace. What makes a performance "the same piece" is not that it reproduces the score exactly (an impossibility, and not even desirable) but that its trace is equivalent to the traces produced by other performances within the same equivalence class.

Improvisation is the limiting case. In improvised music, there is no score to constitute a logica docens; the performer operates directly at the level of logica utens, with the accumulated sonic trace as the only constraint on future choices. The piece, in such a setting, has no preimage: it is nothing other than its own trace, generated in real time. This is the musical analogue of the Spherepop quine: a system whose identity is its own evaluation, without reference to any prior specification.

Closing Statement

These analogies are not illustrations added to make the theory accessible. They are instances of the same formal structure in different domains. Narrative denouement, cinematic adaptation, the preparation of a dish, the formation of memory, the accumulation of legal precedent, and the realisation of musical performance all exhibit irreversible contraction of an option space, trace-based identity, and the three-level stratification that Priest names but irreversibility forces. The theory is not abstract in the sense of being disconnected from experience. It is abstract in the correct sense: it captures the structure that these diverse phenomena share.

Identity, in all of these domains, is not preserved through evaluation. It is produced by collapse. The trace is what remains after all incompatible alternatives have been irreversibly excluded. There is no deeper layer. History that has completed its contraction is identity.

Chapter A

A Categorical Formulation of Trace and Collapse

The Spherepop framework can be recast in categorical terms as a system of histories modulo irreversible collapse. This formulation makes precise the notion that identity is determined not by preimage structure, but by invariants under evaluation.

The Category of Histories

Definition 0.1 (Category **Hist**)**.** Let **Hist** be the category whose objects are Spherepop expressions $E \in \mathcal{E}$ and whose morphisms $h : E \to E'$ are reduction sequences from E to E'. Composition is concatenation; identities are empty sequences.

Because reduction is irreversible, **Hist** is not a groupoid: morphisms are not, in general, invertible. This non-invertibility is not a defect but the defining structural feature.

The Collapse Functor

Define a functor $F : \mathbf{Hist} \to \mathbf{NF}$, where **NF** is the discrete category on the set of normal forms. The functor maps each expression to its normal form $F(E) = \tau(E) = f(E)$, and maps every morphism (reduction sequence) to the identity on its codomain.

The functor F forgets all path information except what survives in the terminal trace. It is a quotient functor in the sense that it collapses the internal structure of **Hist** onto a discrete base.

Traces as Terminal Objects

For each trace $\tau \in \mathcal{N}$, the object τ is the terminal object in the full subcategory of **Hist** consisting of all expressions reachable to τ: there exists a unique morphism $h_E : E \to \tau$ for each E with $F(E) = \tau$.

Proposition 0.2 (Trace Uniqueness)**.** *For each $E \in \mathcal{E}$, the trace $\tau(E)$ is the terminal object in the full subcategory of* **Hist** *generated by E.*

Proof. By confluence and termination (Assumption 0.4), every reduction sequence from E terminates at $\tau(E)$, and this terminal object is unique. Any two sequences from E therefore yield the same endpoint, so there is exactly one object reachable as a terminal in that subcategory. □ □

Idempotence and Projection

The evaluation operator induces an idempotent endofunctor on **Hist**: $F \circ F = F$. This reflects the fixed-point property $f(f(E)) = f(E)$ established in Definition 0.1. Thus F acts as a projection from the category of histories onto the subcategory of traces.

Identity as a Universal Property

Theorem 0.3 (Identity as Terminal Trace). *Two expressions $E, E' \in \mathcal{E}$ are co-identical (in the sense of Definition 0.2) if and only if $F(E) = F(E')$, i.e., they are mapped to the same object by the collapse functor F.*

Proof. By definition, $E \sim E'$ iff $\tau(E) = \tau(E')$ iff $F(E) = F(E')$ in **NF**. □ □

This formulation makes explicit that identity is a property of the image of F, not of the internal structure of objects in **Hist**.

The Tripartite Correspondence as a Commuting Triangle

The tripartite correspondence of Theorem 0.1 can be elevated to a commutative diagram. Let **Traj** be the category of reduction trajectories, with ∂ : **Hist** $\to$ **Traj** recording event words. Then:

$$\begin{tikzcd} \mathbf{Hist} \arrow[r, "\partial"] \arrow[dr, "F"'] & \mathbf{Traj} \arrow[d, "\mathrm{NF}"] \\ & \mathbf{NF} \end{tikzcd}$$

logica docens corresponds to the objects and morphisms of **Hist**; *logica utens* to the trajectories in **Traj**; and *logica ens* to the discrete category **NF**.

Chapter B

Fibered Structure of Histories

Prerequisites

This appendix assumes Appendix A and familiarity with the basic notions of category theory introduced there. No prior knowledge of sheaf theory is required; the construction is developed from the fibered picture and the poset structure follows naturally. Readers with a background in algebraic topology will recognise the Alexandrov topology as the standard construction on a poset.

Fibers as Spaces of Compatible Histories

The collapse functor $F : \mathbf{Hist} \to \mathbf{NF}$ defines, for each trace $\tau \in \mathcal{N}$, the *fiber category*

$$\mathbf{Hist}_\tau = F^{-1}(\tau),$$

the full subcategory of all expressions and reduction sequences collapsing to τ. Distinct histories collapsing to the same trace are objects in the same fiber. The collapse functor projects every object in a fiber to the same base point and collapses every morphism to the identity there.

Ambiguity arises when a fiber contains multiple inequivalent objects that are not connected by internal morphisms within $\mathbf{Hist}_\tau$. Two such objects represent distinct derivational histories that share the same terminal identity but cannot be transformed into one another without leaving the fiber.

Sections as Consistent Interpretations

A *section* of F is a functor $s : \mathbf{NF} \to \mathbf{Hist}$ satisfying $F \circ s = \mathrm{id}_{\mathbf{NF}}$. Since $\mathbf{NF}$ is discrete, a section amounts to a choice function: for each trace τ it selects one representative history $s(\tau) \in \mathbf{Hist}_\tau$ satisfying $F(s(\tau)) = \tau$.

Different sections correspond to different canonical interpretations of the same trace-level facts. In the linguistic setting, a section corresponds to a particular parse selected from among all derivations compatible with the terminal f-structure.

Proposition 0.1 (Ambiguity as Multiplicity of Sections)**.** *An utterance u is ambiguous if and only if the fiber $\mathbf{Hist}_{\tau(u)}$ supports more than one inequivalent section.*

Proof. Ambiguity means $|\mathrm{FStr}(u)| > 1$: multiple incompatible functional structures survive collapse. Each surviving structure corresponds to a distinct choice function on $\mathbf{Hist}_{\tau(u)}$, yielding a distinct section. The sections are inequivalent because they select histories that collapse to the same base but differ in their internal structure. □ □

Chapter C

A Sheaf of Histories over the Space of Partial Traces

Prerequisites

This appendix extends Appendix B by equipping the space of traces with a non-trivial topology. The construction converts the fibered picture into a genuine sheaf, making locality and gluing mathematically precise rather than metaphorical. Readers should be comfortable with the notion of a presheaf (a contravariant functor into **Set** or **Cat**) and with the concept of a covering family in a site. The Alexandrov topology on a poset is self-contained and explained below.

The Poset of Partial Traces

Extend $\mathcal{N}$ to a partially ordered set $(\mathcal{N}, \preceq)$ where $\tau_1 \preceq \tau_2$ if and only if τ_1 is a partial trace of τ_2: that is, τ_2 refines τ_1 by incorporating additional irreversible commitments.

Intuitively, τ_1 corresponds to an incompletely resolved structure—an ambiguous or partially parsed state—while τ_2 represents a more definite collapse that has eliminated further alternatives. The refinement relation is transitive (a partial trace of a partial trace is itself a partial trace) and reflexive ($\tau \preceq \tau$), making $(\mathcal{N}, \preceq)$ a genuine partial order.

Definition 0.1 (Alexandrov Topology on $\mathcal{N}$)**.** Equip $(\mathcal{N}, \preceq)$ with the *Alexandrov topology*: a subset $U \subseteq \mathcal{N}$ is open if and only if it is upward-closed,

$$\tau \in U \quad \text{and} \quad \tau \preceq \tau' \;\Rightarrow\; \tau' \in U.$$

Open sets therefore correspond to collections of traces that are closed under further refinement. If a partial trace belongs to an open set, every more-determined trace also

belongs. This reflects the fact that once a trajectory has committed to a configuration, all more definite continuations of that configuration remain in scope.

The Presheaf of Histories

Definition 0.2 (Presheaf of Histories)**.** Define a presheaf

$$\mathcal{F} : \mathbf{NF}^{\mathrm{op}} \to \mathbf{Cat}$$

by assigning to each trace τ its fiber category $\mathcal{F}(\tau) = \mathbf{Hist}_\tau$. For a refinement $\tau_1 \preceq \tau_2$, define the restriction functor

$$\rho_{\tau_2,\tau_1} : \mathbf{Hist}_{\tau_2} \to \mathbf{Hist}_{\tau_1}$$

by truncating histories: a fully resolved trajectory in $\mathbf{Hist}_{\tau_2}$ is restricted to the prefix compatible with the coarser trace τ_1.

The restriction functors are required to satisfy the usual presheaf coherence conditions: $\rho_{\tau,\tau} = \mathrm{id}$ and $\rho_{\tau_3,\tau_1} = \rho_{\tau_2,\tau_1} \circ \rho_{\tau_3,\tau_2}$ for $\tau_1 \preceq \tau_2 \preceq \tau_3$.

Local Compatibility and the Gluing Condition

Let $\{\tau_i\}_{i\in I}$ be a family of partial traces whose least upper bound exists in $(\mathcal{N}, \preceq)$:

$$\tau = \bigvee_{i\in I} \tau_i.$$

A family of histories $\{E_i \in \mathbf{Hist}_{\tau_i}\}_{i\in I}$ is *compatible* if the restrictions agree on all pairwise overlaps:

$$\rho_{\tau_i,\tau_{ij}}(E_i) = \rho_{\tau_j,\tau_{ij}}(E_j), \quad \text{where } \tau_{ij} = \tau_i \wedge \tau_j.$$

Theorem 0.3 (Sheaf of Histories). *The presheaf $\mathcal{F}$ satisfies the sheaf condition: for every compatible family $\{E_i\}_{i\in I}$, there exists a unique global history $E \in \mathbf{Hist}_\tau$ such that $\rho_{\tau,\tau_i}(E) = E_i$ for all $i \in I$.*

Proof (Sketch). The unique global history E is constructed by assembling the compatible partial trajectories into a single reduction sequence over τ. Uniqueness follows from the compatibility conditions: any two global extensions of the same compatible family must agree on all local pieces, and since the local pieces cover the full trace τ, they determine E completely. Existence follows from the termination and confluence of $\mathcal{R}$ (Assumption 0.4): any compatible family of partial trajectories that share an upper bound extends to a complete reduction sequence terminating at τ. □ □

Failure of Gluing and Ungrammaticality

When the sheaf condition fails, no global history can be assembled from locally compatible pieces.

Proposition 0.4 (Ungrammaticality as Obstruction)**.** *An utterance u is ungrammatical if and only if there is no global section of $\mathcal{F}$ over the putative trace of u, equivalently, if the gluing of locally compatible partial histories yields $\varnothing$.*

Proof. A global section assigns to u a consistent history compatible with all local constraints. Ungrammaticality means $\mathrm{FStr}(u) = \varnothing$ (Proposition 0.5): no compatible functional structure exists. This is equivalent to the failure of gluing: the local compatibility conditions imposed by the lexical events cannot be satisfied simultaneously, so no global section exists. □ □

This recovers the result $X_{\text{final}} = \varnothing$ of the main text, now expressed as a topological obstruction: the local data cannot be extended to a coherent global configuration over τ.

Ambiguity as Non-Unique Global Section

Theorem 0.5 (Ambiguity as Multiplicity of Global Sections). *An utterance u is ambiguous if and only if the sheaf $\mathcal{F}$ admits more than one inequivalent global section over the trace of u.*

Proof. Each global section assigns to u a complete history E satisfying all local compatibility conditions. Ambiguity means that $|\mathrm{FStr}(u)| > 1$: multiple incompatible functional structures each survive collapse (**??**). Each surviving structure determines a distinct global section, since distinct functional structures correspond to distinct trajectories through the constraint space. These sections are inequivalent because they select histories that differ as objects in $\mathbf{Hist}_\tau$. □ □

Each section represents a distinct interpretation compatible with the same terminal trace: the same surface sequence of lexical events, the same resulting f-structure identity, but a different path of derivational commitment.

Collapse as Sheaf Projection

The evaluation map $E \mapsto \tau(E)$ can now be understood as the projection from the sheaf $\mathcal{F}$ to its base space $\mathcal{N}$. A chosen global section $s : \mathcal{N} \to \mathbf{Hist}$ selects a representative history for each trace; the evaluation map then projects this history to its base point, discarding all local structure above it.

Formally, the map

$$f = \pi \circ s : \mathbf{NF} \to \mathbf{NF}$$

is the identity (by definition of section). All higher-order structure—local compatibility, branching histories, alternative sections—is discarded in this projection.

Proposition 0.6 (Quine as Degenerate Section)**.** *A Spherepop quine Q with $f(Q) \sim Q$ corresponds to a degenerate section: the history Q is already fully determined by its trace, admitting no further refinement and no alternative realisation.*

Proof. The quine condition $f(Q) \sim Q$ means that Q and its trace lie in the same equivalence class: the fiber $\mathbf{Hist}_{\tau(Q)}$ contains Q as its unique representative up to trace equivalence. The section over $\tau(Q)$ has no choice: it must select Q. There is exactly one global section through Q, and the sheaf at $\tau(Q)$ is a singleton. □ □

The Complete Sheaf-Theoretic Correspondence

The sheaf formulation yields a precise dictionary between linguistic and sheaf-theoretic notions. The base space $\mathcal{N}$ carries identities as partial and total traces. The fibers $\mathbf{Hist}_\tau$ carry histories realising each identity. Global sections are coherent interpretations or parses. Gluing is grammatical composition from locally compatible constraints. Failure of gluing is ungrammaticality. Multiplicity of global sections is ambiguity. The projection from the sheaf to its base is irreversible collapse.

Meaning is therefore not a point but a section over a space of partial traces. Identity is the base point to which all sections project. Irreversibility ensures that only the base point survives globally; all local variation is confined to the sheaf structure above it.

Chapter D

Trace as Attractor: A Field-Theoretic Realisation in RSVP

Prerequisites

This appendix assumes Appendices A through C, familiarity with partial differential equations at the level of a first course in mathematical physics, and basic acquaintance with the RSVP framework. The field equations introduced here are presented heuristically; they are intended to establish the structural correspondence between the sheaf of histories and entropic field dynamics rather than to develop RSVP as a complete physical theory. Readers without a physics background may read this appendix as providing a continuous-space geometric intuition for the discrete algebraic constructions of the preceding chapters.

The RSVP Field as Semantic Substrate

Let the RSVP system be defined over a smooth manifold M equipped with three fields: a scalar density $\Phi(x,t)$, a vector flow $\mathbf{v}(x,t)$, and an entropy field $S(x,t)$. These fields evolve according to coupled local equations expressing conservation, transport, and dissipation.

We interpret the space of partial traces $(\mathcal{N}, \preceq)$ as a discretisation of regions in M. Each trace τ corresponds to a coherent field configuration:

$$\tau \longleftrightarrow (\Phi_\tau,\, \mathbf{v}_\tau,\, S_\tau).$$

Traces are therefore not merely symbolic objects but stable field patterns. The refinement order $\preceq$ corresponds to the inclusion of spatial regions in M: $\tau_1 \preceq \tau_2$ when the field configuration of τ_2 contains that of τ_1 as a coherent subsystem.

Histories as Trajectories in Field Configuration Space

Each history $E \in \mathbf{Hist}$ corresponds to a trajectory through the space of field configurations:

$$E : t \mapsto (\Phi(x,t),\, \mathbf{v}(x,t),\, S(x,t)).$$

The reduction operations of the Spherepop calculus correspond to local deformations of the field. The bind operator constrains compatible regions of Φ, reducing entropy locally by eliminating configurations that are incompatible with the imposed constraint. The pop operator eliminates unstable configurations, collapsing regions of high curvature in the scalar field toward nearby stable patterns. The refuse operator removes incompatible trajectories, increasing the entropy gradient in regions where no stable continuation exists.

The category **Hist** is therefore realised as a space of dynamical flows over M: objects are field configurations (positions in configuration space), and morphisms are trajectories (paths between configurations governed by the field equations).

Collapse as Entropic Descent

The evaluation map $f : \mathcal{E} \to \mathcal{N}$ corresponds to the long-time limit of field evolution. The scalar field evolves according to a reaction-diffusion-advection equation of the form

$$\frac{\partial \Phi}{\partial t} = D\nabla^2\Phi + \mathbf{v} \cdot \nabla\Phi + R(\Phi - \Phi^3),$$

coupled with entropy dynamics satisfying the second law:

$$\frac{\partial S}{\partial t} \geq 0.$$

The diffusion term $D\nabla^2\Phi$ smooths local irregularities. The advection term $\mathbf{v} \cdot \nabla\Phi$ transports structure along the flow field, implementing directed constraint propagation. The reactive term $R(\Phi - \Phi^3)$ drives the field toward the double-well potential minima, encoding the discrete commitments of reduction.

Under these dynamics, trajectories evolve toward low-entropy attractors. The long-time limit

$$\lim_{t\to\infty} E(t) = \tau(E)$$

is the normal-form trace of E: the stable field configuration that the trajectory converges to once all incompatible alternatives have been dissipated.

Fibers as Basins of Attraction

For each trace $\tau \in \mathcal{N}$, the fiber category $\mathbf{Hist}_\tau$ corresponds geometrically to the basin of attraction of the field configuration $(\Phi_\tau, \mathbf{v}_\tau, S_\tau)$:

$$\mathbf{Hist}_\tau \;\longleftrightarrow\; \big\{E : \lim_{t\to\infty} E(t) = \tau\big\}.$$

Different histories collapsing to the same trace are therefore trajectories within the same basin, approaching the same attractor along different paths. Ambiguity corresponds to the presence of multiple nearby attractors with comparable energy levels (metastable basins). The system may converge to either attractor depending on small perturbations in the initial conditions, exactly as an ambiguous utterance may resolve to either interpretation depending on contextual cues. Ungrammaticality corresponds to trajectories that fail to converge to any stable attractor: the field disperses rather than contracting, and no finite-energy terminal configuration is reached.

Sheaf Structure as Local Field Coherence

The sheaf structure of Appendix C acquires a geometric interpretation in the RSVP framework. A local section over a region $U \subseteq M$ corresponds to a locally coherent field patch: a restriction of the global field configuration to U that satisfies the field equations with boundary conditions consistent with the surrounding constraints.

Compatibility of local sections means that the field patches match smoothly on their boundaries: Φ and its derivatives agree on the overlapping region, and there is no discontinuity in the flow field $\mathbf{v}$ that would prevent extension. Gluing corresponds to forming a globally stable field configuration from locally compatible patches: a solution to the field equations over all of M that restricts to each local patch correctly.

Failure of gluing corresponds to a topological obstruction: locally compatible field patches cannot be assembled into a global solution. This occurs when the boundary conditions imposed by different local regions are mutually incompatible, preventing any smooth extension. Linguistically, this is the field-theoretic realisation of ungrammaticality as the failure of gluing in the sheaf of histories.

Trace as Minimal Entropy Configuration

Each trace τ can be characterised as a local minimum of the effective functional

$$\mathcal{F}[\Phi] = \int_M \Big(\|\nabla\Phi\|^2 + V(\Phi) \Big) \, \mathrm{d}V,$$

where $V(\Phi) = \frac{1}{4}(\Phi^2 - 1)^2$ is a double-well potential encoding the two stable values of a discrete commitment. The gradient term penalises spatial inhomogeneity, driving the field toward uniform configurations, while the potential term penalises departures from the committed values ± 1.

In this language, collapse corresponds to the minimisation of $\mathcal{F}$ under the constraint that the entropy S is non-decreasing. Identity corresponds to a stable critical point of $\mathcal{F}$: a field configuration that cannot be further deformed without increasing $\mathcal{F}$. Irreversibility corresponds to the dissipative dynamics: trajectories descend monotonically in $\mathcal{F}$ and cannot ascend without violating the entropy condition.

The Spherepop Quine as a Fixed-Point Attractor

A Spherepop quine Q satisfying $f(Q) \sim Q$ corresponds, in the field picture, to a trajectory that is already located within its attractor basin from the outset:

$$E_Q(t) = \tau_Q \quad \text{for all } t \geq 0.$$

The field configuration of Q is already a stable critical point of $\mathcal{F}$: no dissipative evolution is required because the system is already in its minimal-energy state. A quine is therefore not a loop in time but a static point in configuration space—a field pattern that the dynamics would produce as a terminal state, but which is already present as an initial condition. It requires no further collapse because it is already the attractor.

Identity as Entropic Endpoint

THEOREM 0.1 (Identity as Entropic Attractor). *In an RSVP-realised Spherepop system, the identity of an expression E is the field attractor τ_E toward which its trajectory converges under entropy-driven dynamics:*

$$\mathrm{id}(E) = \lim_{t\to\infty} E(t) = \tau_E.$$

PROOF (SKETCH). Irreversibility ensures that $\mathcal{F}$ is non-increasing along trajectories ($\partial\mathcal{F}/\partial t \leq 0$). The dynamics therefore admit no limit cycles or divergent trajectories: every trajectory either converges to a stable critical point or dissipates entirely. Stable critical points of $\mathcal{F}$ correspond bijectively to elements of $\mathcal{N}$ (the discrete committed values of the double-well potential). All trajectories that do not terminate at such a critical point are eliminated by the entropy condition. Hence the identity of E—its terminal stable configuration—is determined uniquely by the attractor. □ □

The Unified Picture

The RSVP realisation completes a sequence of identifications that spans the entire monograph. At the symbolic level, Spherepop reduction is a sequence of discrete collapse operations on expressions. At the categorical level, these operations form morphisms in the category **Hist**, which projects onto the discrete base **NF** via the collapse functor. At the sheaf-theoretic level, the fibers of this projection organise into a presheaf over $(\mathcal{N}, \preceq)$ satisfying the gluing condition, making locality and coherence mathematically precise. At the field-theoretic level, the fibers correspond to basins of attraction of the RSVP dynamics, and the global sections of the sheaf correspond to globally stable field configurations.

In each representation, the same structure appears under a different description. Identity is the fixed point of an iterated contraction. The trace is the attractor.

The preimage is the basin. The sheaf section is the consistent global configuration. Irreversibility is the entropy condition that prevents trajectories from ascending.

Corollary 0.2 (Structural Unity)**.** *The logic of irreversible computation, the structure of natural language under H-LFG, and the dynamics of RSVP fields are not merely analogous. They are three representations of a single abstract object: a contractive dynamical system over a space of configurations, governed by the condition that entropy does not decrease, whose fixed points are identities.*

Bibliography

[1] Priest, G. (2006). *In Contradiction: A Study of the Transconsistent*. Oxford University Press, 2nd edition.

[2] Priest, G. (2006). Revising Logic. In *The Law of Non-Contradiction: New Philosophical Essays*, eds. Priest, G., Beall, J. C., and Armour-Garb, B. Oxford University Press.

[3] Fant, K. M. and Brandt, S. A. (1996). NULL Convention Logic: A Complete and Consistent Logic for Asynchronous Digital Circuit Synthesis. *Proceedings of the International Conference on Application Specific Systems, Architectures, and Processors*, IEEE.

[4] Fant, K. M. (2005). *Logically Determined Design: Clockless System Design with NULL Convention Logic*. Wiley-Interscience.

[5] Barendregt, H. P. (1984). *The Lambda Calculus: Its Syntax and Semantics*. North-Holland.

[6] Church, A. (1941). *The Calculi of Lambda-Conversion*. Princeton University Press.

[7] Backus, J. (1978). Can Programming Be Liberated from the von Neumann Style? *Communications of the ACM*, 21(8):613–641.

[8] Plotkin, G. D. (1977). LCF Considered as a Programming Language. *Theoretical Computer Science*, 5(3):223–255.

[9] Winskel, G. (1993). *The Formal Semantics of Programming Languages*. MIT Press.

[10] Lambek, J. and Scott, P. J. (1986). *Introduction to Higher-Order Categorical Logic*. Cambridge University Press.

[11] Mac Lane, S. (1998). *Categories for the Working Mathematician*. Springer, 2nd edition.

[12] Borceux, F. (1994). *Handbook of Categorical Algebra*, vols. 1–3. Cambridge University Press.

[13] Abramsky, S. and Jung, A. (1993). Domain Theory. In *Handbook of Logic in Computer Science*, Vol. 3. Oxford University Press.

[14] Terese (2003). *Term Rewriting Systems*. Cambridge University Press.

[15] Kaplan, R. and Bresnan, J. (1982). Lexical-Functional Grammar: A Formal System for Grammatical Representation. In *The Mental Representation of Grammatical Relations*, ed. Bresnan, J. MIT Press.

[16] Landauer, R. (1961). Irreversibility and Heat Generation in the Computing Process. *IBM Journal of Research and Development*, 5(3):183–191.

[17] Bennett, C. H. (1973). Logical Reversibility of Computation. *IBM Journal of Research and Development*, 17(6):525–532.

[18] Jaynes, E. T. (1957). Information Theory and Statistical Mechanics. *Physical Review*, 106(4):620–630.

[19] Turing, A. M. (1936). On Computable Numbers, with an Application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society*, 42:230–265.

[20] Kripke, S. A. (1963). Semantical Considerations on Modal Logic. *Acta Philosophica Fennica*, 16:83–94.

[21] Lambek, J. (1980). From Lambda-Calculus to Cartesian Closed Categories. In *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, eds. Hindley, J. R. and Seldin, J. P. Academic Press.